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CHARACTERIZATIONS OF STABILITY OF ABSTRACT DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we investigate many types of stability, like (uniform stability, exponential stability and h-stability) of the first order dynamic equations of the form

$$\begin{cases} u^{\Delta}(t) = Au(t) + f(t), & t \in \mathbb{T},\ t > t_0 \\ u(t_0) = x \in D(A), \end{cases}$$

and

$$\begin{cases} u^{\Delta}(t) = Au(t) + f(t,u), & t \in \mathbb{T},\ t > t_0 \\ u(t_0) = x \in D(A), \end{cases}$$

in terms of the stability of the homogeneous equation

$$\begin{cases} u^{\Delta}(t) = Au(t), & t \in \mathbb{T},\ t > t_0 \\ u(t_0) = x \in D(A), \end{cases}$$

where f is rd-continuous in $t \in \mathbb{T}$ and with values in a Banach space X, with $f(t,0) = 0$, and A is the generator of a C_0-semigroup $\{T(t) : t \in \mathbb{T}\} \subset L(X)$, the space of all bounded linear operators from X into itself. Here $D(A)$ is the domain of A and $\mathbb{T} \subseteq \mathbb{R}^{\geq 0}$ is a time scale which is an additive semigroup with property that $a - b \in \mathbb{T}$ for any $a, b \in \mathbb{T}$ such that $a > b$. Finally, we give illustrative examples.

1. Introduction and preliminaries

The theory of dynamic equations on time scales was introduced by Stefan Hilger in 1988 [16], in order to unify continuous and discrete calculus [2, 17]. We refer the reader to the very interesting monographs [3, 4] for more details about calculus on time scales. Concepts of stability are defined by various ways and some of these definitions are not adapted to each other. This is mainly due to what kind of exponential function authors used to define the exponential stability of solutions of dynamic equations. Pötzsche [25] gave the definition by the regular exponential function $e^{-p(t-t_0)}$ (p is a positive constant). Dacunha [9] defined the exponential stability in terms of $e_{-p}(t, t_0)$

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(p is a positive constant and $-p \in \mathcal{R}^+$) and LIU [18] introduced the definition by the use of the generalized time scale exponential functions $e_{\ominus p}(t, t_0)$. Du and Tien [11] characterized the exponential and uniformly exponential stability for linear dynamic equations via solvability of non-regressive non-homogeneous dynamic equations in the space of bounded rd-continuous functions. Choi, Koo and Im [8] investigated the h-stability for nonlinear perturbed dynamic system

$$z^{\Delta}(t) = A(t)z(t) + g(t, z(t)),$$

where $g \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$ by using concept of Bihari type inequality on time scales and the unified time scale quadratic Lyapunov functions. Doan, Kalauch and Siegmund [10] established necessary and sufficient conditions for the existence of uniform exponential stability and characterized the uniform exponential stability of a system by the spectrum of its matrix. Choi, Goo and Koo [5] investigated the h-stability for dynamic systems with non-regressivity condition in terms of transition matrix. The notion of h-stability was introduced by Pinto [24]. For detailed results about h-stability for linear dynamic equations on time scales, we refer the reader to the papers [5,8]. Choi and Koo [7] studied the stability of solutions for linear dynamic equations on time scales by using the concept of u_∞-quasisimilarity and dynamic inequalities. Mihiţ [20] studied the uniform h-stability of the evolution operators on Banach spaces.

An operator $A : \mathbb{T} \longrightarrow L(X)$, the space of all bounded linear operators from a Banach space X into itself, is called regressive if $I + \mu(t)A(t)$ is invertible for every $t \in \mathbb{T}$, and we say that

$$x^{\Delta}(t) = A(t)x(t),\ t \in \mathbb{T}$$

is regressive if A is regressive. Here $\mu(t)$ is the graininess function on a time scale $\mathbb{T}$. We say that a real valued function $p(t)$ on $\mathbb{T}$ is regressive (resp. positively regressive) if $1 + \mu(t)p(t) \neq 0$ (resp. $1 + \mu(t)p(t) > 0$), $t \in \mathbb{T}$. The family of all regressive functions (resp. positively regressive functions) is denoted by $\mathcal{R}$ (resp. $\mathcal{R}^+$).

It is well-known that if $A \in BC_{rd}\mathcal{R}(\mathbb{T}, L(X))$, the space of all right dense continuous and regressive bounded functions from $\mathbb{T}$ to $L(X)$, then the homogeneous initial value problem (IVP)

$$x^{\Delta}(t) = A(t)x(t),\ t \in \mathbb{T}, \quad x(s) = x_s \in X \tag{1}$$

has the unique solution

$$x(t) = e_A(t, s)x_s,$$

and the non-homogeneous IVP

$$x^{\Delta}(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}, \quad x(s) = x_s \in X \tag{2}$$

has the unique solution

$$x(t) = e_A(t, s)x_s + \int_s^t e_A(t, \sigma(s))f(s)\,\Delta s.$$

Also any solution of the perturbed IVP

$$x^{\Delta}(t) = A(t)x(t) + f(t,x),\ t \in \mathbb{T},\quad x(s) = x_s \in X \tag{3}$$

satisfies the integral equation

$$x(t) = e_A(t,s)x_s + \int_s^t e_A(t,\sigma(s))f(s,x(s))\,\Delta s.$$

Here $e_A(t,s)$ is the exponential operator function. For more details, see [13].

In [14], it was proved that the homogeneous equation

$$\begin{cases} u^{\Delta}(t) = Au(t), & t \in \mathbb{T},\ t > t_0 \\ u(t_0) = x \in D(A), \end{cases} \tag{4}$$

has a unique solution which is given by

$$x(t) = T(t-t_0)x,\ t \geq t_0,$$

when A is the generator of a C_0-semigroup of bounded linear operators $\{T(t) : t \in \mathbb{T}\}$, $\mathbb{T} \subseteq \mathbb{R}^{\geq 0}$ is a time scale which is an additive semigroup with property that $a - b \in \mathbb{T}$ for any $a, b \in \mathbb{T}$ such that $a > b$. Here, $D(A)$ is the domain of A. We dropped the condition of regressiveness and boundedness of A for the existence and uniqueness [3] of solutions of the homogeneous IVP (4). Also, many characterizations of stability of Equation (4) were obtained. Necessary and sufficient conditions for a linear operator A to be the generator of a C_0-semigroup were derived in [15].

In this paper, we establish the existence and uniqueness of solutions of non-homogeneous dynamic equations of the form

$$\begin{cases} u^{\Delta}(t) = Au(t) + f(t), & t \in \mathbb{T},\ t > t_0 \\ u(t_0) = x \in D(A), \end{cases} \tag{5}$$

and we prove that it is given by

$$u(t) = T(t-t_0)x + \int_{t_0}^t T(t-\sigma(s))f(s)\Delta s.$$

Also, we want to go further in stability of dynamic equations. We investigate many types of stability, like (uniform stability, exponential stability and h-stability) of both of Equation (5) and the equation

$$\begin{cases} u^{\Delta}(t) = Au(t) + f(t,u), & t \in \mathbb{T},\ t > t_0 \\ u(t_0) = x \in D(A), \end{cases} \tag{6}$$

in terms of the stability of the homogeneous equation (4).

When we consider the exponential operator function $e_A(t,s)$, we need the concept of regressiveness since $e_A(t,s)$ is defined only for $A(t)$ regressive. The continuous dynamic equation (e.g. ordinary differential equations) are always regressive since $\mathbb{T} = \mathbb{R}$ has the graininess function $\mu(t) \equiv 0$. However, non-regressivity is always possible in discrete dynamic equations (e.g. difference equations). In fact, if there is even one point in $\mathbb{T}$ with non zero graininess, then nonregressivity is possible [19].

Now, we introduce some definitions of strongly continuous semigroups (C_0-semigroups) $T = \{T(t) : t \in \mathbb{T}\} \subset L(X)$, and its generator A, where $\mathbb{T} \subseteq \mathbb{R}^{\geq 0}$ is a time scale which is an additive semigroup with property that $a - b \in \mathbb{T}$ for any $a, b \in \mathbb{T}$ such that $a > b$. See [14, 15, 22].

Definition. A C_0-semigroup T on X is a family of linear bounded operators $\{T(t) : t \in \mathbb{T}\} \subset L(X)$, satisfying

(1) $T(t + s) = T(t)T(s)$ for every $t, s \in \mathbb{T}$ (the semigroup property).
(2) $T(0) = I$, (I is the identity operator on X).
(3) $\lim_{t\to 0^+} T(t)x = x$ (i.e., $T(\cdot)x : \mathbb{T} \longrightarrow X$ is continuous at 0) for each $x \in X$.

If in addition $\lim_{t\to 0^+} \|T(t) - I\| = 0$, then T is called a uniformly continuous semigroup.

Definition. We say that a linear operator A is the generator of T if

$$Ax = \lim_{s \longrightarrow 0^+} \frac{T(\mu(t))x - T(s)x}{\mu(t) - s}, \ x \in D(A), \tag{7}$$

where the domain $D(A)$ of A is the set of all $x \in X$ for which the above limit exists uniformly in t.

We refer the reader to [14, 15] for more details about the properties of a C_0-semigroup T and its generator A.

It is known that when a linear operator A is the generator of a C_0-semigroup of operators $\{T(t) : t \in \mathbb{R}^{\geq 0}\}$ on X, the non-homogeneous IVP

$$\begin{cases} x'(t) = Ax(t) + f(t), \quad t > 0 \\ x(0) = x \in D(A), \end{cases} \tag{8}$$

has the unique solution

$$x(t) = T(t)x + \int_0^t T(t - s)f(s)ds, \ t \in \mathbb{R}^{\geq 0},$$

provided that $f : [0, \infty[\to X$ is continuously differentiable on $[0, \infty[$. See e.g. [12, 23]. The paper is organized as follows. In Section 2, we establish necessary and sufficient conditions for the non-homogeneous IVP (5) to have a unique solution, when A is the generator of a C_0-semigroup T and $f \in C_{rd} = C_{rd}(\mathbb{T}, X)$, the space of rd-continuous functions from $\mathbb{T}$ to X. Section 3 is devoted to investigate the stability, uniform stability, exponential stability and h-stability of the IVPs (5) and (6). Section 4 includes some illustrative examples.

2. The existence and uniqueness of solutions of the non-homogeneous initial value problem

In this section we consider the non-homogeneous IVP

$$\begin{cases} u^{\Delta}(t) = Au(t) + f(t), \quad t \in \mathbb{T} \\ u(0) = x_0, \end{cases} \tag{9}$$

where $f \in C_{rd}$ and A is the generator of a C_0-semigroup T. So, by Theorem 2.4 in [14] the corresponding homogeneous IVP of (9) has a unique solution

$$u(t) = T(t)x_0 \text{ for every initial value } x_0 \in D(A).$$

Definition. A function $u : \mathbb{T} \to X$ is a (classical) solution of (9) on $\mathbb{T}$ if $u \in C^1_{rd}$, $u(t) \in D(A)$ for $t \in \mathbb{T}$ and (9) is satisfied on $\mathbb{T}$.

Definition. If $f \in C_{rd}$ and T is a C_0-semigroup generated by A, then we define

$$\int_0^t T(t - \sigma(s))f(s)\Delta s := \lim_{\acute{t} \to t^-} \int_0^{\acute{t}} T(t - \sigma(s))f(s)\Delta s,\ t \in \mathbb{T}.$$

The limit on the right-hand side exists by Cauchy Criteria.

Lemma 2.1. *If $f \in C_{rd}$, then any solution of* (9) *with initial value $x \in X$ is given by*

$$u(t) = T(t)x + \int_0^t T(t - \sigma(s))f(s)\Delta s. \tag{10}$$

Proof. Let u be a solution of (9). Then the function $\phi(s) = H_t(s)u(s)$ is differentiable for $s \in]0, t[_{\mathbb{T}}$, where $H_t(s) = T(t - s)$ and

$$\begin{aligned}\phi^{\Delta}(s) &= H_t(\sigma(s))u^{\Delta}(s) + H_t^{\Delta}(s)u(s)\\ &= T(t - \sigma(s))[Au(s) + f(s)] - T(t - \sigma(s))Au(s)\\ &= T(t - \sigma(s))f(s). \qquad (11)\end{aligned}$$

Integrating (11) from 0 to t, we get (10). ☐

For every $f \in C_{rd}$, the right-hand side of (10) is a rd-continuous function on $\mathbb{T}$. It is natural to consider it as a generalized solution of (9) even if it is not differentiable and does not strictly satisfy the equation in the sense of Definition 2.1. We therefore introduce the so called mild solutions in the following definition.

Definition. Let $x \in X$ and $f \in C_{rd}$. The function $u \in C_{rd}$ given by

$$u(t) = T(t)x + \int_0^t T(t - \sigma(s))f(s)\Delta s, \quad t \in \mathbb{T},$$

is called the mild solution of the IVP (9) on $\mathbb{T}$.

We start with a general criterion for existence and uniqueness of solutions of the IVP (9).

Theorem 2.2. *Let A be the generator of a C_0-semigroup T, $f \in C_{rd}$ and*

$$v(t) = \int_0^t T(t - \sigma(s))f(s)\Delta s, \quad t \in \mathbb{T}. \tag{12}$$

Assume one of the following conditions is satisfied;

(i) *$v(t)$ is rd-continuously differentiable on $\mathbb{T}$.*

(ii) *$v(t) \in D(A)$ for $t \in \mathbb{T}$ and $Av(t)$ is rd-continuous on $\mathbb{T}$.*

Then the IVP (9) *has a unique solution u on $\mathbb{T}$ for every $x \in D(A)$. Conversely, if* (9) *has a solution u on $\mathbb{T}$ for some $x \in D(A)$, then v satisfies both* (i) *and* (ii).

Proof. If the IVP (9) has a solution u for some $x \in D(A)$, then this solution is given by (10). Consequently $v(t) = u(t) - T(t)x$ is differentiable for $t \in \mathbb{T}$ as the difference of two differentiable functions and $v^{\Delta}(t) = u^{\Delta}(t) - T(t)Ax$ is obviously rd-continuous on $\mathbb{T}$. Therefore (i) is satisfied. Also if $x \in D(A)$, $T(t)x \in D(A)$, $AT(t)x = T(t)Ax$ for $t \in \mathbb{T}$ [14, Theorem 2.2], and thereby $v(t) = u(t) - T(t)x \in D(A)$ for $t \in \mathbb{T}$ and $Av(t) = Au(t) - AT(t)x = u^{\Delta}(t) - f(t) - T(t)Ax$ is rd-continuous on $\mathbb{T}$. Thus also (ii) is satisfied.

On the other hand, for $h > 0$,

$$\begin{aligned}
\frac{T(h)-T(\mu(t))}{h-\mu(t)}v(t) &= \frac{1}{h-\mu(t)}\left[\int_0^t T(t+h-\sigma(s))f(s)\Delta s\right.\\
&\quad \left. - \int_0^t T(t+\mu(t)-\sigma(s))f(s)\Delta s\right]\\
&= \frac{1}{h-\mu(t)}\left[\int_0^{t+h} T(t+h-\sigma(s))f(s)\Delta s\right.\\
&\quad - \int_0^{t+\mu(t)} T(t+\mu(t)-\sigma(s))f(s)\Delta s\\
&\quad - \int_t^{t+h} T(t+h-\sigma(s))f(s)\Delta s\\
&\quad \left. + \int_t^{t+\mu(t)} T(t+\mu(t)-\sigma(s))f(s)\Delta s\right]\\
&= \frac{v(t+h)-v(t+\mu(t))}{h-\mu(t)}\\
&\quad - \frac{1}{h-\mu(t)}\int_t^{t+h} T(t+h-\sigma(s))f(s)\Delta s\\
&\quad + \frac{1}{h-\mu(t)}\mu(t)f(t). \qquad (13)
\end{aligned}$$

It is clear that the right-hand side of (13) has the limit $v^{\Delta^+}(t) - f(t)$ as $h \to 0^+$. If $v(t)$ is rd-continuously differentiable on $\mathbb{T}$, then it follows from (13) that $v(t) \in D(A)$ for $t \in \mathbb{T}$, $t > 0$ and $Av(t) = v^{\Delta}(t) - f(t)$. This implies that $u(t) = T(t)x + v(t)$ is the solution of the IVP (9) for $x \in D(A)$. If $v(t) \in D(A)$ it follows from (13) that $v(t)$ is differentiable from the right at t and the right derivative $v^{\Delta^+}(t)$ of v satisfies $v^{\Delta^+}(t) = Av(t) + f(t)$. Since $v^{\Delta^+}(t)$ is rd-continuous, $v(t)$ is rd-continuously differentiable and $v^{\Delta}(t) = Av(t) + f(t)$. Again, we obtain $u(t) = T(t)x + v(t)$ is the solution of (9) for $x \in D(A)$. $\square$

As a consequence of the previous result, we can see the following:

(i) In the continuous case $\mathbb{T} = \mathbb{R}^{\geq 0}$ [23], the solution (10) yields
$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

(ii) In the discrete case $\mathbb{T} = h\mathbb{Z}^{\geq 0}$, $h > 0$, the solution (10) yields
$$u(t) = T(t)x + \sum_{i=0}^{t-1} T(t-ih)f(ih).$$

3. Stability of abstract initial value problems

In this section, we study many types of stability, like (uniform stability, exponential stability and h-stability) of the non-homogeneous IVP
$$CP(f) : \begin{cases} u^{\Delta}(t) = Au(t) + f(t), & t \in \mathbb{T},\ t > t_0 \\ u(t_0) = x \in D(A), \end{cases}$$
in terms of the stability of homogeneous IVP
$$CP(0) : \begin{cases} u^{\Delta}(t) = Au(t), & t \in \mathbb{T},\ t > t_0 \\ u(t_0) = x \in D(A), \end{cases}$$
where $f \in C_{rd}$ and A is the generator of a C_0-semigroup T. In [14], we studied many types of stability of $CP(0)$.

The definitions of the types of stability of the dynamic equations of the form
$$x^{\Delta}(t) = F(t,x),\ x(t_0) = x_0 \in X,\ t, t_0 \in \mathbb{T} \tag{14}$$
are presented, where $F : \mathbb{T} \times X \to X$ is rd-continuous in the first argument with $F(t,0) = 0$. See [5,6,9,14,20,21].

Definition. Eq. (14) is said to be stable if, for every $t_0 \in \mathbb{T}$ and for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that, for any two solutions $x(t) = x(t, t_0, x_0)$ and $\overline{x}(t) = x(t, t_0, \overline{x}_0)$ of Eq. (14), the inequality $\|x_0 - \overline{x}_0\| < \delta$ implies $\|x(t) - \overline{x}(t)\| < \epsilon$ for all $t \geq t_0$, $t \in \mathbb{T}$.

Definition. Eq. (14) is said to be uniformly stable if, for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ independent on any initial point t_0 such that, for any two solutions $x(t) = x(t, t_0, x_0)$ and $\overline{x}(t) = x(t, t_0, \overline{x}_0)$ of Eq. (14), the inequality $\|x_0 - \overline{x}_0\| < \delta$ implies $\|x(t) - \overline{x}(t)\| < \epsilon$ for all $t \geq t_0$, $t \in \mathbb{T}$.

Definition. Eq. (14) is said to be exponentially stable if there exist $\alpha > 0$ with $-\alpha \in \mathcal{R}^+$ and $\gamma : \mathbb{T} \times \mathbb{R}^{\geq 0} \to \mathbb{R}^+$ which is rd-continuous in the first argument and continuous in the second argument such that, any solution $x(t) = x(t, t_0, x_0)$ of Eq. (14) satisfies $\|x(t)\| \leq \gamma(t_0, \|x_0\|)e_{-\alpha}(t, t_0)$ for all $t \geq t_0$, $t \in \mathbb{T}$.

Definition. Eq. (14) is said to be uniformly exponentially stable if γ is independent of $t_0 \in \mathbb{T}$.

Definition. Let $h : \mathbb{T} \to \mathbb{R}$ be a positive bounded function. We say that Eq. (14) is h-stable if there exists $\gamma : \mathbb{T} \times \mathbb{R}^{\geq 0} \to \mathbb{R}^{+}$ which is rd-continuous in the first argument and continuous in the second argument such that any solution $x(t) = x(t, t_0, x_0)$ of Eq. (14) satisfies

$$\|x(t, t_0, x_0)\| \leq \gamma(t_0, \|x_0\|)h(t)h(t_0)^{-1},\ t \geq t_0;$$

(here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition. Eq. (14) is called uniformly h-stable if γ is independent of $t_0 \in \mathbb{T}$.

Theorem 3.1. *The following statements are equivalent*:

(i) $CP(0)$ *is stable.*
(ii) *For every* $\epsilon > 0$ *and* $t_0 \in \mathbb{T}$ *there exists* $\delta = \delta(t_0, \epsilon) > 0$ *such that for any solution* $x(t) = x(t, t_0, x_0)$ *of* $CP(0)$*, we have*

$$\|x_0\| < \delta \Longrightarrow \|x(t)\| < \epsilon;$$

(iii) $\{\|T(t - t_0)\| : t \in \mathbb{T}, t \geq t_0\}$ *is bounded for every* $t_0 \in \mathbb{T}$.
(iv) $CP(0)$ *is uniformly stable.*
(v) *There exists* $\gamma > 0$ *such that for every* $t_0 \in \mathbb{T}$ *and for any solution* $x(t) = x(t, t_0, x_0)$ *of* $CP(0)$*, we have*

$$\|x(t)\| \leq \gamma\|x_0\|,\ t \geq t_0,\ t \in \mathbb{T}.$$

(vi) $CP(f)$ *is uniformly stable, for every* $f \in C_{rd}$.
(vii) $CP(f)$ *is stable, for every* $f \in C_{rd}$.

Proof. See [14, Lemmas 4.1, 4.2 and Theorem 4.3] for the proof of the equivalence (i)-(iv).

(iv)$\Rightarrow$(v) Assume that $CP(0)$ is uniformly stable. Let $\epsilon > 0$. There exists $\delta = \delta(\epsilon)$ independent on any initial point $t_0 \in \mathbb{T}$ such that for any two solutions $x(t) = x(t, t_0, x_0)$ and $\overline{x}(t) = x(t, t_0, \overline{x}_0)$ of $CP(0)$ with initial values $x_0, \overline{x}_0 \in D(A)$, the inequality $\|x_0 - \overline{x}_0\| < \delta$ implies $\|x(t) - \overline{x}(t)\| < \epsilon$, $t \geq t_0$, $t \in \mathbb{T}$.

Now, let $\epsilon = 1$. There is $\delta > 0$ such that for any $t_0 \in \mathbb{T}$ and for any solution $y(t) = y(t, t_0, y_0)$ of $CP(0)$, we have

$$\|y_0\| < \delta \Longrightarrow \|y(t)\| < 1,\ t \geq t_0,\ t \in \mathbb{T}.$$

Let $t_0 \in \mathbb{T}$, $0 \neq x_0 \in D(A)$ and $y_0 = \frac{\delta x_0}{2\|x_0\|}$. We have $\|y_0\| < \delta$, which implies $\|y(t, t_0, y_0)\| < 1$, namely, $\|\frac{\delta}{2\|x_0\|}T(t - t_0)x_0\| < 1$, and so $\|T(t - t_0)x_0\| < \frac{2}{\delta}\|x_0\|$. Take $\gamma = \frac{2}{\delta}$. Consequently,

$$\|x(t)\| \leq \gamma\|x_0\|,\ t \geq t_0, t \in \mathbb{T}.$$

(v)$\Rightarrow$(vi) Suppose that there is $\gamma > 0$ such that for any solution $x(t) = x(t, t_0, x_0)$ of $CP(0)$, we have

$$\|T(t - t_0)x_0\| \leq \gamma\|x_0\|.$$

Let $\epsilon > 0$. Take $\delta = \frac{\epsilon}{\gamma}$. For any $t_0 \in \mathbb{T}$, $x_0, \overline{x}_0 \in D(A)$ such that $\|x_0 - \overline{x}_0\| < \delta$, we have

$$\begin{aligned}\|x_f(t) - \overline{x}_f(t)\| &= \|T(t-t_0)(x_0 - \overline{x}_0)\| \leq \gamma\|x_0 - \overline{x}_0\| \\ &= \frac{\epsilon}{\delta}\|x_0 - \overline{x}_0\| < \epsilon,\ t \geq t_0,\ t \in \mathbb{T}.\end{aligned}$$

Therefore, $CP(f)$ is uniformly stable.

(vi)$\Longrightarrow$(vii) This implication can be obtained directly by the definition.

(vii)$\Longrightarrow$(i) This implication can be obtained directly, taking $f \equiv 0$. □

Theorem 3.2. *The following statements are true.*

(i) $CP(0)$ *is exponentially stable if and only if there exists* $\alpha > 0$ *with* $-\alpha \in \mathcal{R}^+$ *and there exists* $\gamma \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ *such that*

$$\|T(t)\| \leq \gamma(t_0)e_{-\alpha}(t + t_0, t_0),\ t \in \mathbb{T}.$$

(ii) $CP(0)$ *is uniformly exponentially stable if and only if the exists* $\alpha > 0$ *with* $-\alpha \in \mathcal{R}^+$ *and there exists a constant* $\gamma > 0$ *such that for any* $t_0 \in \mathbb{T}$,

$$\|T(t)\| \leq \gamma e_{-\alpha}(t + t_0, t_0),\ t \in \mathbb{T}.$$

Proof. See [14, Theorem 6.3] for the proof of (i). The proof of (ii) can be preformed in a similar way. □

In the following result we show that the exponential stability of $CP(0)$ is a sufficient condition for the boundedness of $CP(f)$, where $f \in BC_{rd}$.

Theorem 3.3. *If* $CP(0)$ *is exponentially stable, then for every* $f \in BC_{rd}$, *the solution* $x_f(\cdot)$ *of* $CP(f)$ *belongs to* BC_{rd}.

Proof. Assume $CP(0)$ is exponentially stable. Then there exist $\alpha > 0$ with $-\alpha \in \mathcal{R}^+$ and $\gamma \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that

$$\|T(t)\| \leq \gamma(t_0)e_{-\alpha}(t + t_0, t_0),\ t \geq t_0,\ t_0,\ t \in \mathbb{T}.$$

For every function $f \in BC_{rd}$, the solution of $CP(f)$ with initial value $x_0 \in D(A)$ is given by

$$x_f(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t - \sigma(s))f(s)\Delta s.$$

This implies

$$\|x_f(t)\| \leq \|T(t-t_0)x_0\| + \|\int_{t_0}^{t} T(t - \sigma(s))f(s)\Delta s\|.$$

We have

$$\begin{aligned}\|\int_{t_0}^{t} T(t-\sigma(s))f(s)\Delta s\| &\leq \int_{t_0}^{t} \|T(t-\sigma(s)\|\|f(s)\|\Delta s \\ &\leq \gamma\|f\|_\infty \int_{t_0}^{t} e_{-\alpha}(t, \sigma(s))\Delta s\end{aligned}$$

$$= \frac{\gamma\|f\|_\infty}{\alpha}(1 - e_{-\alpha}(t, t_0)) \leq \frac{\gamma\|f\|_\infty}{\alpha}.$$

Therefore,

$$\|x_f(t)\| \leq \gamma e_{-\alpha}(t, t_0)\|x_0\| + \frac{\gamma\|f\|_\infty}{\alpha}.$$

Hence, noting that $e_{-\alpha}(t, t_0) \to 0$ as $t \to \infty$ [18], it follows the boundedness of $x_f(\cdot)$. □

Theorem 3.4. *Let A be the generator of a C_0-semigroup T on X. Then the following statements are equivalent.*

(i) $\int_0^\infty \|T(s)x\|^p \Delta s < \infty$, $1 \leq p < \infty$ *for every* $x \in X$.

(ii) $\lim_{t\to\infty} \|T(t)x\| = 0$ *for every* $x \in X$.

(iii) $CP(0)$ *is uniformly exponentially stable.*

Proof. (i)⇒(ii) Since T is a C_0-semigroup on $\mathbb{T}$, then there are constants $M \geq 1$ and $\omega > 0$ such that $\|T(t)\| \leq Me_\omega(t, 0)$ (see [15]). We assume that $\lim_{t\to\infty} \|T(t)x\| \neq 0$. Then there are $x \in X$, $\delta > 0$ and $t_j \to \infty$ such that $\|T(t_j)x\| \geq \delta$. We can assume that $t_{j+1} - t_j > \frac{1}{\omega}$. Set $I_j = [t_j - \frac{1}{\omega}, t_j]_{\mathbb{T}}$. Then the intervals $\{I_j\}$ are disjoint. Indeed, let $x \in I_j \cap I_{j+1}$, then

$$t_j - \frac{1}{\omega} \leq x \leq t_j, t_{j+1} - \frac{1}{\omega} \leq x \leq t_{j+1}.$$

Hence, $t_{j+1} - t_j - \frac{1}{\omega} \leq 0$, this is a contradiction. Also, for $t \in I_j$ we have $\|T(t)x\| \geq \frac{\delta}{Me}$. Indeed,

$$\delta \leq \|T(t_j)x\| \leq \|T(t_j - t)\|\|T(t)x\|,$$

it follows that $\|T(t)x\| \geq \frac{\delta}{\|T(t_j - t)\|}$. But $\|T(t_j - t)\| \leq Me_\omega(t_j, t) \leq Me^{\omega(t_j - t)}$ $\leq Me$. Then $\|T(t)x\| \geq \frac{\delta}{Me}$. Therefore,

$$\int_0^\infty \|T(t)x\|^p \Delta t \geq \sum_{j=1}^\infty \int_{t_j - \frac{1}{\omega}}^{t_j} \|T(t)x\|^p \Delta t \geq (\frac{\delta}{Me})^p \sum_{j=1}^\infty \frac{1}{\omega} = \infty,$$

this is a contradiction. Therefore $\lim_{t\to\infty} \|T(t)x\| = 0$ for every $x \in X$.

(ii)⇒(iii) Condition (ii) and the uniform boundedness Theorem insure the boundedness of $\{\|T(t)\| : t \in \mathbb{T}\}$. To show (iii), we assume that $\|T(t)\| \leq M$, $t \in \mathbb{T}$ for some $M \geq 1$. Let $0 < \eta < \frac{1}{M}$. For $x \in X$ define $\theta_x(\eta)$ by

$$\theta_x(\eta) = \sup\{t \in \mathbb{T} : \|T(s)x\| \geq \eta\|x\|,\ s \in [0, t]_{\mathbb{T}}\}.$$

Since $\|T(t)x\| \to 0$ as $t \to \infty$, $\theta_x(\eta)$ is finite and positive for every $x \in X$. Therefore, $\theta_x(\eta) < t_1$. For $t \in \mathbb{T}_{t_1}$, we have

$$\|T(t)x\| \leq \|T(t - t_1)\|\|T(t_1)x\| \leq M\eta\|x\|. \tag{15}$$

Set $\beta = M\eta$, which is less than 1. Inequality (15) implies that

$$\|T(t)\| \leq \beta < 1,\ t \in \mathbb{T}_{t_1}.$$

Finally, fix $t_2 \in \mathbb{T}_{t_1}$, and let $\nu = \frac{-1}{t_2}\log\beta > 0$. Let $t \in \mathbb{T}$, then $t = nt_2 + s$ for some $n \in \mathbb{Z}^{\geq 0}$, and $s \in [0, t_2]_{\mathbb{T}}$. In view of $e^{\nu t} = \beta^{-\frac{t}{t_2}}$, we obtain

$$\begin{aligned}\|T(t)\| &\leq \|T(s)\|\|T(nt_2)\| \leq M\|T(t_2)\|^n \leq M\beta^n = \frac{M}{\beta}\beta^{n+1} \leq M_3\beta^{\frac{t}{t_2}} \\ &= M_3e^{-\nu t} \leq M_3e_{\ominus\nu}(t,0) \leq M_3e_{-\alpha}(t,0),\end{aligned}$$

where $\alpha = \frac{\nu}{1+\nu\mu}$, $M_3 \geq \frac{1}{\eta}$.

(iii)$\Rightarrow$(i) Assume that $CP(0)$ is uniformly exponentially stable. Then by Theorem 3.2(ii), there exists $\alpha > 0$ with $-\alpha \in \mathcal{R}^+$ and there exists $\gamma > 0$ such that for any $t_0 \in \mathbb{T}$, $\|T(t)\| \leq \gamma e_{-\alpha}(t + t_0, t_0) \leq \gamma e^{-\alpha t}$, $t \in \mathbb{T}$ see [18]. Therefore, for $1 \leq p < \infty$

$$\int_0^\infty \|T(s)x\|^p \Delta s \leq \gamma^p \|x\|^p \int_0^\infty e^{-\alpha p s}\Delta s < \infty. \qquad \square$$

In the following result we establish a necessary and sufficient condition for $CP(0)$ to be h-stable.

Theorem 3.5. *Let $h : \mathbb{T} \to \mathbb{R}$ be a positive bounded function on $\mathbb{T}$. $CP(0)$ is h-stable if and only if there exists $\gamma \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that*

$$\|T(t - t_0)\| \leq \gamma(t_0)h(t)h(t_0)^{-1},\ t \geq t_0,\ t \in \mathbb{T}.$$

Proof. Let $CP(0)$ be h-stable. Then there is $\gamma_1 \in C_{rd}(\mathbb{T} \times \mathbb{R}^{\geq 0}, \mathbb{R}^+)$ such that for any solution $x(t) = T(t - t_0)x$ of $CP(0)$ with initial value $x \in D(A)$, we have

$$\|T(t - t_0)x\| \leq \gamma_1(t_0, \|x\|)h(t)h(t_0)^{-1},\ t \geq t_0,\ t \in \mathbb{T}.$$

Using the density of $D(A)$ in X and Corollary 2.3 in [14], we obtain

$$\|T(t - t_0)x\| \leq \gamma_1(t_0, \|x\|)h(t)h(t_0)^{-1},\ x \in X,\ t \geq t_0, t \in \mathbb{T}.$$

This implies that

$$\|T(t - t_0)\| \leq \gamma(t_0)h(t)h(t_0)^{-1},\ t \geq t_0,\ t \in \mathbb{T},$$

where $\gamma(t_0) = \gamma_1(t_0, 1)$. Conversely, assume that there exists $\gamma \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that

$$\|T(t - t_0)\| \leq \gamma(t_0)h(t)h(t_0)^{-1},\ t \geq t_0,\ t \in \mathbb{T}.$$

Let $x(t, t_0, x_0) = T(t - t_0)x_0$ be any solution of $CP(0)$ with initial value $x_0 \in D(A)$. Then

$$\begin{aligned}\|x(t)\| &= \|T(t - t_0)x_0\| \\ &\leq \|T(t - t_0)\|\|x_0\| \\ &\leq \gamma(t_0)\|x_0\|h(t)h(t_0)^{-1},\ t \geq t_0,\ t \in \mathbb{T}.\end{aligned}$$

Therefore, $CP(0)$ is h-stable. $\square$

Choi, Goo and Koo in [5] extended the concept of h-stability introduced by Pinto [24] to dynamic equations. They investigated the h-stability for dynamic equations $CP(f)$ when $A, f \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$, $n \in \mathbb{N}$ and $M_n(\mathbb{R})$ is the family of all $n \times n$ real matrices, with the nonregressivity condition on A. Du and Tien in [11] studied the exponential stability for the perturbed equation

$$x^{\Delta}(t) = A(t)x(t) + f(t,x),\ t \in \mathbb{T}, \tag{16}$$

when $A(\cdot) \in C_{rd}(\mathbb{T}^+, L(X))$ and $f(t,x) : \mathbb{T}^+ \times X \to X$ is rd-continuous in the first argument with $f(t,0) = 0$. Now, we extend these results concerning the h-stability for Eq. (16) when $A(t) = A$ is the generator of T.

The solution of the equation

$$x^{\Delta}(t) = Ax(t) + f(t,x),\ x(t_0) = x_0,\ t \ge t_0,\ t, t_0 \in \mathbb{T}, \tag{17}$$

through $(t_0, x_0 \in D(A))$ satisfies

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-\sigma(s))f(s,x(s))\Delta s. \tag{18}$$

Theorem 3.6. *Let $h : \mathbb{T} \to \mathbb{R}$ be a positive bounded function on $\mathbb{T}$. If the following conditions are satisfied*

(i) *There is $\gamma \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that*

$$\|T(t-t_0)\| \le \gamma(t_0)h(t)h(t_0)^{-1},\ t \ge t_0,\ t \in \mathbb{T},$$

(ii) $\|f(t,x)\| \le L\|x\|$ *for all* $t \in \mathbb{T}$,

(iii) *There exists $\beta = \beta(t_0) \ge 0$ such that $\int_{t_0}^{\infty} \frac{\gamma(\sigma(s))h(s)}{h(\sigma(s))}\Delta s \le \beta < \infty$,*

then Eq. (17) *is h-stable.*

Proof. Let $x(t) = T(t-t_0)x_0$ be a solution of $CP(0)$ with initial value $x_0 \in D(A)$, we have

$$\|T(t-t_0)x_0\| \le \gamma(t_0)\|x_0\|h(t)h(t_0)^{-1},\ \ t \ge t_0,\ \ t \in \mathbb{T}.$$

For any $t_0 \in \mathbb{T}$, $t \ge t_0$, the solution of Eq. (17) satisfies

$$\begin{aligned}\|x(t)\| &\le \|T(t-t_0)x_0\| + \int_{t_0}^{t} \|T(t-\sigma(s))f(s,x(s))\|\Delta s \\ &\le \gamma(t_0)\|x_0\|h(t)h(t_0)^{-1} + Lh(t)\int_{t_0}^{t} \gamma(\sigma(s))h(\sigma(s))^{-1}\|x(s)\|\Delta s.\end{aligned}$$

Dividing by $h(t) > 0$ on both sides,

$$\frac{\|x(t)\|}{h(t)} \le \gamma(t_0)\frac{\|x_0\|}{h(t_0)} + L\int_{t_0}^{t} \frac{\gamma(\sigma(s))h(s)}{h(\sigma(s))}\frac{\|x(s)\|}{h(s)}\Delta s.$$

By using Gronwall's inequality on time scales [1], we obtain

$$\frac{\|x(t)\|}{h(t)} \le \gamma(t_0)\frac{\|x_0\|}{h(t_0)}e_{L\frac{\gamma(\sigma(s))h(s)}{h(\sigma(s))}}(t,t_0)$$

$$\begin{aligned}
&\leq \gamma(t_0)\frac{\|x_0\|}{h(t_0)}\exp(L\int_{t_0}^{t}\frac{\gamma(\sigma(s))h(s)}{h(\sigma(s))}\Delta s)\\
&\leq \gamma(t_0)\frac{\|x_0\|}{h(t_0)}\exp(L\int_{t_0}^{\infty}\frac{\gamma(\sigma(s))h(s)}{h(\sigma(s))}\Delta s)\\
&\leq \gamma(t_0)\frac{\|x_0\|}{h(t_0)}e^{L\beta}
\end{aligned}$$

for all $t \geq t_0$, t, $t_0 \in \mathbb{T}$. Thus

$$\|x(t)\| \leq d(t_0, \|x_0\|)h(t)h(t_0)^{-1},\ t \geq t_0,$$

where $d(t_0, \|x_0\|) = \gamma(t_0)\|x_0\|e^{L\beta(t_0)}$, $t_0 \in \mathbb{T}$. Therefore Eq. (17) is h-stable. □

Corollary 3.7. *Let $h : \mathbb{T} \to \mathbb{R}$ be a positive function such that both of $h(t)$ and $h(t)/h(\sigma(t))$ are bounded functions. If the following conditions are satisfied*

(i) *There is $\gamma \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that*

$$\|T(t - t_0)\| \leq \gamma(t_0)h(t)h(t_0)^{-1},\ t \geq t_0,\ t \in \mathbb{T},$$

(ii) *$\|f(t, x)\| \leq L\|x\|$ for all $t \in \mathbb{T}$,*

(iii) *There exists $\beta = \beta(t_0) \geq 0$ such that $\int_{t_0}^{\infty}\gamma(\sigma(s))\Delta s \leq \beta < \infty$,*

then Eq. (17) is h-stable.

Theorem 3.8. *Let $h : \mathbb{T} \to \mathbb{R}$ be a positive bounded function on $\mathbb{T}$. If the following conditions are satisfied*

(i) *There is $\gamma \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that*

$$\|T(t - t_0)\| \leq \gamma(t_0)h(t)h(t_0)^{-1},\ t \geq t_0,\ t \in \mathbb{T},$$

(ii) *There exists $\beta = \beta(t_0) \geq 0$ such that $\int_{t_0}^{\infty}\frac{\|f(s)\|\gamma(\sigma(s))}{h(\sigma(s))}\Delta s \leq \beta < \infty$,*

then $CP(f)$ is h-stable.

Proof. Let $x(t)$ be a solution of $CP(f)$ with initial value $x_0 \in D(A)$. Then it satisfies

$$\begin{aligned}
\|x(t)\| &\leq \|T(t - t_0)x_0\| + \int_{t_0}^{t}\|T(t - \sigma(s))f(s)\|\Delta s\\
&\leq \gamma(t_0)\|x_0\|h(t)h(t_0)^{-1} + h(t)\int_{t_0}^{t}\frac{\|f(s)\|\gamma(\sigma(s))}{h(\sigma(s))}\Delta s\\
&\leq d(t_0, \|x_0\|)h(t)h(t_0)^{-1},
\end{aligned}$$

where $d(t_0, \|x_0\|) = \gamma(t_0)\|x_0\| + \beta(t_0)h(t_0)$, $t_0 \in \mathbb{T}$. Therefore, $CP(f)$ is h-stable. □

Corollary 3.9. *Suppose that $CP(0)$ is uniformly h-stable. Then $CP(f)$ is uniformly h-stable if there exists a positive constant β such that for all $t_0 \in \mathbb{T}$,*

$$\int_{t_0}^{\infty}\frac{\|f(s)\|}{h(\sigma(s))}\Delta s \leq \beta.$$

Remark 3.10. (1) If $h(t) = e_{-\alpha}(t, 0)$ for some positive α with $-\alpha \in \mathcal{R}^+$, then the uniform h-stability of $CP(0)$ coincides with the uniform exponential stability of $CP(0)$.

(2) If $CP(0)$ is uniformly h-stable with $h(t) = e_{-\alpha}(t, 0)$ for some positive α with $-\alpha \in \mathcal{R}^+$ and $\int_{t_0}^{\infty} \frac{\|f(s)\|}{1-\alpha\mu(s)} e_{-\alpha}(t_0, s)\Delta s < \infty$ for each $t_0 \in \mathbb{T}$, then $CP(f)$ is uniformly exponentially stable.

Theorem 3.11. *If the following conditions are satisfied*:

(i) $CP(0)$ *is uniformly* h*-stable,*

(ii) $\|f(t, x)\| \le l(t)\|x\|$, $l \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$,

(iii) *There exists* $\beta \ge 0$ *such that* $\int_{t_0}^{\infty} \frac{l(s)h(s)}{h(\sigma(s))}\Delta s \le \beta$,

then Eq. (17) *is uniformly* h*-stable.*

Proof. Let $CP(0)$ be uniformly h-stable. Then there is a constant $\gamma > 0$ such that for any solution $x(t) = T(t-t_0)x_0$ of $CP(0)$ with initial value $x_0 \in D(A)$, we have

$$\|T(t - t_0)x_0\| \le \gamma\|x_0\|h(t)h(t_0)^{-1},\ t \ge t_0,\ t \in \mathbb{T}. \tag{19}$$

Using formula (18) with inequality (19) and condition (ii), we obtain

$$\begin{aligned}\|x(t)\| &\le \|T(t - t_0)x_0\| + \int_{t_0}^{t} \|T(t - \sigma(s))f(s, x(s))\|\Delta s\\ &\le \gamma\|x_0\|h(t)h(t_0)^{-1} + \gamma h(t) \int_{t_0}^{t} l(s)h(\sigma(s))^{-1}\|x(s)\|\Delta s.\end{aligned}$$

Dividing by $h(t) > 0$ on both sides,

$$\frac{\|x(t)\|}{h(t)} \le \gamma\frac{\|x_0\|}{h(t_0)} + \gamma \int_{t_0}^{t} \frac{l(s)h(s)}{h(\sigma(s))}\frac{\|x(s)\|}{h(s)}\Delta s.$$

By using Gronwall's inequality on time scales [1], we get

$$\begin{aligned}\frac{\|x(t)\|}{h(t)} &\le \gamma\frac{\|x_0\|}{h(t_0)} e_{\gamma\frac{l(s))h(s)}{h(\sigma(s))}}(t, t_0)\\ &\le \gamma\frac{\|x_0\|}{h(t_0)} e^{\gamma\beta}\end{aligned}$$

for all $t \ge t_0$, t, $t_0 \in \mathbb{T}$. Thus

$$\|x(t)\| \le d_1\|x_0\|h(t)h(t_0)^{-1},\ t \ge t_0,$$

where $d_1 = \gamma e^{\gamma\beta} > 0$. Therefore Eq. (17) is uniformly h-stable. □

4. Illustrative examples

Example 4.1. Consider the IVP

$$\begin{cases} u^{\Delta}(t) = Au(t) + f(t), & t \in \mathbb{T}\\ u(0) = x_0 \in D(A), \end{cases} \tag{20}$$

where $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -2 \end{smallmatrix}\right)$, $f(t) = \left(\begin{smallmatrix} 2t-2 \\ 0 \end{smallmatrix}\right)$, $t \in \mathbb{T}$ and $\mathbb{T} = \frac{1}{2}\mathbb{Z}^{\geq 0}$. The matrix A is non-regressive and it is the generator of the C_0-semigroup

$$T(t) = (I + \frac{1}{2}A)^{2t}, \ t \in \mathbb{T},$$

then

$$T(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ t \in \frac{1}{2}\mathbb{N}.$$

Therefore Eq. (20) has the unique solution given by

$$\begin{aligned} u(t) &= T(t)x_0 + \int_0^t T(t-\sigma(s))f(s)\Delta s, \ x_0 \in \mathbb{R}^2 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_0 + \sum_{s=0}^{t-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2s-2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_0 + \begin{pmatrix} \sum_{s=0}^{t-1} 2s-2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_0 + \begin{pmatrix} t^2-3t \\ 0 \end{pmatrix}. \end{aligned}$$

Consequently, $\|T(t)\| = 1$, $t \in \mathbb{T}$ which implies that the system (20) is uniformly stable.

Example 4.2. Consider the IVP (20) with $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $f(t) = \left(\begin{smallmatrix} e_1(t,0) \\ e_{-1}(t,0) \end{smallmatrix}\right)$, $t \in \mathbb{T} = \mathbb{Z}^{\geq 0}$. The operator A is non-regressive and it is the generator of the C_0-semigroup

$$T(n) = (I + A)^n = 2^n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ n \in \mathbb{Z}^{\geq 0}.$$

Then Eq. (20) has the unique solution given by

$$\begin{aligned} u(t) &= T(t)x_0 + \int_0^t T(t-\sigma(s))f(s)\Delta s, \ x_0 \in \mathbb{R}^2 \\ &= 2^t \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x_0 + \sum_{s=0}^{t-1} 2^{t-s-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^t \\ 0 \end{pmatrix} \\ &= 2^t \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x_0 + 2^{2t-1} \sum_{s=0}^{t-1} 2^{-s} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 2^t \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x_0 + 2^t(2^t-1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Consequently, $\|T(t)\| = 2^{t+1}$, which is unbounded. Therefore Eq. (20) is not stable.

Example 4.3. Consider the IVP

$$CP(f): \begin{cases} u^{\Delta}(t) = Au(t) + f(t), \quad t_0 < t, \ t \in \mathbb{T} \\ u(t_0) = x_0, \end{cases}$$

where $A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & -2 \end{pmatrix}$ and $f(t) = \begin{pmatrix} e_{-1}(t,0) \\ 0 \end{pmatrix}$, $t \in \mathbb{T} = \frac{1}{2}\mathbb{Z}^{\geq 0}$. The operator A is non-regressive and it is the generator of the C_0-semigroup

$$T(t-t_0) = (I + \frac{1}{2}A)^{2(t-t_0)} = (\frac{1}{4})^{2(t-t_0)} \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}^t = (\frac{3}{4})^{2(t-t_0)} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix},$$

$t > t_0, \ t_0, t \in \mathbb{T}$. Thus $\|T(t-t_0)\| = \frac{\sqrt{10}}{3}(\frac{3}{4})^{2(t-t_0)} = \frac{\sqrt{10}}{3} e_{-\frac{1}{2}}(t, t_0)$. Therefore $CP(0)$ is uniformly stable and uniformly exponentially stable. Consequently, $CP(f)$ is uniformly stable. Now, take $h(t) = e_{-\frac{1}{2}}(t,0), \ \gamma > \frac{\sqrt{10}}{3}$. We have

$$\|T(t-t_0)\| \leq \gamma e_{-\frac{1}{2}}(t,0) e_{-\frac{1}{2}}(0,t_0) = \gamma h(t) h(t_0)^{-1}, \ t > t_0, \ t, t_0 \in \mathbb{T}.$$

Therefore $CP(0)$ is uniformly h-stable. Consequently, $CP(f)$ is uniformly h-stable. Indeed, we have

$$\begin{aligned} \sum_{s\in\mathbb{T}} \frac{\|f(s)\|}{h(\sigma(s))} &= \sum_{s\in\mathbb{T}} \frac{\|f(s)\|}{(1-\frac{1}{2}\mu(s)) e_{-\frac{1}{2}}(s,0)} \\ &\leq \sum_{s\in\mathbb{T}} \frac{4}{3} \frac{e_{-1}(s,0)}{e_{-\frac{1}{2}}(s,0)} \\ &= \sum_{s\in\mathbb{T}} \frac{4}{3} e_{-\frac{2}{3}}(s,0) \\ &= \frac{4}{3} \sum_{s\in\mathbb{T}} (1-\frac{1}{3})^{2s} \leq \frac{4}{3} \sum_{s=0}^{\infty} (\frac{4}{9})^s = \frac{12}{5}. \end{aligned}$$

Hence CP(f) is uniformly exponentially stable by Remark 3.10(2).

Example 4.4. Consider the perturbed initial value problem:

$$x^{\Delta}(t) = Ax(t) + q(t)x(t), \ t \in \mathbb{T} = \frac{1}{2}\mathbb{Z}^{\geq 0}, \tag{21}$$

where $A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & -2 \end{pmatrix}$ and q is a rd-continuous nonnegative function that satisfies $\sum_{s\in\mathbb{T}} q(s) < +\infty$. We can see that condition (iii) of Theorem 3.11 holds. Indeed, we have

$$\begin{aligned} \sum_{s\in\mathbb{T}} \frac{q(s)h(s)}{h(\sigma(s))} &= \sum_{s\in\mathbb{T}} \frac{q(s) e_{-\frac{1}{2}}(s,0)}{e_{-\frac{1}{2}}(\sigma(s),0)} \\ &= \sum_{s\in\mathbb{T}} \frac{q(s) e_{-\frac{1}{2}}(s,0)}{(1-\frac{1}{2}(\frac{1}{2})) e_{-\frac{1}{2}}(s,0)} \\ &= \frac{4}{3} \sum_{s\in\mathbb{T}} q(s) < +\infty. \end{aligned}$$

Also, conditions (i)-(ii) of Theorem 3.11 hold. Therefore, Eq. (21) is h-stable.

Example 4.5. Consider the perturbed IVP

$$x^{\Delta}(t) = Ax(t) + f(t, x(t)),\ t \in \mathbb{Z}^{\geq 0}, \tag{22}$$

where $A : l_2 \to l_2$ is an infinite matrix defined by $A = (a_{ij})_{i,j\in\mathbb{N}}$, where

$$a_{ij} = \begin{cases} 0, & i \neq j \\ -1 + \sqrt{\frac{(-1)^{i-1}}{(i-1)!}} & i = j, \end{cases}$$

and $f(t, x(t)) : \mathbb{Z}^{\geq 0} \times l_2 \to l_2$ is defined by $f(t, x(t)) = (\frac{1}{2})^t x(t)$. As usual $l_2 = \{(x_n) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$. The operator A is the generator of the C_0-semigroup $T(k) = (b_{ij}^k)_{i,j\in\mathbb{N}}$, where

$$b_{ij} = \begin{cases} 0, & i \neq j \\ \sqrt{\frac{(-1)^{i-1}}{(i-1)!}} & i = j. \end{cases}$$

So, $\|T(k)\| \leq e^{\frac{1}{2}}$. Therefore, $CP(0)$ is uniformly h-stable with a constant function $h = e^{\frac{1}{2}}$ and $\gamma = e^{\frac{1}{2}}$. Also, we can see that condition (iii) of Theorem 3.11 holds. Indeed, we have

$$\sum_{s\in\mathbb{T}} \frac{(\frac{1}{2})^s h(s)}{h(\sigma(s))} = \sum_{s\in\mathbb{T}} (\frac{1}{2})^s \leq \sum_{s=0}^{\infty} (\frac{1}{2})^s = \frac{1}{1-\frac{1}{2}} = 2.$$

Also, conditions (i)-(ii) in Theorem 3.11 hold. Therefore, Eq. (22) is uniformly h-stable.

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