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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A CLASS OF NONLOCAL PROBLEMS WITH DIRICHLET BOUNDARY CONDITION

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Abstract. In this article we are concerned with some non-local problems of Kirchhoff type with Dirichlet boundary condition in Orlicz-Sobolev spaces. A result of the existence of infinitely many solutions is established using variational methods and Ricceri's critical points principle modified by Bonanno.

1. Introduction

In this article the following Kirchhoff type problem in Orlicz-Sobolev space is studied.

$$\begin{cases} -M(\int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x)\mathrm{div}(a(|\nabla u|)\nabla u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $M : [0, +\infty) \to \mathbb{R}$ is a continuous function such that there exists a positive number m with $M(t) \geq m$ for all $t \geq 0$. Notice that if $\varphi(t) = p|t|^{p-2}t$ and $\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s$ for all $t \in \mathbb{R}$, then problem (1.1) becomes the well-known p-Kirchhoff type equation

$$\begin{cases} -M(\int_\Omega |\nabla u|^p\,\mathrm{d}x)\Delta_p u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

proposed by Kirchhoff, see [19]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the length changes of the string produced by transverse vibrations. Since the first equation in (1.2) contains an integral over Ω, it is no longer a pointwise identity, and

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therefore it is often called a nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population identity, see [9]. The parameters in (1.3) have the following meanings: h is the cross-section area, E is the Young modulus, ρ is the mass density, L is the length of the string, and P_0 is the initial tension. In recent years, p-Kirchhoff type problems have been studied by many researchers, we refer to [2, 8, 14, 21, 22, 27, 28] in which the authors have used different techniques to obtain the existence of solutions for (1.2). This problem in the case when $p(\cdot)$ is a continuous function, has also been studied in many papers, see for instance [6, 10, 13, 15, 16]. Also the problem (1.1) is studied in [18] with different boundary condition. Assume that $a : (0, +\infty) \to \mathbb{R}$ is a function such that the mapping, defined by

$$\varphi(t) := \begin{cases} a(|t|)t & t \neq 0, \\ 0 & t = 0, \end{cases} \tag{1.4}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. We can refer to [7, 17,23,24] for some nonlinear and nonhomogeneous versions of the problem (1.1) (when $M(t) \equiv 1$), which have been studied in Orlicz-Sobolev spaces. Motivated by the works above, we study the existence of weak solutions for problem (1.1), which is an extension from the previous studies on nonlocal problems in classical Sobolev spaces and on nonhomogeneous problems in Orlicz-Sobolev spaces.

2. Preliminaries

Now we introduce the spaces needed to study the problem (1.1), and give a brief review of some concepts and facts of Orlicz and Orlicz-Sobolev spaces, which are useful for our aim. We refer the readers for more details to [1, 7, 11, 12, 23–25].

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary $\partial\Omega$. $f, g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, and $\lambda > 0$ and $\mu \geq 0$ are two parameters. For $\varphi : \mathbb{R} \to \mathbb{R}$ defined in (1.4), set

$$\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s \qquad \forall t \in \mathbb{R},$$

on which we will impose some suitable condition later.

We see that Φ is a young function, i.e., $\Phi(0) = 0$, Φ is convex, and $\lim_{t\to\infty} \Phi(t) = +\infty$. Furthermore since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0, \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty.$$

The function Φ is then called an N-function. The function Φ^* defined by

$$\Phi^*(t) = \int_0^t \varphi^{-1}(s)\,\mathrm{d}s, \qquad \forall t \in \mathbb{R},$$

is called the complementary function of Φ and it satisfies,

$$\Phi^*(t) = \sup\{st - \Phi(s) : s \geq 0\}, \qquad \forall t \geq 0.$$

We observe that the function Φ^* is also an N-function and the following young's inequality holds

$$st \leq \Phi(s) + \Phi^*(t), \qquad \forall s, t \geq 0.$$

The Orlicz class defined by the N-function Φ is the set

$$K_\Phi(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable}; \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x < \infty \right\},$$

and the Orlicz space $L_\Phi(\Omega)$ is then defined as the linear hull of the set $K_\Phi(\Omega)$. The space $L_\Phi(\Omega)$ is a Banach space under the following Luxemburg norm,

$$\|u\|_\Phi := \inf \left\{ k > 0; \int_\Omega \Phi\left(\frac{u(x)}{k}\right)\,\mathrm{d}x \leq 1 \right\},$$

or the equivalent Orlicz norm

$$\|u\|_{L\Phi} := \sup \left\{ \left| \int_\Omega u(x)v(x)\,\mathrm{d}x \right|; v \in K_{\Phi^*}(\Omega); \int_\Omega \Phi^*(|v(x)|)\,\mathrm{d}x \leq 1 \right\}.$$

The Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is the space defined by

$$W^{1,\Phi}(\Omega) := \left\{ u \in L_\Phi(\Omega); \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega); i = 1, 2, \ldots, N \right\},$$

and it is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi.$$

Now, we introduce the Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$ which can be renormed by the equivalent norm

$$\|u\| := \||\nabla u|\|_\Phi.$$

Assume that Φ satisfies the following hypotheses;

$$1 < \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty, \tag{2.1}$$

$$N < \varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}, \tag{2.2}$$

we also need the following condition

$$\text{the function } t \to \Phi(\sqrt{t}) \text{ is convex for all } t \in [0, \infty). \tag{2.3}$$

Assumption (2.1) is equivalent with the fact that Φ and Φ^* both satisfy the Δ_2-condition (at infinity) [23]. Actually Δ_2-condition for Φ assures that both $L_\Phi(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are separable and Δ_2-condition for Φ and (2.3) assure that $L_\Phi(\Omega)$ is a uniformly convex space and thus a reflexive Banach space [23]. Consequently the Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ is also a reflexive Banach space.

Proposition 2.1 (see [7,23,24]). *Let $u \in W_0^{1,\Phi}(\Omega)$. Then;*

$$\|u\|^{\varphi^0} \leq \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x \leq \|u\|^{\varphi_0}; \quad \text{if} \quad \|u\| < 1,$$

$$\|u\|^{\varphi_0} \leq \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x \leq \|u\|^{\varphi^0}; \quad \text{if} \quad \|u\| > 1.$$

Lemma 2.1. *Let $u \in W_0^{1,\Phi}(\Omega)$ and*

$$\int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x \leq r \tag{2.4}$$

for some $0 < r < 1$. Then $\|u\| < 1$.

Proof. Arguing as in [4], by the definition,

$$\|u\| = \inf\left\{k > 0; \int_\Omega \Phi\left(\frac{|\nabla u(x)|}{k}\right)\,\mathrm{d}x \leq 1\right\}$$

for every $u \in W_0^{1,\Phi}(\Omega)$. Then (2.4) implies $\|u\| \leq 1$.

We first observe that

$$\Phi(t) \geq \tau^{\varphi^0}\Phi\left(\frac{t}{\tau}\right) \qquad \forall t > 0 \text{ and } \tau \in (0,1). \tag{2.5}$$

Arguing by contradiction, assume that there exists $u \in W_0^{1,\Phi}(\Omega)$ with $\|u\| = 1$, and such that (2.4) holds. Let us take $\xi \in (0,1)$. Using relation (2.5) we have

$$\int_\Omega \Phi(|\nabla u(x)|)\mathrm{d}x \geq \xi^{\varphi^0}\int_\Omega \Phi(|\nabla v(x)|)\,\mathrm{d}x, \tag{2.6}$$

where $v(x) := \frac{u(x)}{\xi}$ for all $x \in \Omega$. We have $\|v\| = \frac{1}{\xi} > 1$. By Proposition 2.1, we deduce that

$$\int_\Omega \Phi(|\nabla v(x)|)\,\mathrm{d}x \geq \|v\|^{\varphi_0} > 1. \tag{2.7}$$

Relation (2.6) and (2.7) show that

$$\int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x \geq \xi^{\varphi^0}.$$

Letting $\xi \nearrow 1$ in the above inequality we obtain

$$\int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x \geq 1,$$

that contradicts condition (2.4). $\square$

Lemma 2.2 ([5, Remark 2.1]). *Let $u \in W_0^{1,\Phi}(\Omega)$ be such that $\|u\| = 1$. Then*

$$\int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x = 1.$$

It follows from condition (2.2) and [11, Lemma D.2], that the Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ is continuously embedded in $W^{1,\varphi_0}(\Omega)$, and since $\varphi_0 > N$, by [20, Theorem 3.2.5] one has $W^{1,\varphi_0}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$. Hence we have the compact embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ and there exists a constant $c > 0$ such that:

$$\|u\|_\infty \leq c\|u\| \tag{2.8}$$

for all u in $W_0^{1,\Phi}(\Omega)$. In the rest of this section we recall the following multiple critical points theorem due to G. Bonanno [3] which can be regarded as supplements of the variational principle of Ricceri [26].

Proposition 2.2. *Let X be a reflexive real Banach space, let $J, I : X \to \mathrm{R}$ be two Gâteaux differentiable functionals such that J is sequentially weakly lower semicontinuous, strongly continuous and coercive, and I is sequentially weakly upper semicontinuous. For every $r > \inf_X J$, put*

$$\varphi(r) := \inf_{J(u)<r} \frac{\sup_{J(v)<r} I(v) - I(u)}{r - J(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \textit{and} \quad \delta := \liminf_{r\to(\inf_X J)^+} \varphi(r).$$

Then the following properties hold:

(a) *for every $r > \inf_X J$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional*

$$h_\lambda := J - \lambda I$$

to $J^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of h_λ in X.

(b) *if $\gamma < +\infty$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds either,*
  - (b_1) *h_λ possesses a global minimum, or*
  - (b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of h_λ such that*

$$\lim_{n\to+\infty} J(u_n) = +\infty.$$

(c) *if $\delta < +\infty$, then for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds either*:
  - (c_1) *there is a global minimum of J which is a local minimum of h_λ, or,*
  - (c_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of h_λ which weakly converges to a global minimum of J, with*

$$\lim_{n\to+\infty} J(u_n) = \inf_{u\in X} J(u).$$

3. Multiple solutions

In this section we shall state and prove the existence of a sequence of pairwise distinct weak solutions for the problem (1.1).

Definition 3.1. A function $u \in W_0^{1,\Phi}(\Omega) := X$ is said to be a weak solution for the problem (1.1) if

$$\begin{aligned} &M\left(\int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x\right)\int_\Omega a(|\nabla u(x)|)\nabla u(x)\nabla v(x)\,\mathrm{d}x \\ &-\lambda\int_\Omega f(x,u)v(x)\,\mathrm{d}x - \mu\int_\Omega g(x,u)v(x)\,\mathrm{d}x = 0 \end{aligned}$$

for all $v \in X$.

Throughout this paper, we use

$$F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s, \qquad (x,s) \in \bar{\Omega}\times\mathbb{R},$$

and

$$\hat{M}(t) := \int_0^t M(s)\,\mathrm{d}s, \qquad t \geq 0.$$

For fixed $x_0 \in \Omega$, set $D > 0$ such that $\overline{B(x_0,D)} \subseteq \Omega$, where $B(x_0,D)$ denotes the ball with center at x_0 and radius D. Let

$$k := \frac{2}{D}\omega(D^N - (\frac{D}{2})^N), \tag{3.1}$$

where $\omega = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}$ and Γ is the Gamma function which defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}\,\mathrm{d}z, \quad \forall t > 0.$$

Here, our main result is represented as the following theorem.

Theorem 3.1. *Assume that there exist a point $x_0 \in \Omega$ and values $D, \tau > 0$ such that $\overline{B(x_0,D)} \subseteq \Omega$,*

$$\lim_{t\to 0^+}\frac{\Phi(t)}{t^{\varphi^0}} < \tau, \tag{3.2}$$

and

$$A < \frac{m}{c^{\varphi^0}}B, \tag{3.3}$$

where k is as in (3.1) *and*

$$A := \liminf_{\xi\to 0^+}\frac{\int_\Omega \sup_{|t|\leq\xi}F(x,t)\,\mathrm{d}x}{\xi^{\varphi^0}}, \quad B := \limsup_{\xi\to 0^+}\frac{\int_{B(x_0,\frac{D}{2})}F(x,\xi)\,\mathrm{d}x}{\hat{M}(\mathrm{meas}(\Omega)\tau\xi^{\varphi^0}k^{\varphi^0})}.$$

Moreover, let $F(x,t) \geq 0$ for every $(x,t) \in \Omega\times\mathbb{R}^+$. Then for each $\lambda \in (\frac{1}{B}, \frac{m}{c^{\varphi^0}A})$ and for every L^1-Carathéodory function $g : \bar{\Omega}\times\mathbb{R}\to\mathbb{R}$ that $G(x,t) :=$

$\int_0^t g(x,\xi)\,\mathrm{d}\xi$ for every $(x,t) \in \bar{\Omega} \times \mathbb{R}$ is a non-negative function and satisfies the condition

$$G_0 := \frac{c^{\varphi^0}}{m} \limsup_{\xi \longrightarrow 0^+} \frac{\int_\Omega \sup_{|t|\le\xi} G(x,t)\,\mathrm{d}x}{\xi^{\varphi^0}} < +\infty, \tag{3.4}$$

and for every $\mu \ge 0$ with $\mu < \mu_{g,\lambda} := \frac{1}{G_0}(1 - \frac{\lambda c^{\varphi^0} A}{m})$; there exists a sequence of pairwise distinct weak solutions for the problem (1.1) *which strongly converges to zero in $W_0^{1,\Phi}(\Omega)$, if $G_0 > 0$.*

We now introduce the functionals $J, I : X \to \mathbb{R}$:

$$J(u) = \hat{M}\left(\int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x\right),$$

$$I(u) = \int_\Omega \Big[F(x,u(x)) + \frac{\mu}{\lambda} G(x,u(x))\Big]\,\mathrm{d}x,$$

and

$$h_\lambda(u) = J(u) - \lambda I(u),$$

where μ and λ are two positive constants and g is a function which satisfies condition (3.4).

It is well known that I is a Gateaux differentiable functional and sequentially weakly upper semicontinuous whose Gateaux derivative at the point $u \in X$ is the functional $I'(u) \in X^*$, give by

$$I'(u)(v) = \int_\Omega \Big(f(x,u(x)) + \frac{\mu}{\lambda} g(x,u(x))\Big) v(x)\,\mathrm{d}x$$

for every $v \in X$.

Moreover, J is a Gateaux differentiable functional whose Gateaux derivative at the point $u \in X$ is the functional $J'(u) \in X^*$, given by

$$J'(u)(v) = M\left(\int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x\right) \int_\Omega (a(|\nabla u(x)|)\nabla u(x) \nabla v(x))\,\mathrm{d}x$$

for every $v \in X$.

Lemma 3.1. *J is coercive and sequentially weakly lower semicontinuous.*

Proof. Since $M(t) \ge m$ for all $t \ge 0$, we have

$$J(u) \ge m \int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x.$$

The above inequality and Proposition 2.1, show that for any $u \in X$ with $\|u\| > 1$ we have $J(u) \ge m\|u\|^{\varphi_0}$ which follows $\lim_{\|u\| \to +\infty} J(u) = +\infty$, i.e., J is coercive.

Let $\{u_n\} \subseteq X$ be a sequence such that $u_n \rightharpoonup u$ in X. By [23] the map $u \mapsto \int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x$ is weakly lower semicontinuous, i.e.,

$$\int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega \Phi(|\nabla u_n(x)|)\,\mathrm{d}x. \tag{3.5}$$

From (3.5) and since $\hat{M}$ is continuous and monotone, we have

$$\begin{aligned}\liminf_{n\to\infty} J(u_n) &= \liminf_{n\to\infty} \hat{M}\left(\int_\Omega \Phi(|\nabla u_n(x)|)\,\mathrm{d}x\right)\\ &\geq \hat{M}\left(\liminf_{n\to\infty}\int_\Omega \Phi(|\nabla u_n(x)|)\,\mathrm{d}x\right)\\ &\geq \hat{M}\left(\int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x\right)\\ &= J(u)\end{aligned}$$

namely, J is sequentially weakly lower semicontinuous. □

Proof of Theorem 3.1. We apply the part (c) of Proposition 2.2, to prove Theorem 3.1. Fix $\lambda \in (\frac{1}{B}, \frac{m}{c^{\varphi^0}A})$ and let g be a function which satisfies (3.4), then one has $\mu_{g,\lambda} > 0$, since $\lambda < \frac{m}{c^{\varphi^0}A}$. Now fix $\mu \in [0, \mu_{g,\lambda})$ and put $w_1 := \frac{1}{B}$ and $w_2 := \frac{m}{c^{\varphi^0}A} \cdot \frac{1}{1+\frac{\mu}{\lambda}\cdot\frac{m}{c^{\varphi^0}A}G_0}$. If $G_0 = 0$ clearly $w_1 = \frac{1}{B}$, $w_2 = \frac{m}{c^{\varphi^0}A}$, and $\lambda \in (w_1, w_2)$. If $G_0 \neq 0$, we see that $\frac{\lambda c^{\varphi^0}A}{m} + \mu G_0 < 1$, since $\mu < \mu_{g,\lambda}$, therefore $\frac{m}{c^{\varphi^0}A} \cdot \frac{1}{1+\frac{\mu}{\lambda}\cdot\frac{m}{c^{\varphi^0}A}G_0} > \lambda$, i.e., $\lambda < w_2$. Note that $\lambda > \frac{1}{B}$, hence one has $\lambda \in (w_1, w_2)$.

Clearly, the weak solutions of the problem (1.1) are exactly the solutions of the equation $h'_\lambda(u) = 0$.

Now we show that $\delta < +\infty$, where δ is defined in Proposition 2.2. Let $\xi > 0$ and put $r = m(\frac{\xi}{c})^{\varphi^0}$, from (2.8) one has,

$$|u(x)| \leq \|u\|_\infty \leq c\|u\|, \qquad u \in X.$$

taking Proposition 2.1 and Lemma 2.1 into account, we have:

$$J^{-1}(]-\infty, r[) \subseteq \left\{u \in X; \|u\| \leq \frac{\xi}{c}\right\} \subseteq \{u \in X; |u(x)| \leq \xi \text{ for all } x \in \Omega\}$$

and it follows that:

$$\sup_{u\in J^{-1}(]-\infty,r[)} I(u) < \int_\Omega \sup_{|t|\leq\xi}\left[F(x,t) + \frac{\mu}{\lambda}G(x,t)\right]\,\mathrm{d}x.$$

Hence, taking into account that $J(0) = I(0) = 0$, one has

$$\begin{aligned}\varphi(r) &= \inf_{u\in J^{-1}(]-\infty,r[)} \frac{\sup_{v\in J^{-1}(]-\infty,r[)} I(v) - I(u)}{r - J(u)}\\ &\leq \frac{\sup_{v\in J^{-1}(]-\infty,r[)} I(v)}{r}\\ &\leq \frac{\int_\Omega \sup_{|t|\leq\xi}[F(x,t) + \frac{\mu}{\lambda}G(x,t)]\,\mathrm{d}x}{m(\frac{\xi}{c})^{\varphi^0}}\\ &\leq \frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t)\,\mathrm{d}x}{m(\frac{\xi}{c})^{\varphi^0}} + \frac{\mu}{\lambda}\frac{\int_\Omega \sup_{|t|\leq\xi} G(x,t)\,\mathrm{d}x}{m(\frac{\xi}{c})^{\varphi^0}}.\end{aligned}$$

Moreover, it follows from assumption (3.3) that $A < +\infty$, i.e.,

$$\liminf_{\xi\to 0^+} \frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\,\mathrm{d}x}{\xi^{\varphi^0}} < +\infty. \tag{3.6}$$

Hence, from (3.3), (3.4) and (3.6), one has

$$\begin{aligned}
\delta \le\ & \liminf_{r\to 0^+} \varphi(r) \\
\le\ & \liminf_{\xi\to 0^+} \frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\,\mathrm{d}x}{m(\frac{\xi}{c})^{\varphi^0}} \\
& + \limsup_{\xi\to 0^+} \frac{\mu}{\lambda} \frac{\int_\Omega \sup_{|t|\le\xi} G(x,t)\,\mathrm{d}x}{m(\frac{\xi}{c})^{\varphi^0}} \\
=\ & \liminf_{\xi\to 0^+} \frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\,\mathrm{d}x}{m(\frac{\xi}{c})^{\varphi^0}} + \frac{\mu}{\lambda} G_0 < +\infty.
\end{aligned} \tag{3.7}$$

Also, since G is non-negative one has;

$$\begin{aligned}
\limsup_{\xi\to 0^+} \frac{\int_\Omega [F(x,\xi) + \frac{\mu}{\lambda} G(x,\xi)]\,\mathrm{d}x}{\widehat{M}(meas(\Omega)\tau\xi^{\varphi^0}k^{\varphi^0})} &\ge \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x}{\widehat{M}(meas(\Omega)\tau\xi^{\varphi^0}k^{\varphi^0})} \\
&\ge \limsup_{\xi\to 0^+} \frac{\int_{B(x_0,\frac{D}{2})} F(x,\xi)\,\mathrm{d}x}{\widehat{M}(meas(\Omega)\tau\xi^{\varphi^0}k^{\varphi^0})}.
\end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned}
&\lambda \in (\omega_1, \omega_2) \\
&\subseteq \left(\frac{1}{\limsup_{\xi\to 0^+} \frac{\int_\Omega [F(x,\xi)+\frac{\mu}{\lambda}G(x,\xi)]\,\mathrm{d}x}{\widehat{M}(meas(\Omega)\tau\xi^{\varphi^0}k^{\varphi^0})}}, \frac{1}{\liminf_{\xi\to 0^+} \frac{\int_\Omega \sup_{|t|\le\xi}[F(x,t)+\frac{\mu}{\lambda}G(x,t)]\,\mathrm{d}x}{m(\frac{\xi}{c})^{\varphi^0}}} \right) \\
&\subseteq (0, \frac{1}{\delta}).
\end{aligned}$$

Now we can apply Proposition 2.2 part (c) for fixed λ, and show that 0, that is the unique global minimum of J, is not a local minimum of the functional h_λ. Let $\{d_n\}$ be a real sequence of positive numbers such that $d_n \to 0$ ad $n \to \infty$ and

$$B = \lim_{n\to\infty} \frac{\int_{B(x_0,\frac{D}{2})} F(x,d_n)\,\mathrm{d}x}{\hat{M}(\mathrm{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0})}. \tag{3.9}$$

Let $\{v_n\} \subseteq X$ be a sequence defined by

$$v_n(x) := \begin{cases} 0 & x \in \Omega \setminus B(x_0, D), \\ d_n & x \in B(x_0, \frac{D}{2}), \\ \frac{2d_n}{D}(D - |x - x_0|) & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}). \end{cases}$$

Then we have,

$$
\begin{aligned}
J(v_n) &= \hat{M}\left(\int_\Omega \Phi(|\nabla v_n|)\,\mathrm{d}x\right)\\
&= \hat{M}\left(\int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} \Phi(\frac{2}{D}\omega(D^N-(\frac{D}{2})^N)d_n)\,\mathrm{d}x\right)\\
&= \hat{M}\left(\int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} \Phi(kd_n)\,\mathrm{d}x\right). \qquad (3.10)
\end{aligned}
$$

Moreover from (3.2) and since $\lim_{n\to\infty} kd_n = 0$, there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that $kd_n \in (0, \eta)$ and

$$\Phi(kd_n) < \tau k^{\varphi^0} {d_n}^{\varphi^0}, \quad \forall n \geq n_0.$$

Knowing that $\hat{M}$ is monotone (increasing) and G is nonnegative; we have

$$J(v_n) = \hat{M}\left(\int_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} \Phi(kd_n)\,\mathrm{d}x\right) \leq \hat{M}\left(\mathrm{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0}\right) \tag{3.11}$$

and,

$$I(v_n) \geq \int_{B(x_0,\frac{D}{2})} F(x,d_n)\,\mathrm{d}x. \tag{3.12}$$

If $B < +\infty$, let $\varepsilon \in (\frac{1}{\lambda B}, 1)$ and $\varepsilon' := (1-\varepsilon)B > 0$. According to (3.9), there exists n_ε such that

$$\frac{\int_{B(x_0,\frac{D}{2})} F(x,d_n)\,\mathrm{d}x}{\hat{M}(\mathrm{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0})} - B > (\varepsilon - 1)B,$$

and then,

$$\int_{B(x_0,\frac{D}{2})} F(x,d_n)\,\mathrm{d}x > \varepsilon B \hat{M}\left(\mathrm{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0}\right) \tag{3.13}$$

for all $n \geq n_\varepsilon$. Hence, from (3.10)-(3.13) we have;

$$
\begin{aligned}
h_\lambda(v_n) = J(v_n) - \lambda I(v_n) &\leq \hat{M}(\mathrm{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0}) - \lambda\varepsilon B\hat{M}\left(\mathrm{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0}\right)\\
&= (1-\lambda\varepsilon B)\hat{M}\left(\mathrm{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0}\right) < 0
\end{aligned}
$$

for every $n \geq \max\{n_0, n_\varepsilon\}$. On the other hand if $B = +\infty$ let $\sigma > \frac{1}{\lambda}$, from (3.9) there exists n_σ such that,

$$\int_{B(x_0,\frac{D}{2})} F(x,d_n)\,\mathrm{d}x > \sigma\hat{M}\left(\mathrm{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0}\right), \quad \forall n \geq n_\sigma$$

and,

$$\begin{aligned} h_\lambda(v_n) &= J(v_n) - \lambda I(v_n) \\ &\leq \hat{M}(\text{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0}) - \lambda\sigma\hat{M}\left(\text{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0}\right) \\ &= (1-\lambda\sigma)\hat{M}\left(\text{meas}(\Omega)\tau d_n^{\varphi^0} k^{\varphi^0}\right) < 0 \end{aligned}$$

for every $n \geq \max\{n_0, n_\sigma\}$. Hence $h_\lambda(v_n) < 0$ for every n large enough. It shows that 0, is not a local minimum of h_λ, since $h_\lambda(0) = J(0) - \lambda I(0) = 0$. Then owing to the fact that 0 is the unique global minimum of J, so there exists a sequence $\{u_n\} \subset X$ of pairwise distinct critical points of h_λ such that $\lim_{n\to\infty} \|u_n\| = 0$. $\square$

Remark 3.1. As a special case by putting $M \equiv 1$, $\mu = 0$ and $f(x,u) = h(x)f^*(u)$ in (1.1) the problem is converted to

$$\begin{cases} -\text{div}(a(|\nabla u|)\nabla u) = \lambda h(x) f^*(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which is studied in [5] and the existence of infinitely many solutions for it is proved.

By an idea in [18], we present the following example which satisfies the assumptions in Theorem 3.1.

Example 3.1. For $N = 3$, let $M(t) = 1 + 2t$ for every $t \geq 0$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\text{meas}(\Omega) = 1$. We consider the following functions which satisfy our results.

$$f(t) := \begin{cases} t^4(5 - 5\sin(\ln|t|) - \cos(\ln|t|)) & t \neq 0, \\ 0 & t = 0, \end{cases}$$

$$\varphi(t) := \begin{cases} \dfrac{|t|^3 t}{\log(1+|t|)} & t \neq 0, \\ 0 & t = 0, \end{cases}$$

and

$$g(t) = t^5 e^{-t}(6 - t)$$

for every $t \in \mathbb{R}$.

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