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ON SOME PROPERTIES OF J-CLASS OPERATORS

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Abstract. The notion of hypercyclicity was localized by J-sets and in this paper, we will investigate for an equivalent condition through the use of open sets. Also, we will give a J-class criterion, that gives conditions under which an operator belongs to the J-class of operators.

1. Introduction

Let X be a Banach space over the field $\mathcal{C}$ of complex numbers. In what follows, $\mathcal{N}$ denotes the set of all positive integers and T stands for a bounded linear operator acting on X, i.e., $T \in B(X)$. For any subset D of X, $Orb(T, D)$ denotes the orbit of D under T and is defined by $Orb(T, D) = \{T^n x : x \in D,\ n = 0, 1, 2, \ldots\}$. If $D = \{x\}$ and $Orb(T, x)$ is dense in X, then the operator T is hypercyclic and the vector x is a hypercyclic vector for T. In this case the underlying Banach space X should be separable. If $D = \{\lambda x : \lambda \in \mathcal{C}\}$ and $Orb(T, D)$ is dense in X, then the operator T is supercyclic and the vector x is a supercyclic vector for T. Also, T is called topologically transitive if for every pair of nonempty open sets U, V of X there exists a nonnegative integer n such that $T^n(U) \cap V \neq \emptyset$. It is well known that an operator T is hypercyclic if and only if it is topologically transitive ([5]).

The fundamental tool in the development of hypercyclicity is known as the *Hypercyclicity Criterion* which was developed by Kitai ([15]) and independently by Getner and Shapiro ([11]).

Theorem 1.1 (Hypercyclicity Criterion)**.** *Let T be an operator on X. If there are dense subsets $X_0, Y_0 \subset X$, an increasing sequence $\{n_k\}$ of positive integers, and mappings $S_{n_k} : Y_0 \longrightarrow X$, such that*

(i) *for any $x \in X_0$, $T^{n_k} x \longrightarrow 0$,*
(ii) *for any $y \in Y_0$, $S_{n_k} y \longrightarrow 0$,*
(iii) *for any $y \in Y_0$, $T^{n_k} S_{n_k} y \longrightarrow y$,*

then T is hypercyclic.

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Definition 1.2. Let T be an operator. For every $x \in X$ the set

$$J(x) = \{y \in X : \text{there exist a strictly increasing sequence of positive integers } \{k_n\} \text{ and a sequence } \{x_n\} \subset X \text{ such that } x_n \to x \text{ and } T^{k_n} x_n \longrightarrow y\},$$

denotes the *extended (prolongational) limit set of x under T*, and an operator T is called a J-class operator provided there exists a non-zero vector $x \in X$ so that $J(x) = X$. In this case, x is called a J-class vector for T.

In this paper, we give a description of J-sets by using open sets. Also, we state and prove a criterion, called *J-class Criterion*, for characterizing the J-class vectors and operators. Some examples satisfying the *J-class Criterion* are given and we end the paper by raising two open questions.

For some sources on these topics one can see [1–21].

2. On the J-class operators and J-class criterion

In what follows, the symbol U_x denotes an open neighborhood $U \subset X$ of vector $x \in X$ in the norm-topology of X and in order to formulate the arguments involving topological transitive maps, we give the following definitions and theorems.

Definition 2.1. Let $T \in B(X)$. Then for any subsets $A, B \subseteq X$, the *return set from A to B* is denoted by $N_T(A, B)$ that is defined by

$$N_T(A, B) = \{n \geq 0 : T^{-n}(A) \cap B \text{ is nonempty subset of } X\}.$$

Definition 2.2. An operator T is called topologically transitive if for any pair U, V of nonempty open subsets of X, $N_T(U, V) \neq \emptyset$.

Theorem 2.3 ([14]). *An operator T is topologically transitive if and only if for any pair U, V of nonempty open subsets of X, the return set $N_T(U, V)$ is an infinite set.*

The notion of J-sets is well known in the theory of topological dynamics ([4]). If one wants to work on general non-separable Banach spaces and the dynamical behavior of the iterates of an operator T, the suitable substitute of hypercyclicity is the J-sets.

Definition 2.4. Let T be an operator acting on X and $x \in X$. Then we use the notation $A(x)$ to define

$$A(x) = \{y \in X : \forall U_x, \forall U_y,\ \exists n \in \mathcal{N},\ \text{s.t. } T^n(U_x) \cap U_y \neq \emptyset\}.$$

In the following example we will show it can be happen that $A(x) \neq J(x)$ for some $x \in X$. Then, under a theorem, we will give conditions under which $A(x) = J(x)$.

Example 2.5. We will show it can be happen that $A(x) \neq J(x)$ for some $x \in X$. First note that

$$\{T^n x : n \geq 1\} \subseteq A(x)$$

for all $x \in X$. Now consider the operator $T = \frac{1}{2}B$ where B is the backward shift operator on $\ell^2(\mathcal{N})$, the space of square summable sequences, and consider vector $x \in \ell^2(\mathcal{N})$ such that $Tx \neq 0$, then $Tx \in A(x)$. On the other hand for every strictly increasing sequence of positive integers $\{k_n\}$ and every sequence $\{x_n\} \subset X$, if $x_n \longrightarrow x$, then $T^{k_n} x_n \longrightarrow 0$ and we get $J(x) = \{0\}$, therefore $J(x) \neq A(x)$.

Note that the inclusion $J(x) \subseteq A(x)$ is obvious and in the following theorem we investigate conditions under which $J(x) = A(x)$ for some $x \in X$.

Theorem 2.6. *Suppose that there exist $x_0 \in X$ and $\varepsilon > 0$ such that $x \in J(x)$ for all $x \in B(x_0, \varepsilon)$. Then $J(x_0) = A(x_0)$.*

Proof. Consider $0 < \varepsilon < 1$ and $x_0 \in X$ such that $x \in J(x)$ for all $x \in B(x_0, \varepsilon)$. Also, let k_n be the smallest positive integer such that

$$T^{k_n} B(x_0, \frac{\varepsilon}{n}) \cap B(y, \frac{1}{n}) \neq \emptyset.$$

Note that there exists a positive integer $m \geq k_n$ such that

$$B(x_0, \frac{\varepsilon}{n+1}) \cap T^{-m} B(y, \frac{1}{n+1}) \neq \emptyset.$$

Now by continuity of T there exists a nonempty open set W_{n+1} satisfying

$$W_{n+1} \subset B(x_0, \frac{\varepsilon}{n+1}) \cap T^{-m} B(y, \frac{1}{n+1}).$$

We have $x \in J(x) \subset A(x)$ for all x in W_{n+1}. Fix $x' \in W_{n+1}$, and put $U_{x'} = W_{n+1}$ in the definition of $A(x')$. Then there exists $n_0 > 0$ satisfying $T^{n_0}(W_{n+1}) \cap W_{n+1} \neq \emptyset$. Set $W'_{n+1} = W_{n+1} \cap T^{-n_0}(W_{n+1})$, then $T^{n_0}(W'_{n+1}) \subset W_{n+1}$. Thus $T^{n_0+m}(W'_{n+1}) \subset T^m(W_{n+1}) \subset B(y, \frac{1}{n+1})$. But $W'_{n+1} \subset W_{n+1} \subset B(x_0, \frac{\varepsilon}{n+1})$, hence

$$T^{n_0+m} B(x_0, \frac{\varepsilon}{n+1}) \cap B(y, \frac{1}{n+1}) \neq \emptyset.$$

Now, set $k_{n+1} = m + n_0 > k_n$ and choose $x_{n+1} \in B(x_0, \frac{\varepsilon}{n+1})$. Therefore, we can find a strictly increasing sequence $\{k_n\}$ of positive integers such that $x_n \longrightarrow x_0$ and $T^{k_n} x_n \longrightarrow y$ from which we conclude that $A(x) \subseteq J(x)$. The converse inclusion is obvious and so the proof is complete. ☐

Theorem 2.7. *Let T be an operator on X and $x \in X$. Then $J(x) = J^{tra}(x)$ where*

$$J^{tra}(x) := \{y \in X : \forall U_x, \forall U_y,\ \forall N \in \mathcal{N},\ \exists n \geq N,\ s.t.\ T^n(U_x) \cap U_y \neq \emptyset\}.$$

Proof. Let us prove that $J^{tra}(x) \subseteq J(x)$, since the converse inclusion is obvious. Let $y \in J^{tra}(x)$ and consider U_x, U_y as two neighborhoods of x and y, respectively. Choose $\varepsilon > 0$ such that

$$B(x,\varepsilon) \subset U_x; B(y,\varepsilon) \subset U_y.$$

Assume that for $n = 1, 2, \dots, j$, there exist vectors $x_n \in B(x, \frac{\varepsilon}{n})$ and also there exist integers k_n such that $k_{n-1} < k_n$ and $T^{k_n} x_n \in B(y, \frac{\varepsilon}{n})$. Consider the open balls $B(x, \frac{\varepsilon}{j+1})$ and $B(y, \frac{\varepsilon}{j+1})$. Clearly, there exists $k_{j+1} \geq k_j + 1$ such that

$$T^{k_{j+1}} B(x, \frac{\varepsilon}{j+1}) \cap B(y, \frac{\varepsilon}{j+1}) \neq \emptyset.$$

Now we can choose $x_{j+1} \in B(x, \frac{\varepsilon}{j+1})$ and $y_{j+1} = T^{k_{j+1}} x_{j+1} \in B(y, \frac{\varepsilon}{j+1})$. So by the induction we can construct the sequences $\{x_n\}$ and $\{k_n\}$ with desired properties, i.e., $x_n \longrightarrow x$ and $T^{k_n} x_n \longrightarrow y$. □

It is well known that for a vector $x \in X$, if $J^{tra}(x)$ is the whole underlying space, then T is a J-class operator.

Not only the hypercyclicity criterion is one of the fundamental tools in the hypercyclicity development, but also it is the best known sufficient condition to ensure that an operator is hypercyclic. So, we are interested in proposing the following *J-class criterion* and in order to state it, the following lemmas will be needed.

Lemma 2.8. *Let T be an operator acting on X and $\{k_n\}$ be a fixed strictly increasing sequence of positive integers. Also, let Y be a dense subset of X such that for every $y \in Y$, there exists a sequence $\{w_n\} \in X$ satisfying $w_n \longrightarrow 0$ and $T^{k_n} w_n \longrightarrow y$. Then for every $x \in X$, there exists a sequence $\{w_n\} \in X$ such that $w_n \longrightarrow 0$ and $T^{k_n} w_n \longrightarrow x$.*

Proof. If $x \in X \backslash Y$, then there exists a sequence $\{y_m\} \in Y$ such that $y_m \longrightarrow x$ as $m \longrightarrow \infty$. By assumption for every m, there exists a sequence $\{w'_{m_n}\} \in X$ such that $w'_{m_n} \longrightarrow 0$ and $T^{k_n} w'_{m_n} \longrightarrow y_m$ as $n \longrightarrow \infty$. Now consider the open ball $B(x, \frac{1}{2})$ and the smallest positive integer m_1 such that $y_{m_1} \in B(x, \frac{1}{2})$. Hence there exists a sequence $\{w'_{m_1,n}\} \subset X$ such that

$$w'_{m_1,n} \longrightarrow 0;\ T^{k_n} w'_{m_1,n} \longrightarrow y_{m_1}.$$

Therefore, there exists a positive integer n_1 such that

$$T^{k_{n_1}} w'_{m_1,n_1} \in B(x, \frac{1}{2}; w'_{m_1,n_1} \in B(0, \frac{1}{2}).$$

Set $w_1 = w_2 = \cdots = w_{n_1 - 1} = 0$ and $w_{n_1} = w'_{m_1,n_1}$.

Proceeding in the same way, consider the open ball $B(x, \frac{1}{2^2})$ and the smallest positive integer $m_2 > m_1$ such that $y_{m_2} \in B(x, \frac{1}{2^2})$. Hence, there exists a sequence $\{w'_{m_2,n}\} \subset X$ satisfying

$$w'_{m_2,n} \longrightarrow 0;\ T^{k_n} w'_{m_2,n} \longrightarrow y_{m_2}.$$

Therefore, there exists a positive integer $n_2 > n_1$ such that

$$T^{k_{n_2}} w'_{m_2,n_2} \in B(x, \frac{1}{2^2}); w'_{m_2,n_2} \in B(0, \frac{1}{2^2}).$$

Set $w_{n_1+1} = \cdots = w_{n_2-1} = 0$ and $w_{n_2} = w'_{m_2,n_2}$.

Proceeding inductively, we find a sequence $\{w_n\} \subset X$ such that $w_n \longrightarrow 0$ and $T^{k_n} w_n \longrightarrow x$. This completes the proof. □

It is well known that J-sets are closed, so that if Y is a dense subset of X and $Y \subset J(0)$, then trivially $J(0) = X$. Note that this differs with Lemma 2.8, because we considered that $\{k_n\}$ is a fixed strictly increasing sequence of positive integers.

Definition 2.9. The notation $J'(x)$ is defined as follows: For $T \in B(X)$ and $x \in X$ we define

$$J'(x) := \{y \in X;\ \forall U_x,\ \forall U_y,\ \forall U_0,\ \forall N \in \mathcal{N},\ \exists n \geq N,\ T^n(U_x) \cap U_0 \neq \emptyset,\ T^n(U_0) \cap U_y \neq \emptyset\}.$$

Lemma 2.10. *Let $T \in B(X)$ and $x \in X$. Then $J'(x) \subseteq J(x)$.*

Proof. Suppose $y \in J'(x)$, $W'_1 = B(0, \frac{1}{2})$ and set

$$U_x := W'_1 + x; U_y := W'_1 + y.$$

Since $y \in J'(x)$, there exists an integer $k_1 \geq 1$ such that $T^{k_1}(U_x) \cap W'_1 \neq \emptyset$ and $T^{k_1}(W'_1) \cap U_y \neq \emptyset$. Hence, there exist vectors $x'_1,\ y'_1,\ x''_1,\ y''_1$ in W'_1 such that

$$T^{k_1}(x'_1 + x) = y'_1;\ T^{k_1}(x''_1) = y''_1 + y.$$

Therefore, we get $T^{k_1}(x + x'_1 + x''_1) = y + y'_1 + y''_1$. Note that

$$x_1 := x + x'_1 + x''_1 \in B(x, 1);\ y_1 := y + y'_1 + y''_1 \in B(y, 1).$$

Proceeding in the same way, the sequences $\{x_n\} \subset X$, $\{y_n\} \subset X$ and $\{k_n\} \subset \mathcal{N}$ can be found such that for all $n \geq 2$, $k_n > (k_{n-1} + 1)$, $x_n \in B(x, \frac{1}{n})$ and $y_n \in B(y, \frac{1}{n})$, and $T^{k_n} x_n \longrightarrow y$ as $n \to \infty$. Thus indeed $y \in J(x)$ and the proof is complete. □

In the following theorem, we state and prove a *J-class Criterion* which gives sufficient conditions for an operator to be a J-class operator.

Theorem 2.11 (J-class Criterion)**.** *Let T be an operator on an infinite dimensional Banach space X and let there exist a dense subset $Y \subseteq X$. If for $x \in X$ there exists a strictly increasing sequence $\{k_n\}$ of positive integers satisfying the conditions*:

(i) *there exists a sequence $\{x_n\} \subset X$ such that $x_n \longrightarrow x$ and $T^{k_n} x_n \longrightarrow 0$,*
(ii) $\forall y \in Y,\ \exists \{w_n\} \subset X$ *such that* $w_n \longrightarrow 0$, $T^{k_n} w_n \longrightarrow y$,

then x is a J-class vector for T.

Proof. Let $x_n \longrightarrow x$, we will show that x is a J-class vector for operator T. Consider V, U_x, U_0 as a non-void open subset of X, neighborhoods of x and the zero vector, respectively. Consider an integer $N \geq 1$ and choose a vector $y \in V$ and $\varepsilon > 0$ such that:

$$B(y,\varepsilon) \subset V,\ B(x,\varepsilon) \subset U_x,\ B(0,\varepsilon) \subset U_0.$$

By assumption and Lemma 2.10, there exists an integer $n \geq N$ such that:

$$T^n(B(x,\varepsilon)) \cap B(0,\varepsilon) \neq \emptyset,$$

and

$$T^n(B(0,\varepsilon)) \cap B(y,\varepsilon) \neq \emptyset.$$

This implies that $y \in J'(x)$. Since V is an arbitrary non-void open subset of X, $J'(x)$ is the whole underlying space and Lemma 2.10 implies that x is a J-class vector for operator T. □

Although operators satisfying the Hypercyclicity Criterion obviously satisfy the J-class Criterion, but in the following example we show directly that the conditions of J-class Criterion are consistent.

Example 2.12. Consider the weighted backward shift operator T on $\ell^1(\mathcal{N})$ given by

$$T(x^1, x^2, \ldots) = (2x^2, \frac{3}{2}x^3, \frac{4}{3}x^4, \ldots).$$

Also, denote the set of finite sequences with elements from $Q+iQ$ by Y. Clearly, there exist $\{N_j\}_j \subset \mathcal{N}$ and $\{x_n\}_n$ in Y such that for all $j \geq 1$,

$$x_j = (x^1, x^2, \ldots, x^{N_j}, 0, \ldots),\ x_{j+1} = (x^1, x^2, \ldots, x^{N_{j+1}}, 0, \ldots),\ N_j < N_{j+1},$$

and $x_n \to 0$. Hence $\{N_k\}$ is a strictly increasing sequence of positive integers such that if $j \geq N_k$, then $T^j x_k = 0$.

Now consider $y = (y^1, y^2, \ldots, y^m, 0, \ldots) \in Y$ and for all $n \geq 1$ set

$$w_n(y) = (\underbrace{0, 0, \ldots, 0}_{n\text{-times}}, \frac{1}{n+1}y^1, \ldots, \frac{m}{n+m}y^m, 0, \ldots).$$

Thus we get

$$\|w_n(y)\| = \sum_{k=1}^{m} |\frac{k}{k+n}y^k| \leq \frac{m}{n+1}\|y\|,$$

so $\{w_n(y)\} \subset \ell^1(\mathcal{N})$ and $w_n(y) \to 0$ as $n \to \infty$. Note that for all $n \geq 1$ and all $(x^1, x^2, x^3, \ldots) \in \ell^1(\mathcal{N})$ we have

$$T^n(x^1, x^2, x^3, \ldots) = \left((n+1)x^{n+1}, (\frac{n+2}{2})x^{n+2}, (\frac{n+3}{3})x^{n+3}, \ldots\right).$$

Hence for all $k \geq 1$ we obtain

$$T^{N_k} w_{N_k}(y) = \left((N_k+1) \times (\frac{1}{N_k+1}y^1), \ldots, (\frac{N_k+m}{m}) \times (\frac{m}{N_k+m})y^m, 0, \ldots\right).$$

Therefore $w_{N_k}(y) \to 0$ and $T^{N_k} w_{N_k}(y) \to y$ as $k \to \infty$. So the operator T satisfies the J-class Criterion.

Note that any non-separable Banach spaces can not support hypercyclic operators, but this is not true for J-class operators. For example in [9] it is shown that the non-separable Banach space $\ell^\infty(\mathcal{N})$ admits a J-class operator. In the following example we show that the Banach space $\ell^\infty(\mathcal{N})$ supports operator satisfying the J-class Criterion.

Example 2.13. Consider the backward shift operator B on $\ell^\infty(\mathcal{N})$. Then for every $\lambda \in \mathcal{C}$ with $|\lambda| > 1$, the operator $T = \lambda B$ holds in the J-class Criterion. For this denote the space $\ell^\infty(\mathcal{N})$ by Y and fix $y = (y^1, y^2, \ldots) \in Y$. For every $n \geq 1$, define

$$w_n = (\underbrace{0, 0, \ldots, 0}_{n\text{-times}}, \frac{y^1}{\lambda^n}, \frac{y^2}{\lambda^n}, \ldots);\ x_n = (0, 0, \ldots).$$

Obviously, $\{w_n\} \subset \ell^\infty(\mathcal{N})$ and

$$T^n x_n \to 0,\ w_n \to 0,\ T^n w_n \to y.$$

Therefore, the operator T satisfies the J-class Criterion and $J(0) = \ell^\infty(\mathcal{N})$.

Theorem 2.14. *Let T be an operator on X. The following conditions are equivalent*:

(i) *T satisfies the J-class criterion with respect to x,*
(ii) *$J'(x) = X$.*

Proof. It is trivial that (ii) implies (i). To prove that (i) implies (ii), let $y \in X$ and suppose that U_y, U_0 are two neighborhoods of y and the zero vector, respectively. Consider $\varepsilon > 0$ such that:

$$B(0, \varepsilon) \subset U_0,\ B(y, \varepsilon) \subset U_y.$$

Note that the relation $0 \in X = J(0)$ implies that there exists an integer $n_1 \geq 0$ such that $w_{n_1} \in B(0, \varepsilon)$ and $T^{k_{n_1}} w_{n_1} \in B(0, \varepsilon)$. Also, since $y \in J(0) = X$, there exists an integer $n_2 \geq 0$ such that $w_{n_2} \in B(0, \varepsilon)$ and $T^{k_{n_2}} w_{n_1} \in B(y, \varepsilon)$. Now set $n = \max(n_1, n_2)$, hence we obtain:

$$T^{k_n}(U_0) \cap U_0 \neq \emptyset,\ T^{k_n}(U_0) \cap U_y \neq \emptyset.$$

This implies that $y \in J'(x)$ and so (ii) holds. □

Theorem 2.15. *Let T be an operator on X. If $(x, 0)$ is a J-class vector for $T \times T$, then $J'(x) = X$.*

Proof. Fix $N \geq 1$ and let $y \in X$. Suppose U_0, U_x, U_y are neighborhoods of 0, x, y, respectively, and consider $\varepsilon > 0$ such that:

$$B(0, \varepsilon) \subset U_0,\ B(x, \varepsilon) \subset U_x,\ B(y, \varepsilon) \subset U_y.$$

Since $J(x,0) = X \times X$, there exist a strictly increasing sequence of positive integers k_n and a sequence $\{(x_n, w_n)\} \subset X \times X$ such that $(x_n, w_n) \to (x, 0)$ and $T^{k_n} \times T^{k_n}(x_n, w_n) \longrightarrow (0, x)$. Hence there exists an integer $n_1 \geq N$ such that $x_{n_1} \in B(x, \varepsilon)$ and $T^{k_{n_1}} x_{n_1} \in B(0, \varepsilon)$, and so

$$T^{k_{n_1}}(U_x) \cap U_0 \neq \emptyset.$$

On the other hand since $(x, y) \in X \times X = J(x, 0)$, thus there exist a strictly increasing sequence of positive integers k_n and a sequence $\{(x_n, w_n)\} \subset X \times X$ such that $(x_n, w_n) \to (x, 0)$ and $T^{k_n} \times T^{k_n}(x_n, w_n) \longrightarrow (x, y)$. So, there exists an integer $n_2 \geq N$ such that $w_{n_2} \in B(0, \varepsilon)$ and $T^{k_{n_2}} w_{n_2} \in B(y, \varepsilon)$. This implies that

$$T^{k_{n_2}}(U_0) \cap U_y \neq \emptyset.$$

If $n = \max(n_1, n_2)$, then the above discussion shows that $J'(x) = X$. □

The paper is ended with four questions on the J-class operators.

It is well known that any powers of hypercyclic operators are hypercyclic with the same hypercyclic vectors ([1]). A similar question for the J-class operators can be given as follows:

Question 2.16. Let X be a Banach space and $T \in B(X)$. If T is a J-class operator, is T^n also a J-class operator for every $n \geq 2$? If so, what is the relation between their J-class vectors?

The first example of a hypercyclic operator whose adjoint is also hypercyclic was found by Salas ([19]). Later he showed that every separable Banach space with separable dual space, supports such an operator ([20]). A similar question can be posed as follows:

Question 2.17. Let X be a Banach space. Does there exist a non-hypercyclic operator $T \in B(X)$ such that both T and T^* are J-class operators?

If the converse of Theorem 2.15 is also true, then the following theorem is an immediate consequence of Theorem 2.14 and Theorem 2.15.

Theorem 2.18. *Let T be an operator on X. The following are equivalent*:

(i) *T satisfies the J-class criterion with respect to x,*
(ii) $(x, 0)$*, is a J-class vector $T \times T$,*
(iii) $J'(x) = X$.

So the third question is as follows:

Question 2.19. Let T be an operator on X. If $J'(x) = X$, can we say that $(x, 0)$ is a J-class vector for $T \times T$.

In Definition 1.2, if $k_n = n$ for all n, then we use the notation $J^{mix}(x)$ instead of $J(x)$. Inspired by the J-sets and the J^{mix}-sets, we introduce the following new definition:

Definition 2.20. Let T be an operator on X. For every $x \in X$ the set

$$\{(y_1, y_2) \in X \times X : \text{there exist a strictly increasing sequence of positive integers } \{k_n\} \text{ and sequence } \{x_n^{\,i}\} \subset X \text{ such that } x_n^{\,i} \to x \text{ and } T^{k_n} x_n^{\,i} \longrightarrow y_i,\ i = 1, 2\}$$

is denoted by $J^{\,wmix}(x)$. We say that a non-zero vector x is a $J^{\,wmix}$-class vector if $J^{\,wmix}(x) = X \times X$. Also, in this case we say that T is a $J^{\,wmix}$-class operator.

Obviously for an operator T and a vector $x \in X$ we have:

$$J^{\,mix}(x) \times J^{\,mix}(x) \subset J^{\,wmix}(x) \subset J(x) \times J(x).$$

Thus, $J^{\,mix}$-class vectors are $J^{\,wmix}$-class vectors, and the second ones are J-class vectors (and, correspondingly, for operators).

Question 2.21. If T and x satisfy in the J-class Criterion, can we say that x is a $J^{\,mix}$-class vector for T?

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