Commun. Korean Math. Soc. **34** (2019), No. 1, pp. 127–136
https://doi.org/10.4134/CKMS.c180082
pISSN: 1225-1763 / eISSN: 2234-3024

NORMALITY ON JACOBSON AND NIL RADICALS

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Abstract. This article concerns the normal property of elements on Jacobson and nil radicals which are generalizations of commutativity. A ring is said to be *right njr* if it satisfies the normal property on the Jacobson radical. Similarly a ring is said to be *right nunr* (resp., *right nlnr*) if it satisfies the normal property on the upper (resp., lower) nilradical. We investigate the relations between right duo property and the normality on Jacobson (nil) radicals. Related examples are investigated in the procedure of studying the structures of right njr, nunr, and nlnr rings.

Throughout this article every ring is associative with identity unless otherwise stated. Let R be a ring. $J(R)$, $N^*(R)$, $N_*(R)$, and $N(R)$ denote the Jacobson radical, the upper nilradical, the lower nilradical, and the set of all nilpotent elements in R, respectively. We also use *nilpotent* for a nilpotent element for simplicity. Note that $N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. $U(R)$ denotes the group of all units in R. Let $n \geq 2$. Denote the n by n full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $T_n(R)$); and write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. E_{ij} denotes the matrix with (i, j)-entry 1 and elsewhere 0. Let $R[x]$ (resp., $R[[x]]$) be the polynomial (resp., power series) ring with an indeterminate x over R. The set of all idempotents in R is denoted by $I(R)$.

In this article we will study the normal property of elements on Jacobson radicals, upper nilradicals, and lower nilradicals. These works are clearly generalizations of commutative rings. We first consider the following rings which provide the motivation for our study.

Example 0.1. We follow the ring construction in [8, Example 1.2] and adapt it for our purpose. Let S be a ring and $R_n = D_{2^n}(S)$, where n is a positive integer. Define a map

$$\sigma : R_n \to R_{n+1} \text{ by } A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Received March 6, 2018; Accepted May 18, 2018.
2010 *Mathematics Subject Classification.* 16N20, 16N40, 16U80.
Key words and phrases. right njr, right nunr, right nlnr, Jacobson radical, upper nilradical, lower nilradical, Abelian ring, right duo ring, reduced ring, matrix ring, polynomial ring.

Then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Set $R = \cup_{n=1}^{\infty} R_n$.

(1) Let S be a semiprime (i.e., $N_*(S) = 0$) ring. Then $N_*(R) = 0$ by [9, Theorem 2.2(1)], and furthermore we have

$$0 = N_*(R) \subsetneq N^*(R) = \{(a_{ij}) \in R \mid a_{ii} \in N^*(S)\} \subseteq J(R).$$

(2) Let S be a ring with $N_*(S) \neq 0$ and $N_*(S)^k = 0$ for some $k \geq 2$. Set $T = \cup_{i=1}^{\infty} D_{2^n}(N_*(S))$. Then T is an ideal of R with $T^k = 0$, and R/T is isomorphic to

$$R/T = \bigcup_{n=1}^{\infty} D_{2^n}(S/N_*(S)).$$

So R/T is semiprime by [9, Theorem 2.2(1)] because $S/N_*(S)$ is semiprime, and $N_*(R) = T$ follows. This implies

$$0 \neq T = N_*(R) \subsetneq N^*(R) = \{(a_{ij}) \in R \mid a_{ii} \in N^*(S)\} \subseteq J(R).$$

(3) Let S_0 be a semiprimitive (i.e., $J(S_0) = 0$) domain and $S = S_0[[x]]$. Then $J(S) = xS = xS_0[[x]]$ and moreover we have

$$0 = N_*(R) \subsetneq N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0\} \subsetneq \{(a_{ij}) \in R \mid a_{ii} \in J(S)\} = J(R),$$

by help of the argument in (1).

A ring R is usually called *reduced* if $N(R) = 0$. A ring is usually called *Abelian* if every idempotent is central. Reduced rings are easily shown to be Abelian. Observe that $D_n(R)$ is Abelian over an Abelian ring R by [7, Lemma 2], where $n \geq 1$.

Lemma 0.2. *Let R be a ring.*

(1) [1, Theorem 3] $N_*(R[x]) = N_*(R)[x]$.

(2) [1, Theorem 1] $J(R[x]) = I[x]$, *where $I = J(R[x]) \cap R$ is a nil ideal of R.*

1. Normality on Jacobson radicals

In this section we consider the normality of elements on Jacobson radicals. A ring R shall be said to satisfy the *right normal on Jacobson radical* (simply, said to be *right njr*) if $J(R)a \subseteq aJ(R)$ for all $a \in R$. Left njr rings can be defined analogously, and a ring is called *njr* if it is both right and left njr. It is evident that a ring R is njr if and only if $aJ(R) = J(R)a$ for all $a \in R$.

Following von Neumann [10], a ring R is said to be *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. Such kind of ring is also called *von Neumann regular* by Goodearl [4]. It is shown that R is regular if and only if every principal right (left) ideal of R is generated by an idempotent in [4, Theorem 1.1]. From this result we can obtain that regular rings are semiprimitive.

The n by n full matrix rings over regular rings are also regular by [4, Lemma 1.6]. Semiprimitive rings are clearly njr. But njr rings need not be Abelian as can be seen by $\mathrm{Mat}_n(R)$ with $n \geq 2$ over any regular ring R. In fact, $J(\mathrm{Mat}_n(R)) = 0$, and $\mathrm{Mat}_n(R)$ is non-Abelian because $E_{11}E_{12} = E_{12} \neq 0 = E_{12}E_{11}$ and $E_{11} \in I(\mathrm{Mat}_n(R))$. In the following we consider a condition under which given njr rings can be Abelian.

Proposition 1.1. *Let R be a ring with $N(R) \subseteq J(R)$. Then if R is right (or left) njr, then R is Abelian.*

Proof. Let R be njr and assume on the contrary that there exist $e \in I(R)$ and $a \in R$ such that $ea(1-e) \neq 0$. Note $ea(1-e) \in N(R)$. Since R is right njr and $N(R) \subseteq J(R)$, we get $ea(1-e) = [ea(1-e)](1-e) = (1-e)b$ for some $b \in J(R)$. This yields

$$0 \neq ea(1-e) = e[ea(1-e)] = e[(1-e)b] = 0,$$

a contradiction. Thus R is Abelian. The left case is proved similarly. □

We apply Proposition 1.1 to show that the upper triangular matrix rings need not be right (resp., left) njr. Consider $T_n(R)$ with $n \geq 2$ over any ring R with $N(R) \subseteq J(R)$. As $J(T_n(R))$ contains $N(T_n(R)) = \{(a_{ij}) \in T_n(R) \mid a_{ii} \in N(R) \text{ for all } i\}$, $T_n(R)$ cannot be right (resp., left) njr by Proposition 1.1 because $T_n(R)$ is non-Abelian. In fact, $E_{1n} = E_{1n}E_{nn} \in J(T_n(R))E_{nn}$ (resp., $E_{1n} = E_{11}E_{1n} \in E_{11}J(T_n(R))$) is not contained in

$$E_{nn}J(T_n(R)) = \{(a_{ij}) \in T_n(R) \mid a_{nn} \in J(R) \text{ and } a_{st} = 0 \text{ for all } s,t \text{ with } (s,t) \neq (n,n)\}$$

(resp.,

$$J(T_n(R))E_{11} = \{(a_{ij}) \in T_n(R) \mid a_{11} \in J(R) \text{ and } a_{st} = 0 \text{ for all } s,t \text{ with } (s,t) \neq (1,1)\}).$$

The converse of Proposition 1.1 need not hold by the following.

Example 1.2. (1) Let S be a reduced ring in Example 0.1. Then every R_n is an Abelian ring in which every idempotent is of the form $fE_{11} + \cdots + fE_{2^n,2^n}$ in R_n for some $f \in I(S)$, by [7, Lemma 2]; hence $R = \cup_{i=1}^{\infty} R_n$ is also Abelian because f is central in S. Observe that

$$N(R) = N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0\} \subseteq J(R),$$

noting $R/N^*(R) \cong S$.

Consider $E_{12}, E_{23} \in J(R)$. Then $E_{13} = E_{12}E_{23} \in J(R)E_{23}$ cannot be contained in $E_{23}J(R)$ because the first row of every matrix in $E_{23}J(R)$ is zero. Thus R is not right njr.

(2) Let K be a field of characteristic zero and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Let R be the factor ring $A/(ba-ab-1)$, called the first Weyl algebra over K, where $(ba-ab-1)$ is the ideal of A generated by $ba - ab - 1$. It is well-known that R is a simple domain. Write $\bar{r} = r + (ba - ab - 1)$ for $r \in A$. Consider $R[[x]]$. Then $N(R[[x]]) = 0$ and

$R[[x]]$ is Abelian. Moreover $J(R[[x]]) = xR[[x]]$. Assume that $R[[x]]$ is right njr. Consider $\bar{b}x \in J(R[[x]])$ and $\bar{a} \in R[[x]]$. Note $(\bar{b}x)\bar{a} = (\bar{b}\bar{a})x = (\bar{a}\bar{b}+1)x$. Since $R[[x]]$ is right njr, $\bar{b}x\bar{a} = \bar{a}f(x)$ for some $f(x) \in J(R[[x]])$. It then follows that $f(x) = \alpha x$ for some $\alpha \in R$. This yields

$$(\bar{a}\bar{b}+1)x = (\bar{b}x)\bar{a} = \bar{a}f(x) = (\bar{a}\alpha)x,$$

entailing $\bar{a}\bar{b}+1 = \bar{a}\alpha$. Thus we get $\bar{a}(\alpha - \bar{b}) = 1$, but this equality is impossible. Therefore $R[[x]]$ is not right njr. We can conclude similarly that $R[[x]]$ is not left njr.

Consider the ring R in Example 0.1 over a division ring S. Then

$$J(R) = N(R) = N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0\} \text{ and } R/J(R) \cong S.$$

So R is a local ring with nil $J(R)$. But R is not right njr by the argument in Example 0.1(1).

Proposition 1.3. *Let R be a local ring with $J(R)^2 = 0$. Then R is njr.*

Proof. Let $a \in R$. Suppose $a \in U(R)$ (i.e., $a \notin J(R)$). Then, for every $b \in J(R)$, $ab = aba^{-1}a \in J(R)a$ because $aba^{-1} \in J(R)$. Suppose $a \in J(R)$. Then $aJ(R) = 0 = J(R)a$ because $J(R)^2 = 0$. Thus R is right njr. The proof of left njr can be done similarly. □

$D_2(R)$ is njr by Proposition 1.3 when R is a division ring. Based on this result, one may ask whether $D_2(R)$ is right njr over a domain R. But the answer is negative by the following.

Example 1.4. (1) Let R be the simple domain in Example 1.2(2), and consider $D_2(R[[x]])$ over the ring $R[[x]]$ which is neither right nor left njr. Write $P = D_2(R[[x]])$. Observe first

$$J(R[[x]]) = xR[[x]] \text{ and } J(P) = \left\{ \begin{pmatrix} f & g \\ 0 & f \end{pmatrix} \mid f \in xR[[x]] \text{ and } g \in R[[x]] \right\}.$$

Consider $\left(\begin{smallmatrix} \bar{b}x & 0 \\ 0 & \bar{b}x \end{smallmatrix}\right) \in J(P)$ and $\left(\begin{smallmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{smallmatrix}\right) \in P$. Assume that $\left(\begin{smallmatrix} \bar{b}x & 0 \\ 0 & \bar{b}x \end{smallmatrix}\right)\left(\begin{smallmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} f & g \\ 0 & f \end{smallmatrix}\right)$ for some $\left(\begin{smallmatrix} f & g \\ 0 & f \end{smallmatrix}\right) \in J(P)$. Then we get $\bar{b}\bar{a}x = \bar{a}f$, but the existence of f is impossible by the argument in Example 1.2(2). Thus P is not right njr. Furthermore P is not left njr via a similar method.

(2) We consider a kind of subring of P in (1). Let R be the simple domain in Example 1.2(2), and consider $Q = D_2(R[x])$. Then $J(R[x]) = 0$ by Lemma 0.2(2), and $J(Q) = \left(\begin{smallmatrix} 0 & R[x] \\ 0 & 0 \end{smallmatrix}\right)$. Consider $\left(\begin{smallmatrix} 0 & \bar{b}x \\ 0 & 0 \end{smallmatrix}\right) \in J(Q)$ and $\left(\begin{smallmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{smallmatrix}\right) \in Q$. Assume that $\left(\begin{smallmatrix} 0 & \bar{b}x \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & g \\ 0 & 0 \end{smallmatrix}\right)$ for some $\left(\begin{smallmatrix} 0 & g \\ 0 & 0 \end{smallmatrix}\right) \in J(Q)$. Then we get $\bar{b}\bar{a}x = \bar{a}g$, but the existence of g is impossible by the argument in Example 1.2(2). Thus Q is not right njr. Similarly Q is shown to be not left njr.

Following Feller [3], a ring R is said to be *right duo* if every right ideal of R is two-sided. This is equivalent to the condition that $Rr \subseteq rR$ for all $r \in R$. A *left duo* ring is defined analogously. A ring is called *duo* if it is both right and left duo. Right (left) duo rings are easily shown to be Abelian. It is also proved simply that if a ring is right duo then so is each of its homomorphic images. One can see more useful results and examples for one-sided duo rings in [2, 11]. Let R be a right duo ring and $a^n = 0$ for some $a \in N(R)$. Then $Ra \subseteq aR$ and so we obtain

$$\begin{aligned}(RaR)^n &= Ra(Ra)(Ra)(Ra)\cdots(Ra)(Ra)R \subseteq Ra(aR)(aR)(aR)\cdots(aR)(aR)\\ &= Ra^2(Ra)(Ra)\cdots(Ra)(Ra)R \subseteq Ra^2(aR)(aR)\cdots(aR)(aR)\\ &= Ra^3(Ra)\cdots(Ra)(Ra)R \subseteq \cdots \subseteq Ra^nR = 0.\end{aligned}$$

This implies $N(R) = N^*(R) = N_*(R)$. This result is also valid for left duo rings, via a similar method. We will use this fact freely.

The simple domain R, in Example 1.2(2), is neither right nor left duo; and $D_2(R)$ is also neither right nor left njr.

Theorem 1.5. (1) *Let R be a ring. If $D_2(R)$ is right* (*resp., left*) *njr, then R is right* (*resp., left*) *duo.*

(2) *Let R be a semiprimitive ring. If R is right* (*resp., left*) *duo, then $D_2(R)$ is right* (*resp., left*) *njr.*

Proof. We apply the proof of [6, Theorem 1.2]. (1) Note first

$$J(D_2(R)) = \left\{ \begin{pmatrix} f & g \\ 0 & f \end{pmatrix} \mid f \in J(R) \text{ and } g \in R \right\},$$

and suppose that $D_2(R)$ is right njr. Let $0 \neq a \in R$. Given any $b \in R$, we will show $ba = ac$ for some $c \in R$. Consider $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in J(D_2(R))$ and $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \in D_2(R)$. Since $D_2(R)$ is right njr, $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} d & c \\ 0 & d \end{smallmatrix}\right)$ for some $\left(\begin{smallmatrix} d & c \\ 0 & d \end{smallmatrix}\right) \in J(D_2(R))$. Then $ba = ac$ and so R is right duo. The proof for the left case is similar.

(2) Note first $J(D_2(R)) = \left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right)$. Suppose that R is right duo. Consider $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in J(D_2(R))$ and $\left(\begin{smallmatrix} a & c \\ 0 & a \end{smallmatrix}\right) \in D_2(R)$. Since R is right duo, $ba = ad$ for some $d \in R$. So we have

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ad \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}.$$

But $\left(\begin{smallmatrix} 0 & d \\ 0 & 0 \end{smallmatrix}\right) \in J(D_2(R))$, and so $D_2(R)$ is right njr. The proof for the left case is similar. $\square$

In Theorem 1.5(2), observe that if the semiprimitive ring R is right or left duo then R is reduced. In fact, since R is right or left duo, we have $N_*(R) = N^*(R) = N(R) \subseteq J(R)$; hence $N(R) = 0$ because $J(R) = 0$.

Next we study the right njr property of polynomial rings.

Theorem 1.6. *Let R be a ring such that $N^*(R) \subseteq J(R[x])$. Suppose that $R[x]$ is right njr. Then $ab = ba$ for all $a \in R$ and $b \in N^*(R)$. Especially $N^*(R)$ is a commutative ring.*

Proof. Observe first that $J(R[x]) = N^*(R)[x]$ by Lemma 0.2(2) because $N^*(R) \subseteq J(R[x])$. Suppose that $R[x]$ is right njr. We apply the proof of [6, Theorem 2.9(1)]. Let $a \in R$ and $0 \neq b \in N^*(R)$. Then $b \in J(R[x])$.

Since $R[x]$ is right njr, $b(a+x) = (a+x)(b_0 + b_1x + \cdots + b_nx^n)$ for some $b_0 + b_1x + \cdots + b_nx^n \in J(R[x])$. But the degree of $b(a+x)$ is 1 and $n = 0$ follows, entailing $b_0 + b_1x + \cdots + b_nx^n = b_0$. Note $b_0 \in N^*(R)$. Furthermore $b(a+x) = (a+x)b_0$ implies $ba = ab_0$ and $b = b_0$. Thus $ba = ab$. Moreover, letting $a \in N^*(R)$, we conclude that $N^*(R)$ is a commutative ring. □

By help of Theorem 1.5, we can say that the right njr property does not go up to polynomial rings.

Example 1.7. Let R be a noncommutative division ring, and consider $D_2(R)$. Since R is semiprimitive and duo, $D_2(R)$ is njr by Theorem 1.5(2). Next consider $D_2(R)[x]$.

Assume that $D_2(R)[x]$ is right njr. Then, from the isomorphism $D_2(R)[x] \cong D_2(R[x])$, we obtain that $R[x]$ is right duo by Theorem 1.5(1). It then follows that $R[x]$ is commutative by [5, Lemma 3]. This is contrary to noncommutativity of R. Therefore $D_2(R)[x]$ is not right njr, in spite of $D_2(R)$ being right njr.

Example 1.7 also provides a counterexample for asking whether $R[x]$, over a ring R, is right njr if $N^*R) \subseteq J(R[x])$ and $N^*(R)$ is commutative. Let $T = D_2(R)$ in Example 1.7. Then $N^*(T)^2 = 0$ since $N^*(T) = \left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right)$; hence $N^*(T) \subseteq J(T[x])$ and $N^*(T)$ is commutative. But $T[x]$ is not right njr.

2. Normality on upper nilradicals

In this section we consider the normality of elements on upper nilradicals. A ring R shall be said to satisfy the *right normal on upper nilradical* (simply, said to be *right nunr*) if $N^*(R)a \subseteq aN^*(R)$ for all $a \in R$. Left nunr rings can be defined analogously, and a ring is called *nunr* if it is both right and left nunr. It is evident that a ring R is nunr if and only if $aN^*(R) = N^*(R)a$ for all $a \in R$.

Proposition 2.1. *Let R be a ring with $N(R) = N^*(R)$. Then if R is right* (*or left*) *nunr, then R is Abelian.*

Proof. The proof is done by using $N^*(R)$ in place of $J(R)$ in the proof of Proposition 1.1. □

We apply Proposition 2.1 to show that the upper triangular matrix rings need not be right nunr. Consider $T_n(R)$ with $n \geq 2$ over any ring R with $N(R) = N^*(R)$. Since $N^*(T_n(R)) = N(T_n(R))$, $T_n(R)$ cannot be right nunr by Proposition 2.1 because $T_n(R)$ is non-Abelian. In fact, $E_{1n} = E_{1n}E_{nn} \in N^*(T_n(R))E_{nn}$ is not contained in

$E_{nn}N^*(T_n(R)) = \{(a_{ij}) \in T_n(R) \mid a_{nn} \in N^*(R)$ and $a_{st} = 0$ for all s, t with

$$(s,t) \neq (n,n)\}.$$

$T_n(R)$ is also not left nunr via a similar argument to the case of left njr.

We next observe that right nunr rings need not be right njr. Consider the power series ring $R[[x]]$ in Example 1.2(1). Then $R[[x]]$ is a domain (hence nunr), but $R[[x]]$ is neither right nor left njr.

The converse of Proposition 2.1 need not be true by the following.

Example 2.2. Let R be the simple domain in Example 1.2(2), and consider $D_2(R[x])$. Then $D_2(R[x])$ is Abelian by [7, Lemma 2]. Note first $N^*(D_2(R[x])) = \left(\begin{smallmatrix} 0 & R[x] \\ 0 & 0 \end{smallmatrix}\right) = N(D_2(R[x]))$. The computation is almost similar to one of Example 1.4(2), but we write it for completeness. Consider $\left(\begin{smallmatrix} 0 & \bar{b}x \\ 0 & 0 \end{smallmatrix}\right) \in N^*(D_2(R[x]))$ and $\left(\begin{smallmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{smallmatrix}\right) \in D_2(R[x])$. Assume that $\left(\begin{smallmatrix} 0 & \bar{b}x \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \bar{a} & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & g \\ 0 & 0 \end{smallmatrix}\right)$ for some $\left(\begin{smallmatrix} 0 & g \\ 0 & 0 \end{smallmatrix}\right) \in N^*(D_2(R[x]))$. Then we get $\bar{b}\bar{a}x = \bar{a}g$, but the existence of g is impossible by the argument in Example 1.2(2). Thus $D_2(R[x])$ is not right nunr. Similarly $D_2(R[x])$ is shown to be not left nunr.

Recall that the domain R in Example 2.2 is neither right nor left duo, and that $D_2(R)$ is neither right nor left nunr.

Theorem 2.3. (1) *Let R be a ring. If $D_2(R)$ is right* (*resp., left*) *nunr, then R is right* (*resp., left*) *duo.*

(2) *Let R be a semiprime ring. If R is right* (*resp., left*) *duo, then $D_2(R)$ is right* (*resp., left*) *nunr.*

Proof. We apply the proof of Theorem 1.5. (1) Note first

$$N^*(D_2(R)) = \left\{ \begin{pmatrix} f & g \\ 0 & f \end{pmatrix} \mid f \in N^*(R) \text{ and } g \in R \right\}.$$

Suppose that $D_2(R)$ is right nunr. Let $0 \neq a \in R$. Consider $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in N^*(D_2(R))$ and $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \in D_2(R)$, where b is arbitrary in R. Since $D_2(R)$ is right nunr, $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} d & c \\ 0 & d \end{smallmatrix}\right)$ for some $\left(\begin{smallmatrix} d & c \\ 0 & d \end{smallmatrix}\right) \in N^*(D_2(R))$. Then $ba = ac$ and so R is right duo.

(2) Suppose that R is right duo. Then $N(R) = N_*(R) = N^*(R)$, and so R is reduced because $N_*(R) = 0$. It then follows that $N^*(D_2(R)) = \left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right)$. Consider $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in N^*(D_2(R))$ and $\left(\begin{smallmatrix} a & c \\ 0 & a \end{smallmatrix}\right) \in D_2(R)$. Since R is right duo, $ba = ad$ for some $d \in R$. So we have

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ad \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}.$$

But $\left(\begin{smallmatrix} 0 & d \\ 0 & 0 \end{smallmatrix}\right) \in N^*(D_2(R))$, and so $D_2(R)$ is right nunr. The proof for the left case is similar. □

3. Normality on lower nilradicals

In this section we consider the normality of elements on lower nilradicals. A ring R shall be said to satisfy the *right normal on lower nilradical* (simply, said to be *right nlnr*) if $N_*(R)a \subseteq aN_*(R)$ for all $a \in R$. Left nlnr rings can be defined analogously, and a ring is called *nlnr* if it is both right and left nlnr. It is evident that a ring R is nlnr if and only if $aN_*(R) = N_*(R)a$ for all $a \in R$.

Proposition 3.1. *Let R be a ring with $N(R) = N_*(R)$. Then if R is right (or left) nunr then R is Abelian.*

Proof. The proof is done by using $N_*(R)$ in place of $J(R)$ in the proof of Proposition 1.1. □

We apply Proposition 3.1 to show that the upper triangular matrix rings need not be right nlnr. Consider $T_n(R)$ with $n \geq 2$ over any ring R with $N(R) = N_*(R)$. Since $N_*(T_n(R)) = N(T_n(R))$, $T_n(R)$ cannot be right nlnr by Proposition 3.1 because $T_n(R)$ is non-Abelian. In fact, $E_{1n} = E_{1n}E_{nn} \in N_*(T_n(R))E_{nn}$ is not contained in

$$E_{nn}N_*(T_n(R)) = \{(a_{ij}) \in T_n(R) \mid a_{nn} \in N_*(R) \text{ and } a_{st} = 0 \text{ for all } s, t \text{ with } (s,t) \neq (n,n)\}.$$

$T_n(R)$ is also not left nlnr via a similar argument to the case of left njr.

In the following we observe that right nlnr rings need not be right nunr.

Example 3.2. We follow the ring construction in [8, Example 1.2]. Let S be a ring and $R_n = T_{2^n}(S)$, where n is a positive integer. Define a map

$$\sigma : R_n \to R_{n+1} \text{ by } A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Set $R = \cup_{n=1}^{\infty} R_n$. Suppose that S be a reduced ring. Then R is semiprime by [9, Theorem 2.2(2)]; hence R is nlnr. Next note that

$$N^*(R) = N(R) = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}.$$

However R is neither right nor left nunr by Proposition 2.1 because R is non-Abelian.

The converse of Proposition 3.1 need not be true by applying the argument in Example 2.2.

Theorem 3.3. (1) *Let R be a ring. If $D_2(R)$ is right (resp., left) nlnr, then R is right (resp., left) duo.*

(2) *Let R be a semiprime ring. If R is right (resp., left) duo, then $D_2(R)$ is right (resp., left) nlnr.*

Proof. The proof is done by replacing $N^*(D_2(R))$ by $N_*(D_2(R))$ in the proof of Theorem 2.3. □

The semiprime ring R in Theorem 3.3 is reduced when it is right or left duo. If R is right or left duo then $N(R) = N_*(R)$; hence $N(R) = 0$ because $N_*(R) = 0$.

Next we study the right nlnr property of polynomial rings.

Theorem 3.4. *Let R be a ring. Suppose that $R[x]$ is right njr. Then $ab = ba$ for all $a \in R$ and $b \in N_*(R)$. Especially $N_*(R)$ is a commutative ring.*

Proof. Observe first that $N_*(R[x]) = N_*(R)[x]$ by Lemma 0.2(1). Suppose that $R[x]$ is right nlnr. The proof is similar to one of Theorem 1.6. But we write it for completeness. Let $a \in R$ and $0 \neq b \in N_*(R)$. Then $b \in N_*(R[x])$. Since $R[x]$ is right nlnr, $b(a + x) = (a + x)(b_0 + b_1x + \cdots + b_nx^n)$ for some $b_0 + b_1x + \cdots + b_nx^n \in N_*(R[x])$. Then $b_0 + b_1x + \cdots + b_nx^n = b_0$, noting $b_0 \in N_*(R)$. Furthermore $b(a + x) = (a + x)b_0$ implies $ba = ab_0$ and $b = b_0$. Thus $ba = ab$. Moreover, we obtain that $N_*(R)$ is a commutative ring, letting $a \in N_*(R)$. □

By help of Theorem 3.3, we obtain the following.

Theorem 3.5. *For a ring R the following conditions are equivalent*:
(1) $D_2(R[x])$ *is right nlnr*;
(2) R *is commutative*;
(3) $D_2(R[x])$ *is commutative*;
(4) $D_2(R[x])$ *is right njr*;
(5) $D_2(R[x])$ *is right nunr.*

Proof. (1) $\Rightarrow$ (2). Let $D_2(R[x])$ be right nlnr. Then $R[x]$ is right duo by Theorem 3.3(1); hence $R[x]$ is commutative by [5, Lemma 3], entailing that R is commutative.

The proofs of (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (2) are similar to the proof of (1) $\Rightarrow$ (2), by using Theorem 1.5(1) and Theorem 2.3(1). The directions (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), and (3) $\Rightarrow$ (5) are obvious. □

Since $D_2(R)[x] \cong D_2(R[x])$ over any ring R, we also obtain from Theorem 3.5 then $D_2(R)[x]$ is right nlnr if and only if R is commutative if and only if $D_2(R)[x]$ is commutative if and only if $D_2(R)[x]$ is right njr if and only if $D_2(R)[x]$ is right nunr.

Let R be a ring. If $J(R)$ is nil then right njr is equivalent to right nunr. If $N^*(R) = N_*(R)$, then right nunr is equivalent to nlnr. We do not know the answers of the following.

Questions. (1) Are right njr rings right nunr?
(2) Are right nunr rings right nlnr?

Acknowledgments. The authors thank the referee for very careful reading of the manuscript and valuable suggestions that improved the paper by much. This work was supported by 2-year Research Grant of Pusan National University.

References

[1] S. A. Amitsur, *Radicals of polynomial rings*, Canad. J. Math. **8** (1956), 355–361.
[2] H. H. Brungs, *Three questions on duo rings*, Pacific J. Math. **58** (1975), no. 2, 345–349.

[3] E. H. Feller, *Properties of primary noncommutative rings*, Trans. Amer. Math. Soc. **89** (1958), 79–91.
[4] K. R. Goodearl, *von Neumann Regular Rings*, Monographs and Studies in Mathematics, **4**, Pitman (Advanced Publishing Program), Boston, MA, 1979.
[5] Y. Hirano, C. Y. Hong, J. Y. Kim, and J. K. Park, *On strongly bounded rings and duo rings*, Comm. Algebra **23** (1995), no. 6, 2199–2214.
[6] C. Y. Hong, H. K. Kim, N. K. Kim, T. K. Kwak, and Y. Lee, *One-sided duo property on nilpotents*, (submitted).
[7] C. Huh, H. K. Kim, and Y. Lee, *p.p. rings and generalized p.p. rings*, J. Pure Appl. Algebra **167** (2002), no. 1, 37–52.
[8] S. U. Hwang, Y. C. Jeon, and Y. Lee, *Structure and topological conditions of NI rings*, J. Algebra **302** (2006), no. 1, 186–199.
[9] Y. C. Jeon, H. K. Kim, Y. Lee, and J. S. Yoon, *On weak Armendariz rings*, Bull. Korean Math. Soc. **46** (2009), no. 1, 135–146.
[10] J. V. Neumann, *On regular rings*, Proceedngs of the National Academy of Sciences **22** (1936), 707–713.
[11] G. Thierrin, *On duo rings*, Canad. Math. Bull. **3** (1960), 167–172.

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