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ALGEBRAIC CHARACTERIZATION OF GRAPHICAL DEGREE STABILITY

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Abstract. In this paper, we introduce the *elimination ideal* $I_D(G)$ associated to a simple finite graph G. We obtain the upper bound of Castelnuovo-Mumford regularity of *elimination ideal* for various classes of graphs.

1. Introduction

Let G be a simple finite graph. The degree sequence of a graph is a monotone non-increasing sequence of positive integers. It has been studied extensively, and enjoys a rich literature in combinatorics. One popular chapter of this literature is the characterization of when an integer sequence can be a degree sequence; for example, see [7]. But its intrinsic algebraic properties that records its monotonic behavior is not known. Moreover, the class of monomial ideals of Borel type is important due to its strong connections with stable properties for instance see [6]. Link between Borel type ideals with the combinatorial properties of the graphs is missing for many years.

In this paper, we describe some new terms and connections. A new combinatorial term evolved in this study namely *Graphical Degree Stability* denoted by $\mathrm{Stab}_d(G)$. The graphical degree stability is key to many investigations discussed in this paper. We give a systematic procedure to compute the graphical degree stability, we call it as *Dominating Vertex Elimination Method* (DVE method). We compute the $\mathrm{Stab}_d(G)$ for complete graph (see Proposition 2.7), star graph (see Proposition 2.9), path graph (see Theorem 2.10), cyclic graph (see Theorem 2.12), fan graph (see Proposition 2.14), friendship graph (see Proposition 2.16), wheel graph (see Proposition 2.17) and complete bipartite graph (see Proposition 2.19). We use this concept to introduce the *elimination ideal* $I_D(G)$ of the graph G. The elimination ideal $I_D(G)$ is obtained through *sequential ideals* obtained from a graph by using DVE method. Moreover, we

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compute the upper bound of the Castelnuove-Mumford regularity of elimination ideals for the above mentioned families of graphs.

2. Degree stability of a graph

Throughout in this paper, we assume G to be a finite, simple and connected graph with the degree sequence $(d_1, d_2, \dots, d_n)$. There are many criterions to check whether a given non-increasing sequence of positive integers is graphic or not. Havel-Hakimi criterion (see [4] and [5]) states that a sequence $(d_1, d_2, \dots, d_n)$ of nonnegative integers such that $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic if and only if the sequence $(d_2 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$ is graphic (see [1]). We start with a structural definition associated to the degrees of vertices of a graph.

Definition 2.1. Let G be a simple connected graph on vertex set $v = \{v_1, \dots, v_n\}$. A *dominating vertex* of G is a vertex v_i having degree d_i such that $d_i \geq d_j$ for all $i \neq j$. Moreover, a *dominating set* $D(G)$ of G is the set consisting of all dominating vertices of G.

Remark 2.2. For a simple finite graph G, $D(G)$ is either singleton set or contain vertices having same degree.

Now we define an elementary type of graph.

Definition 2.3. A graph G having at least one isolated vertex is called a *scattered graph*.

Here, arises a natural question.

Question 2.4. How many maximally dominating vertices can be removed recursively from a given graph G without leaving a scattered subgraph?

Giving an answer to this question; we extend the Havel-Hakimi criterion and provide a systematic method named as *Dominating Vertex Elimination Method*.

Dominating Vertex Elimination Method: (DVE Method) For a given simple finite connected graph $G = G_0$ with a dominating set $D(G_0)$. Choose a vertex $v_0 \in D(G_0)$ such that $G_1 = G_0 - \{v_0\}$ is not a scattered graph. Again choose some vertex $v_1 \in D(G_1)$ such that $G_2 = G_1 - \{v_1\}$ is not a scattered graph with a dominating set $D(G_2)$. Repeat the process to get chain of subgraphs of G that is, $G = G_0 \supset G_1 \supset \cdots \supset G_r$. Since G is a finite graph so definitely this chain will stop so $r \leq n - 2$ where $n = |G|$.

Definition 2.5. Let G be a simple connected graph with vertex set $[n]$. By DVE method, we get a chain of subgraphs of G, $G = G_0 \supset G_1 \supset \cdots \supset G_r$, where the vertex set of G_k (for $1 \leq k \leq r$) is $[n - k]$. The maximum number r with the property that for all $i \leq r$, G_i is not a scattered graph, is said to be *graphical degree stability* of the graph G denoted by $\mathrm{Stab}_d(G)$. Moreover, we call this chain as *the sequential chain of subgraphs* of G.

Here, we give an example for more clarification.

Example 2.6. Consider the graph $G = G_0$ as shown in Figure 1. Note that

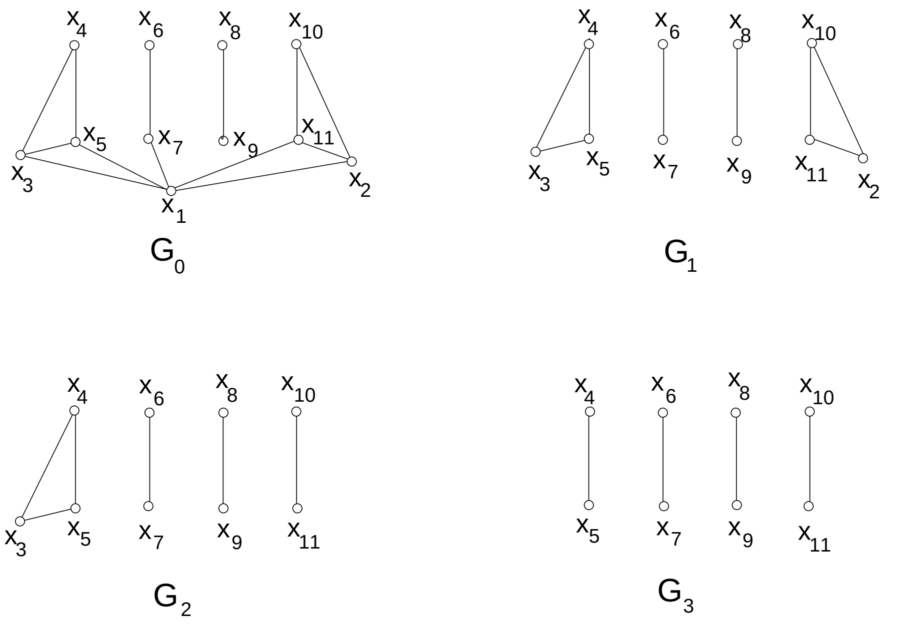

FIGURE 1. Vertex deletion of a graph

the degree sequence of a given graph $G = G_0$ is $(5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1)$, and $D(G_0) = \{x_1\}$. By DVE method removing x_1 from G_0, we have a new graph that is, $G_0 - \{x_1\} = G_1$ with $D(G_1) = \{x_2, x_3, x_4, x_5, x_{10}, x_{11}\}$. Since G_1 is not a scattered graph so again removing the vertex x_2 from G_1, we have a new graph that is, $G_1 - \{x_2\} = G_2$ with $D(G_2) = \{x_3, x_4, x_5\}$. Since G_2 is not a scattered graph so removing x_3 from G_2, we get a new graph that is, $G_2 - \{x_3\} = G_3$ with $D(G_3) = \{x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}$. Now, any more deletion of vertex will leave a scattered subgraph. As we can remove only 3 dominating degree vertices, so $\mathrm{Stab}_d(G) = 3$.

Now we present results regarding the graphical degree stability for some families of graphs.

Proposition 2.7. *The graphical degree stability of complete graph K_n is $n-2$ for $n \geq 3$.*

Proof. We will prove it by using induction on n. Clearly for $n = 3$, $\mathrm{Stab}_d(K_3) = 1$. Suppose the statement is true for $n = k-1$. Note that, removing any vertex from K_k results in K_{k-1}. Thus $\mathrm{Stab}_d(K_k) = k - 2$. □

Example 2.8. Consider the complete graph K_4, applying DVE method on K_4 we get K_3, then we get K_2 (i.e., an edge). We can not proceed further as any vertex deletion from K_2 will yield an isolated vertex. Hence, we have $\mathrm{Stab}_d(K_4) = 2$.

Now we proceed for the star graph S_n with n vertices.

Proposition 2.9. *The graphical degree stability of star graph S_n is 0 for $n \geq 2$.*

Proof. By applying the DVE method on star graph and removing the only dominating vertex from the dominating set of star graph gives us scattered graph, since star graph with n vertices has one vertex of degree $n-1$ and all other vertices of degree one. So $\mathrm{Stab}_d(S_n) = 0$. □

We continue with the path graph P_n with $n \geq 3$.

Theorem 2.10. *The graphical degree stability of a path P_n for $n \geq 3$ is given as;*

$$\mathrm{Stab}_d(P_n) = \begin{cases} \frac{n-3}{3}, & \text{if } n \equiv 0 \pmod 3, \\ \frac{n-4}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{n-2}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Since path P_n has two end vertices having degrees 1 and all other intermediate vertices having degrees 2. So $D(G_0) = D(P_n)$ contains all intermediate vertices. By DVE method, removing any intermediate vertex other then the neighbors of end vertices (as it will immediately give isolated vertices). For maximum number of deletion we remove every third vertex from any one side if it is allowed (that is if it is not the end vertex or neighbor of end vertex) yields sequential chain of subgraphs of G and we will remain with pieces of different lengths of this path graph. If we represent the number of these pieces by e then clearly $\mathrm{Stab}_d(P_n) = e - 1$.

Case 1. *When $n \equiv 0 \pmod 3$.*

Since $n \equiv 0 \pmod 3$ so $n = 3m$. Removing every third vertex will give us m pieces including $m-1$ paths of length one and one path of length 2. This implies $e = m = \frac{n}{3} \Rightarrow \mathrm{Stab}_d(P_n) = \frac{n}{3} - 1 = \frac{n-3}{3}$ *when* $n \equiv 0 \pmod 3$.

Case 2. *When $n \equiv 1 \pmod 3$.*

Since $n \equiv 1 \pmod 3$ so $n = 3m + 1$. Removing every third vertex will definitely give us m pieces including $m-1$ paths of length one and one path of length 3. This implies $e = m = \frac{n-1}{3} \Rightarrow \mathrm{Stab}_d(P_n) = \frac{n-1}{3} - 1 = \frac{n-4}{3}$ *when* $n \equiv 1 \pmod 3$.

Case 3. *When $n \equiv 2 \pmod 3$.*

Since $n \equiv 2 \pmod 3$ so $n = 3m+2$. Like before removing every third vertex will give us $m+1$ pieces of length 1. This implies $e = m + 1 = \frac{n-2}{3} + 1 = \frac{n+1}{3} \Rightarrow \mathrm{Stab}_d(P_n) = e - 1 = \frac{n+1}{3} - 1 = \frac{n-2}{3}$ *when* $n \equiv 2 \pmod 3$. □

Example 2.11. Consider the path P_7. Applying DVE method on P_7 we get two paths P_2 and P_4. We can not proceed further, as further deletion will yield a scattered graph. Therefore, $\mathrm{Stab}_d(P_7) = 1$.

Now we pick the cyclic graph C_n with $n \geq 3$.

Theorem 2.12. *The graphical degree stability of cyclic graph C_n for $n \geq 3$ is given as*;

$$\mathrm{Stab}_d(C_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \pmod 3, \\ \frac{n-1}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{n-2}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Deleting any one vertex from cyclic graph of order n results in path graph of order $n-1$ so $\mathrm{Stab}_d(C_n) = 1 + \mathrm{Stab}_d(P_{n-1})$.
Case 1. *When* $n \equiv 0 \pmod 3$.
Since $n \equiv 0 \pmod 3 \Rightarrow n-1 \equiv 2 \pmod 3$, Now from Proposition 2.10, $\mathrm{Stab}_d(P_{n-1}) = \frac{n-1-2}{3} = \frac{n}{3} - 1 \Rightarrow \mathrm{Stab}_d(C_n) = 1 + \mathrm{Stab}_d(P_{n-1}) = \frac{n}{3}$.
Case 2. *When* $n \equiv 1 \pmod 3$.
Since $n \equiv 1 \pmod 3 \Rightarrow n-1 \equiv 0 \pmod 3$. From previous Proposition 2.10, $\mathrm{Stab}_d(P_{n-1}) = \frac{n-1-3}{3} = \frac{n-4}{3} \Rightarrow \mathrm{Stab}_d(C_n) = 1 + \mathrm{Stab}_d(P_{n-1}) = 1 + \frac{n-4}{3} = \frac{n-1}{3}$.
Case 3. *When* $n \equiv 2 \pmod 3$.
Since $n \equiv 2 \pmod 3 \Rightarrow n-1 \equiv 1 \pmod 3$, Now from Proposition 2.10, $\mathrm{Stab}_d(P_{n-1}) = \frac{n-1-4}{3} = \frac{n-5}{3} \Rightarrow \mathrm{Stab}_d(C_n) = 1 + \mathrm{Stab}_d(P_{n-1}) = 1 + \frac{n-5}{3} = \frac{n-2}{3}$. □

Example 2.13. Consider the cycle C_7, Applying DVE method on C_7 we get a path P_6. Again applying the DVE method we get two paths P_2 and P_3. We can not proceed further as by continuing again we will get a scattered graph. So $\mathrm{Stab}_d(C_7) = 2$.

We proceed with the fan graph F_n containing n-vertices.

Proposition 2.14. *The graphical degree stability of fan graph F_n for $n \geq 2$ is given as*;

$$\mathrm{Stab}_d(F_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \pmod 3, \\ \frac{n-1}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{n-2}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Applying the DVE method on fan graph and deleting the only dominating vertex from the dominating set of fan graph of order n results in path graph of order $n-1$, thus the result followed from Theorem 2.12. □

Example 2.15. Consider the fan graph F_7. Applying DVE method on F_7 we get a path P_6. Again applying the DVE method we get two paths P_2 and P_3. We can not proceed further as by continuing again we will get a scattered graph. So $\mathrm{Stab}_d(F_7) = 2$.

Now we continue with the friendship graph $\mathcal{F}_n$ for $n \geq 2$:

Proposition 2.16. *The graphical degree stability of friendship graph $\mathcal{F}_n$ for $n \geq 2$ is* 1.

Proof. The dominating set for friendship graph consists of the central vertex to which all other vertices are adjacent. By DVE method removing this central vertex from $\mathcal{F}_n$ we obtain paths of length one. So the result followed. $\square$

Now, we pick wheel graph W_n with n vertices. In a wheel graph one vertex has degree $n-1$ and all other vertices have degrees 3.

Proposition 2.17. *The graphical degree stability of wheel graph W_n for $n \geq 4$ is given as*;

$$\mathrm{Stab}_d(W_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \pmod 3, \\ \frac{n+2}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{n+1}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Applying the DVE method on wheel graph and removing the only dominating vertex from the wheel graph of order n results in cyclic graph of order $n-1$, thus $\mathrm{Stab}_d(W_n) = 1 + \mathrm{Stab}_d(C_{n-1})$. Now the result followed from Theorem 2.12. $\square$

Example 2.18. Consider the wheel graph W_7. Applying DVE method on W_7 we get a cycle C_6. By DVE method again, we have a path P_5. Again applying the DVE method we get two P_2 paths. We can not proceed further as by continuing again we will get a scattered graph. So $\mathrm{Stab}_d(W_7) = 3$.

We conclude this section with the complete bipartite graphs.

Proposition 2.19. *The graphical degree stability of complete bipartite graph $K_{m,n}$ is $n-1$ for $m \geq n$.*

Proof. Since $K_{m,n}$ has m vertices of degree n and n vertices of degree m, and $m \geq n$ so by DVE method we can eliminate at most $n-1$ vertices of degree m without having scattered subgraphs. Thus $\mathrm{Stab}_d(K_{m,n}) = n-1$. $\square$

3. Stability properties of the elimination ideal of a graph

Throughout this section, we assume that $S = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over an infinite field K.

Definition 3.1. Let G be a simple connected graph on vertex set $V = \{v_1, \ldots, v_n\}$ with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. We define the *sequential ideal* of G as $Q(G) = (x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n})$.

Definition 3.2. Let G be a simple connected graph on vertex set $V = \{v_1, \ldots, v_n\}$ with graphical degree stability r and with sequential chain of subgraphs

$G = G_0 \supset G_1 \supset \cdots \supset G_r$ and sequential ideals $Q(G_i) = (x_1^{d_{i_1}}, x_2^{d_{i_2}}, \ldots, x_{n-i}^{d_{i_{n-i}}})$. We define the *elimination ideal* of G as,

$$I_D(G) = Q_{G_0} \cap Q_{G_1} \cap \cdots \cap Q_{G_r}.$$

Here follows a direct consequence of the above definition.

Corollary 3.3. *Let G be a simple connected graph on vertex set $V = \{v_1, v_2 \ldots, v_n\}$. Then*

(1) $\dim(S/I_D(G)) = \mathrm{Stab}_d(G)$.
(2) $\mathrm{depth}(S/I_D(G)) = 0$.
(3) $\mathrm{proj}\dim(S/I_D(G)) = n$.

Proof. Let $\mathrm{Stab}_d(G) = r$. By $I_D(G) = \bigcap_{i=1}^r Q_{G_i}$. As $m \in \mathrm{Ass}(S/I_D(G))$, so $m = P_0 \supset P_1 \supset \cdots \supset P_r$ where $P_i = \sqrt{Q_{G_i}}$. Therefore the $\mathrm{depth}(S/I_D(G)) = 0$. Moreover $P_r = (x_1, \ldots, x_{n-r}) \in \mathrm{Min}(S/I_D(G))$ so $\dim(S/I_D(G)) = r$. Hence by Auslander-Buchsbaum, we have $\mathrm{proj}\dim(S/I_D(G)) = n$. □

Here, we recall some elementary definitions and results regarding stable properties of ideals.

Let K be an infinite field, $S = K[x_1, x_2, \ldots, x_n]$, $n \geq 2$ the polynomial ring over K and $I \subset S$ a monomial ideal. Let $G(I)$ be the *minimal set of monomial generators* of I and $\deg(I)$ the *highest degree* of a monomial of $G(I)$. Given a monomial $u \in S$ set $m(u) = \max\{i \mid x_i \mid u\}$ and $m(I) = \max_{u \in G(I)} m(u)$. Also, $I_{\geq t}$ be the ideal generated by the monomials of I of degree $\geq t$. A monomial ideal I is *stable* if for each monomial $u \in I$ and $1 \leq j \leq m(u)$ it follows $\frac{x_j u}{x_{m(u)}} \in I$. If $\beta_{ij}(I)$ are graded Betti numbers of I, then the *Castelnuovo-Mumford regularity* of I is given by $\mathrm{reg}(I) = \max\{j - i : \beta_{ij} \neq 0\}$. Set $q(I) = m(I)(\deg(I) - 1) + 1$. Let $I \subset S$ be a monomial ideal and $I_{\geq q(I)}$ be the ideal generated by the monomials of I of degree $\geq q(I)$.

Definition 3.4. A monomial ideal $I \in S$ is said to be a *Borel-fixed ideal* if $(I : x_t^\infty) = (I : (x_1, \ldots, x_t)^\infty)$ for all $t = 1, \ldots, n$.

Moreover, Herzog, Popescu and Vladoiu in [6] stated that a monomial ideal is of *Borel type* if it fulfill the previous condition. Moreover, they mentioned that a monomial ideal I is of *Borel type*, if and only if for any monomial $u \in I$ and for any $1 \leq j < i \leq n$, there exists an integer $t > 0$ such that $x_j^t u / x_i^{\nu_i(u)} \in I$, where $\nu_i(u) > 0$ is the exponent of x_i in u.

Remark 3.5. If I, J are two ideals of Borel type, then $I + J$, $I \cap J$ and $I \cdot J$ are of Borel type. Also, a quotient ideal of an ideal of Borel type by a monomial ideal is of Borel type.

Here we recall the following result from [3].

Corollary 3.6. *Let I be a monomial ideal and $e \geq \deg(I)$ an integer such that $I_{\geq e}$ is stable. Then $reg(I) \leq e$.*

The bound for regularity of the *elimination ideal* of complete graph K_n is given in the following theorem.

Theorem 3.7. *Let $G = K_n$ be a complete graph. Then* $\operatorname{reg}(I_D(K_n)) \leq (n-1)^2$ *for $n \geq 3$.*

Proof. By Proposition 2.7, we have $\mathrm{Stab}_d(K_n) = n-2,\ n \geq 3$. Suppose for some fixed i term, $a_i = n-i-1$ and $\gamma(G_i) = a_i^2$ for $n \geq 3$ and $0 \leq i \leq n-2$. A sequential ideal of complete graph is of the form $Q_{G_i} = (x_1^{a_i}, x_2^{a_i}, \ldots, x_{n-i}^{a_i})$ for all $0 \leq i \leq n-2$. Therefore, its elimination ideal is $I_D(K_n) = \bigcap_{i=0}^{n-2} Q_{G_i}$. We first consider only the sequential ideals Q_{G_i} where $0 \leq i \leq n-2$ and $n \geq 3$. Now, we show that $Q_{G_i \geq \gamma(G_i)}$ is stable. For this, let $u \in Q_{G_i \geq \gamma(G_i)}$, so $u = v \cdot x_j^{a_i}$ for some $1 \leq j \leq n-i$ and $v \in (x_1, \ldots, x_{n-i})^{\gamma(G_i)-a_i}$ then u belongs to the stable ideal $(x_1, \ldots, x_{n-i})^{\gamma(G_i)}$. Now, we need to prove that $(x_1, \ldots, x_{n-i})^{\gamma(G_i)} \subset Q_{G_i \geq \gamma(G_i)}$. Let $w \in (x_1, \ldots, x_{n-i})^{\gamma(G_i)}$, then $w = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_{n-i}^{\alpha_{n-i}}$ with all $\alpha_t \geq 0$ and $\sum_{t=1}^{n-i} \alpha_t \geq \gamma(G_i)$. That is, if there exists k with $\alpha_k \geq a_i$ for some $1 \leq k \leq n-i$, then we can write $w = x_k^{a_i} \cdot w_1$ $\Rightarrow w \in Q_{G_i \geq \gamma(G_i)}$. Suppose contrary that there does not exist such k, then for all $1 \leq t \leq n$, $\alpha_t < a_i = n-i-1$. Consider the special case, let for all t, $\alpha_t = n-2-i$. Since $\sum_{t=1}^{n-i} \alpha_t \geq a_i^2$ so $\sum_{t=1}^{n-i} n-2-i \geq (n-1-i)^2$ $\Rightarrow (n-i)(n-2-i) \geq (n-1-i)^2 \Rightarrow 0 \geq 1$ which is a contradiction. Hence $Q_{G_i \geq \gamma(G_i)}$ is stable. Due to [1], we have $I_D(K_n) = \bigcap_{i=0}^{n-2} Q_{G_i}$ is stable for $\gamma(G)$ where $\gamma(G) = \max\{\gamma(G_i) | 0 \leq i \leq n-2\}$. Therefore, $I_D(K_n)_{\geq \gamma(G)}$ is stable. Hence by Corollary 3.6 $\operatorname{reg}(I_D(G)) \leq \gamma(G) = (n-1)^2$. □

Remark 3.8. In general, one cannot get $Q_{G_i \geq \gamma(G_i)-1}$ stable when $Q_{G_i} = (x_1^{a_i}, x_2^{a_i}, \ldots, x_{n-i}^{a_i})$ the sequential ideal for complete graph K_n for all $0 \leq i \leq n-2$, $a_i = n-i-1$ and $\gamma(G_i) = a_i^2$ for $n \geq 3$. For example, if $n = 4$ and $I = Q_{G_1} = ({x_1}^2, {x_2}^2, {x_3}^2)$, $\gamma(G_1) = 4$ and clearly $I_{\geq 3}$ is not stable.

Theorem 3.9. *Let $G = S_n$ be a star graph. Then* $\operatorname{reg}(I_D(S_n)) \leq n-1$ *for $n \geq 2$.*

Proof. By Proposition 2.9, we have $\mathrm{Stab}_d(S_n) = 0,\ n \geq 3$. So, $I_D(S_n) = Q_{G_0}$ and $Q_{G_0} = (x_1^{n-1}, x_2, \ldots, x_n)$. We first show that Q_{G_0} has stable ideal $Q_{G_0 \geq \gamma(G)}$ where $\gamma(G) = n-1$. Let $u \in Q_{G_0 \geq \gamma(G)}$, so $u = v \cdot x_j^{a_j}$ for some $1 \leq j \leq n$,

$$a_j = \begin{cases} n-1, & \text{if } j = 1, \\ 1, & \text{if } 2 \leq j \leq n, \end{cases}$$

and $v \in (x_1, \ldots, x_n)^{\gamma(G)-a_j}$ then u belongs to the stable ideal $(x_1, \ldots, x_n)^{\gamma(G)}$.

Now, we only need to prove that $(x_1, \ldots, x_n)^{\gamma(G)} \subset Q_{G_0 \geq \gamma(G)}$. If $w \in (x_1, \ldots, x_n)^{\gamma(G)}$, then $w = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with all $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i \geq \gamma(G)$. That is, if there exists some k with $\alpha_k \geq 1$ where $2 \leq k \leq n$ or $\alpha_1 \geq n-1$, then the result follows. Suppose contrary that there does not exist such k, then for all $2 \leq k \leq n$, $\alpha_k < 1$ and $\alpha_1 < n-1$. That is, $\alpha_k = 0$ for $2 \leq k \leq n$. But $\sum_{i=1}^{n} \alpha_i \geq \gamma(G) = (n-1)$. Therefore $\alpha_1 + \cdots + \alpha_n \geq n-1 \Rightarrow \alpha_1 \geq n-1$, which is a contradiction. Thus $Q_{G_0 \geq \gamma(G)}$ is stable. Therefore, $I_D(S_n)_{\geq \gamma(G)}$ is stable. Hence by Corollary 3.6, we have $\operatorname{reg}(I_D(G)) \leq \gamma(G) = n-1$. $\square$

Proposition 3.10. *Let $G = P_n$, $n \geq 3$ be a path graph. Then its sequential ideal $Q_{G_i}(P_n)$ will be: $Q_{G_i}(P_n) = (x_1^2, x_2^2, \ldots, x_{n-(2+3i)}^2, x_{n-(2+3i)+1}, \ldots, x_{n-i})$ for all $0 \leq i \leq r$, where $r = \operatorname{Stab}_d(P_n)$.*

Proof. The degree sequence of a general path graph $G = G_0 = P_n$ is $d_0 = (\underbrace{2, 2, \ldots, 2}_{n-2}, 1, 1)$. Using DVE method, we obtain G_1 having degree sequence $d_1 = (\underbrace{2, 2, \ldots, 2}_{n-5}, 1, 1, 1, 1)$. Applying DVE method recursively, we get a sequential chain of subgraphs $G = G_0 \supset G_1 \supset \cdots \supset G_r$ having degree sequence $d_i = (\underbrace{2, 2, \ldots, 2}_{n-(2+3i)}, \underbrace{1, 1, \ldots, 1}_{2(i+1)})$ with $n-i$ terms. Thus sequential ideal Q_{G_i} of path graph is $Q_{G_i}(P_n) = (x_1^2, x_2^2, \ldots, x_{n-(2+3i)}^2, x_{n-(2+3i)+1}, \ldots, x_{n-i})$. $\square$

Theorem 3.11. *Let $G = P_n$ be a path graph. Then $\operatorname{reg}(I_D(P_n)) \leq n-1$ for $n \geq 3$.*

Proof. By Proposition 2.10, let $\operatorname{Stab}_d(P_n) = e$, $n \geq 3$. Suppose for some fixed i term, $\gamma(G_i) = n - 3i - 1$ for $n \geq 3$ and $0 \leq i \leq e$. From Proposition 3.10, the sequential ideal of path graph is of the form $Q_{G_i} = (x_1^2, x_2^2, \ldots, x_{n-(2+3i)}^2, x_{n-(2+3i)+1}, \ldots, x_{n-i})$. And its elimination ideal is $I_D(P_n) = \bigcap_{i=0}^{e} Q_{G_i}$ where $e = \operatorname{Stab}_d(P_n)$. We first consider only the sequential ideals Q_{G_i}, where $0 \leq i \leq e$ and $n \geq 3$.

Now, we show that $Q_{G_i \geq \gamma(G_i)}$ is stable. For this, let $u \in Q_{G_i \geq \gamma(G_i)}$, so $u = v \cdot x_j^{a_j}$ for some $1 \leq j \leq n-i$ and $v \in (x_1, \ldots, x_{n-i})^{\gamma(G_i) - a_j}$ where

$$a_j = \begin{cases} 2, & \text{if } 1 \leq j \leq n-(2+3i), \\ 1, & \text{if } n-(2+3i)+1 \leq j \leq n-i, \end{cases}$$

then u belongs to the stable ideal $(x_1, \ldots, x_{n-i})^{\gamma(G_i)}$. Now, we need to prove that $(x_1, \ldots, x_{n-i})^{\gamma(G_i)} \subset Q_{G_i \geq \gamma(G_i)}$. Let $w \in (x_1, \ldots, x_{n-i})^{\gamma(G_i)}$, then $w = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_{n-i}^{\alpha_{n-i}}$ with all $\alpha_t \geq 0$ and $\sum_{t=1}^{n-i} \alpha_t \geq \gamma(G_i)$. That is, if there exists some k with $\alpha_k \geq 2$ for some $1 \leq k \leq n-(2+3i)$ or $\alpha_k \geq 1$ for some $n-(2+3i)+1 \leq k \leq n-i$, then we can write $w = x_k^{a_k} \cdot w_1 \Rightarrow w \in Q_{G_i \geq \gamma(G_i)}$ and the result follows. On contrary suppose that there does not exist such k, then for all $1 \leq k \leq n-(2+3i)$, $\alpha_k < 2$ and for all $n-(2+3i)+1 \leq k \leq n-i$, $\alpha_k < 1$. Since for all $n-(2+3i)+1 \leq k \leq n-i$, $\alpha_k < 1$ so $\alpha_k = 0$, and for all $1 \leq k \leq n-(2+3i)$ $\alpha_k < 2$ so we consider the special case when $\alpha_k = 1$. Since $\sum_{t=1}^{n-i} \alpha_t \geq n-3i-1$ so $\sum_{t=1}^{n-2-3i} \alpha_t \geq (n-3i-1)$

$\Rightarrow n-3i-2 \geq n-3i-1$ as for all $1 \leq k \leq n-(2+3i)$, $\alpha_k = 1$. This implies $1 \geq 2$ which is a contradiction. Hence $Q_{G_i \geq \gamma(G_i)}$ is stable. Now by [1, Proposition 1.1], $I_D(P_n) = \bigcap_{i=0}^{e} Q_{G_i}$ is stable for $\gamma(G)$, where $\gamma(G) = \max\{\gamma(G_i) \,|\, 0 \leq i \leq e\}$. Therefore, $I_D(P_n)_{\geq \gamma(G_0)}$ is stable. Hence by Corollary 3.6, $\operatorname{reg}(I_D(G)) \leq \gamma(G_0) = n-1$. $\square$

Theorem 3.12. *Let $G = C_n$ be a cyclic graph. Then $\operatorname{reg}(I_D(C_n)) \leq n+1$ for $n \geq 3$.*

Proof. By Proposition 2.12, let $\operatorname{Stab}_d(C_n) = e$ for $n \geq 3$. Note that $Q_{G_1}(C_n) = Q_{G_0}(P_{n-1})$ and $Q_{G_2}(C_n) = Q_{G_1}(P_{n-1})$ and so on. We have $Q_{G_i}(C_n) = Q_{G_{i-1}}(P_{n-1})$. By Theorem 3.11, we have $Q_{G_i}(P_{n-1})_{\geq n-3i-2}$ is stable ideal. This implies that $Q_{G_i}(C_n)_{\geq n-3i+1}$ is stable for $1 \leq i \leq e$. Now we show that for $i = 0$, Q_{G_0} is a stable ideal. The sequential ideal Q_{G_0} of cyclic graph is $Q_{G_0} = (x_1^2, x_2^2, \ldots, x_n^2)$. Let $u \in Q_{G_0 \geq \gamma(G_0)}$, where $\gamma(G_0) = n+1$ for $n \geq 3$, so $u = v \cdot x_j^{a_j}$ for some $1 \leq j \leq n$, and $v \in (x_1, \ldots, x_n)^{\gamma(G_0)-2}$, then u belongs to the stable ideal $(x_1, \ldots, x_n)^{\gamma(G_0)}$. Let $w \in (x_1, \ldots, x_n)^{\gamma(G_0)}$, then $w = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with all $\alpha_t \geq 0$ and $\sum_{t=1}^{n} \alpha_t \geq \gamma(G_0)$. That is, we need to prove that, if there exists some k such that $\alpha_k \geq 2$ for some $1 \leq k \leq n$, then we can write $w = x_k^{a_k} \cdot w_1 \Rightarrow w \in Q_{G_0 \geq \gamma(G_0)}$ and the result follows.

On contrary, we suppose that there does not exist such k, then for all $1 \leq k \leq n$, $\alpha_k < 2$. Consider the maximum possibility, suppose for all $1 \leq k \leq n$, $\alpha_k = 1$. Since $\sum_{t=1}^{n} \alpha_t \geq n+1$ so $\sum_{t=1}^{n} \alpha_t \geq n+1 \Rightarrow n \geq n+1$ which is a contradiction. Hence $Q_{G_0 \geq \gamma(G_0)}$ is stable. Now by [1, Proposition 1.1], $I_D(C_n) = \bigcap_{i=0}^{e} Q_{G_i}$ is stable for $\gamma(G)$ where $\gamma(G) = \max\{\gamma(G_i) \,|\, 0 \leq i \leq e\}$. Therefore, $I_D(C_n)_{\geq \gamma(G_0)}$ is stable. Hence by Corollary 3.6, $\operatorname{reg}(I_D(G)) \leq \gamma(G_0) = n+1$. $\square$

Remark 3.13. In general, one cannot get $Q_{G_i \geq \gamma(G_i)-1}$ stable, where Q_{G_i} is the sequential ideals of cyclic graph C_n for all $0 \leq i \leq r$, $r = \operatorname{Stab}_d(C_n)$ for $n \geq 3$, $\gamma(G_i) = n-3i+1$. For example, if $n = 5$ and $I = Q_{G_1} = ({x_1}^2, {x_2}^2, x_3, x_4)$, $\gamma(G_1) = 3$ and clearly $I_{\geq 2}$ is not stable.

Theorem 3.14. *Let $G = F_n$ be a fan graph. Then $\operatorname{reg}(I_D(F_n)) \leq 3n+1$ for $n \geq 2$.*

Proof. By Proposition 2.14, let $\operatorname{Stab}_d(F_n) = e$ for $n \geq 2$. Note that $Q_{G_1}(F_n) = Q_{G_0}(P_{n-1})$ and $Q_{G_2}(F_n) = Q_{G_1}(P_{n-1})$ and so on. We have $Q_{G_i}(F_n) = Q_{G_{i-1}}(P_{n-1})$ for $1 \leq i \leq e$. Since by Theorem 3.11, we have $Q_{G_i}(P_{n-1})_{\geq n-3i-2}$ is stable ideal. This implies that $Q_{G_i}(F_n)_{\geq n-3i+1}$ is stable for $1 \leq i \leq e$. Now we show that for $i = 0$, Q_{G_0} is a stable ideal. The sequential ideal Q_{G_0} of fan graph is $Q_{G_0} = (x_1^{n-1}, x_2^3, \ldots, x_{n-2}^3, x_{n-1}^2, {x_n}^2)$. Let $u \in Q_{G_0 \geq \gamma(G_0)}$, where $\gamma(G_0) = 3n+1$ for $n \geq 3$, so $u = v \cdot x_j^{a_j}$ for some $1 \leq j \leq n$, and

$v \in (x_1, \ldots, x_n)^{\gamma(G_0)-a_j}$ where

$$a_j = \begin{cases} n-1, & \text{if } j = 1, \\ 3, & \text{if } 2 \le j \le n-2, \\ 2, & \text{if } j = n-1, n, \end{cases}$$

then u belongs to the stable ideal $(x_1, \ldots, x_n)^{\gamma(G_0)}$. Now, we need to prove that $(x_1, \ldots, x_n)^{\gamma(G_0)} \subset Q_{G_0 \ge \gamma(G_0)}$. Let $w \in (x_1, \ldots, x_n)^{\gamma(G_0)}$, then $w = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with all $\alpha_t \ge 0$ and $\sum_{t=1}^{n} \alpha_t \ge \gamma(G_0)$. That is, we need to prove that, if there exists some k such that $\alpha_k \ge 3$ for some $2 \le k \le n-2$ or $\alpha_k \ge 2$ for some $k = n-1$ or $k = n$, or $\alpha_1 \ge n-1$, then we will be able to write $w = x_k^{a_k} \cdot w_1 \Rightarrow w \in Q_{G_0 \ge \gamma(G_0)}$ and the result follows.

On contrary, we suppose that there does not exist such k, then for all $2 \le k \le n-2$, $\alpha_k < 3$ and for $n-1 \le k \le n$, $\alpha_k < 2$ and $\alpha_1 \le n-1$. Consider the maximum possibility, so suppose for $k = n-1, n$ we have $\alpha_k = 1$, and for all $2 \le k \le n-2$, $\alpha_k = 2$ and $\alpha_1 = n-2$. Since $\sum_{t=1}^{n} \alpha_t \ge 3n+1$ so $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n \ge 3n+1$. This implies $n-2+2(n-4)+2(1) \ge 3n+1 \Rightarrow 3n-8 \ge 3n+1$ which is a contradiction. Hence $Q_{G_0 \ge \gamma(G_0)}$ is a stable ideal. Now by [1, Proposition 1.1], $I_D(F_n) = \bigcap_{i=0}^{e} Q_{G_i}$ is stable for $\gamma(G)$ where $\gamma(G) = \max\{\gamma(G_i) \mid 0 \le i \le e\}$. Therefore, $I_D(F_n)_{\ge \gamma(G_0)}$ is stable. Hence by Corollary 3.6, $\operatorname{reg}(I_D(G)) \le \gamma(G_0) = 3n+1$. □

Theorem 3.15. *Let $G = W_n$ be a wheel graph. Then* $\operatorname{reg}(I_D(W_n)) \le 3(n-1)$ *for $n \ge 4$.*

Proof. By Proposition 2.16, let $\operatorname{Stab}_d(W_n) = e$ for $n \ge 4$. Note that $Q_{G_i}(W_n) = Q_{G_{i-1}}(C_{n-1})$ for $1 \le i \le e$. Since by Theorem 3.11, we have

$$Q_{G_i}(P_{n-1})_{\ge n-3i-2}$$

is a stable ideal. This implies that $Q_{G_i}(W_n)_{\ge n-3i+3}$ is stable for $1 \le i \le e$. Now we show that for $i = 0$ Q_{G_0} is a stale ideal. The sequential ideal Q_{G_0} of wheel graph is $Q_{G_0} = (x_1^{n-1}, x_2^3, \ldots, x_n^3)$. Let $u \in Q_{G_0 \ge \gamma(G_0)}$, where $\gamma(G_0) = 3n+1$ for $n \ge 4$. So $u = v \cdot x_j^{a_j}$ for some $1 \le j \le n$, and $v \in (x_1, \ldots, x_n)^{\gamma(G_0)-a_j}$ where

$$a_j = \begin{cases} n-1, & \text{if } j = 1, \\ 3, & \text{if } 2 \le j \le n, \end{cases}$$

then u belongs to the stable ideal $(x_1, \ldots, x_n)^{\gamma(G_0)}$. Now, we need to prove that $(x_1, \ldots, x_n)^{\gamma(G_0)} \subset Q_{G_0 \ge \gamma(G_0)}$. Let $w \in (x_1, \ldots, x_n)^{\gamma(G_0)}$, then $w = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with all $\alpha_t \ge 0$ and $\sum_{t=1}^{n} \alpha_t \ge \gamma(G_0)$. That is, we need to prove that, if there exists some k such that $\alpha_k \ge 3$ for some $2 \le k \le n$ or $\alpha_1 \ge n-1$, then we will be able to write $w = x_k^{a_k} \cdot w_1 \Rightarrow w \in Q_{G_0 \ge \gamma(G_0)}$ and the result follows.

On contrary, we suppose that there does not exist such k, then for all $2 \le k \le n$, $\alpha_k < 3$ and $\alpha_1 \le n-1$. Consider the maximum possibility, so suppose for all $2 \le k \le n$, $\alpha_k = 2$ and $\alpha_1 = n-2$. Since $\sum_{t=1}^{n} \alpha_t \ge 3(n-1)$, that is $\alpha_1 + \alpha_2 + \cdots + \alpha_n \geq 3(n-1)$. This implies $n-2+2(n-2) \geq 3n-3$. $\Rightarrow 3n-6 \geq 3n-3$ which is a contradiction. Hence $Q_{G_0 \geq \gamma(G_0)}$ is a stable ideal. Now by [1, Proposition 1.1], $I_D(W_n) = \bigcap_{i=0}^{e} Q_{G_i}$ is stable for $\gamma(G)$ where $\gamma(G) = \max\{\gamma(G_i) \,|\, 0 \leq i \leq e\}$. Therefore, $I_D(W_n)_{\geq \gamma(G_0)}$ is stable. Hence by Corollary 3.6, $\operatorname{reg}(I_D(G)) \leq \gamma(G_0) = 3(n-1)$. □

Theorem 3.16. *Let $G = \mathcal{F}_n$ be a friendship graph. Then* $\operatorname{reg}(I_D(\mathcal{F}_n)) \leq 4n$ *for $n \geq 2$.*

Proof. By Proposition 2.17, let $\operatorname{Stab}_d(\mathcal{F}_n) = 1$ for $n \geq 2$. So $I_D(G) = Q_{G_0} \cap Q_{G_1}$, where $Q_{G_0} = (x_1^{2n}, x_2^2, \ldots, x_{2n+1}^2)$ and $Q_{G_1} = (x_1, x_2, \ldots, x_{2n})$. Clearly Q_{G_1} is a stable ideal for $\gamma(G_1) = 1$. We now show that Q_{G_0} is also a stable ideal for $\gamma(G_0) = 4n$.

The sequential ideal Q_{G_0} of friendship graph is $Q_{G_0} = (x_1^{2n}, x_2^2, \ldots, x_{2n+1}^2)$. Let $u \in Q_{G_0 \geq \gamma(G_0)}$, where $\gamma(G_0) = 4n$ for $n \geq 2$. so $u = v \cdot x_j^{a_j}$ for some $1 \leq j \leq n$, and $v \in (x_1, \ldots, x_{2n+1})^{\gamma(G_0)-a_j}$ where

$$a_j = \begin{cases} 2n, & \text{if } j = 1, \\ 2, & \text{if } 2 \leq j \leq 2n+1, \end{cases}$$

then u belongs to the stable ideal $(x_1, \ldots, x_{2n+1})^{\gamma(G_0)}$. Now, we need to prove that $(x_1, \ldots, x_{2n+1})^{\gamma(G_0)} \subset Q_{G_0 \geq \gamma(G_0)}$. Let $w \in (x_1, \ldots, x_{2n+1})^{\gamma(G_0)}$, then $w = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_{2n+1}^{\alpha_{2n+1}}$ with all $\alpha_t \geq 0$ and $\sum_{t=1}^{2n+1} \alpha_t \geq \gamma(G_0)$. That is, we need to prove that, if there exists some k such that $\alpha_k \geq 2$ for some $2 \leq k \leq 2n+1$ or $\alpha_1 \geq 2n$, then we will be able to write $w = x_k^{a_k} \cdot w_1 \Rightarrow w \in Q_{G_0 \geq \gamma(G_0)}$ and the result follows.

On contrary, we suppose that there does not exist such k, then for all $2 \leq k \leq 2n+1$, $\alpha_k < 2$ and $\alpha_1 \leq 2n$. Consider the maximum possibility, so suppose for all $2 \leq k \leq 2n+1$, $\alpha_k = 1$ and $\alpha_1 = 2n-1$. Since $\sum_{t=1}^{2n+1} \alpha_t \geq 4n$, that is $\alpha_1 + \alpha_2 + \cdots + \alpha_{2n+1} \geq 4n$. This implies $2n-1+1(2n-1) \geq 4n$ $\Rightarrow 4n-2 \geq 4n$ which is a contradiction. Hence $Q_{G_0 \geq \gamma(G_0)}$ is stable ideal. Now by [1, Proposition 1.1], $I_D(G) = \bigcap_{i=0}^{1} Q_{G_i}$ is stable for $\gamma(G)$ where $\gamma(G) = \max\{\gamma(G_i) \,|\, 0 \leq i \leq 1\}$. Therefore, $I_D(G)_{\geq \gamma(G_0)}$ is stable. Hence by Corollary 3.6, $\operatorname{reg}(I_D(G)) = \gamma(G_0) = 4n$. □

Remark 3.17. As the elimination ideal is an ideal of Boreltype. The upper bound of regularity of such ideals were discussed by Ahmad, Anwar in [1] and Cimpoeas in [2]. It is important to note that our obtained bounds are more finer than the one discussed in [1] and [2] for all above cases. It is also worth mentioning that the upper bound obtained above are combinatorial.

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