Commun. Korean Math. Soc. **34** (2019), No. 1, pp. 95–103
https://doi.org/10.4134/CKMS.c180018
pISSN: 1225-1763 / eISSN: 2234-3024

TORSION MODULES AND SPECTRAL SPACES

Hajar Roshan-Shekalgourabi

Abstract. In this paper we study certain modules whose prime spectrums are Noetherian or/and spectral spaces. In particular, we investigate the relationship between topological properties of prime spectra of torsion modules and algebraic properties of them.

1. Introduction

Throughout the article, R is a commutative ring with a nonzero identity and all modules are unitary. We recall some definitions. Let M be an R-module and N be a submodule of M. Then $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and the *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(0_M :_R M)$. If there is no ambiguity, we will write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$). N is said to be *prime* if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$) then $r \in (N : M)$ or $m \in N$. If N is prime, then the ideal $\mathfrak{p} := (N : M)$ is a prime ideal of R. In this case, N is said to be $\mathfrak{p}$-*prime* (see [13, 20]). The set of all prime submodules of an R-module M is called the *prime spectrum* of M and is denoted by $\mathrm{Spec}(M)$. Similarly, the collection of all $\mathfrak{p}$-prime submodules of an R-module M for any $\mathfrak{p} \in \mathrm{Spec}(R)$ is designated by $\mathrm{Spec}_{\mathfrak{p}}(M)$. The set of all prime submodules of M containing N is denoted by $V^*(N)$ (see [21]). Following [16], we define $V(N)$ as $\{P \in \mathrm{Spec}(M) \mid (P : M) \supseteq (N : M)\}$. Set $\mathsf{Z}(M) = \{V(N) : N \leq M\}$ and $\mathsf{Z}^*(M) = \{V^*(N) : N \leq M\}$. Then, the family composed by the $Z(M)$ is the family of closed sets of a topology on $\mathrm{Spec}(M)$, called the *Zariski topology* and denoted by τ. Moreover, if the collection of the sets $Z^*(M)$ is closed under finite union, then this the $Z^*(M)$ are the closed set of another topology, denoted by τ^*. When this is the case, we call the topology τ^* the *quasi-Zariski topology on* $\mathrm{Spec}(M)$ and M is called a *top* module (see [21]).

The concept of prime submodule has led to the development of topologies on the spectrum of modules. Topologies are considered by Duraivel, McCasland,

Received January 15, 2018; Revised June 15, 2018; Accepted August 29, 2018.
2010 *Mathematics Subject Classification.* Primary 13C13, Secondary 13C12, 54D05, 13D45.
Key words and phrases. torsion modules, prime submodules, Zariski topology, spectral space.

Moore, Smith, and Lu in [9, 16, 21]. It is well-known that the Zariski topology on the spectrum of prime ideals of a ring is one of the main tools in algebraic geometry. In the literature, there are many papers devoted to the Zariski topology on the spectrum of modules [1, 2, 6, 10, 18, 22, 26]. Finding relationship between topological properties of prime spectra of modules and algebraic properties of those modules is the subject of many articles. In Section 2, we consider the prime spectrum of modules with different topologies and introduce some conditions under which these are Noetherian or/and spectral spaces.

In the sequel, we recall briefly definitions and basic properties of certain topological spaces that we shall use. Let M be an R-module and N be a submodule of M. Note that $\mathrm{Spec}(\mathbf{0}) = \emptyset$ and that $\mathrm{Spec}(M)$ may be empty for some nonzero R-module M. For example, the *Prüfer group* $\mathbb{Z}_{p^\infty} = \{\alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = (r/p^n)+\mathbb{Z}, r \in \mathbb{Z}, n \in \mathbb{N}_0\}$ as a $\mathbb{Z}$-module has no prime submodule for any prime number p ([15]). Such a module is said to be *primeless*. In the sequel, we always assume that M is not primeless. M is called *primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the *natural map* $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$, defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ for every $P \in \mathrm{Spec}(M)$, is surjective. Finitely generated modules and free modules are primeful (see [16, 17]). The *radical* of N, denoted by $\mathrm{rad}_M(N)$ or briefly $\mathrm{rad}(N)$, is defined to be the intersection of all prime submodules of M containing N. In the case where there are no such prime submodules, $\mathrm{rad}(N)$ is defined as M. If $\mathrm{rad}(N) = N$, we say that N is a *radical submodule* (see [14, 19] and [11]). For an ideal I of R we recall that the *I-torsion submodule of* M is $\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n \in \mathbb{N}\}$ and M is said to be *I-torsion* if $M = \Gamma_I(M)$ (see [8]).

Let H be a proper submodule of an R-module M. We say that H is *maximal* if there is no submodule properly between H and M.

For the reader convenience, we collect several basic facts on prime submodules and prime spectra.

Remark 1.1. Let M be an R-module.

(1) If N is a submodule of M whose residual $(N : M)$ by N is a maximal ideal of R, then N is a prime submodule. In particular, $\mathfrak{m}M$ is a prime submodule of an R-module M for every maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{m}M \neq M$ (see [13, Proposition 2]).

(2) Let $\mathfrak{p}$ be a prime ideal of R and let N be any submodule of M and let $K \in \mathrm{Spec}_{\mathfrak{p}}(M)$. Then $K \cap N = N$ or $K \cap N \in \mathrm{Spec}_{\mathfrak{p}}(N)$ (see [21, Lemma 1.6]).

(3) $V(N) = V((N : M)M) = V^*((N : M)M)$ for every submodule N of M. In particular, $V(IM) = V^*(IM)$ for every ideal I of R (see [16, Result 3]).

(4) If M is a top module, in particular M is a multiplication module, then $\mathrm{Spec}(M)$ is a T_0-space for both the Zariski topology and the quasi-Zariski topology (see [16, Corollary 6.2]).

Remark 1.2. Let M be an R-module. By [16, Theorem 6.1], the following statements are equivalent:

(1) $(\mathrm{Spec}(M), \tau)$ is a T_0-space;

(2) $|\mathrm{Spec}_{\mathfrak{p}}(M)| \leq 1$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$.

Remark 1.3. Let X be a topological space.

(1) Let M be an R-module and Y be a subset of $\mathrm{Spec}(M)$. We will denote the intersection of all elements in Y by $\Im(Y)$ and the *closure* of Y in $\mathrm{Spec}(M)$ w.r.t the (quasi-)Zariski topology by $Cl(Y)$. By [16, Proposition 5.1], we have $V(\Im(Y)) = Cl(Y)$. An element $y \in Y$ is called a *generic point* of Y if $Y = Cl(\{y\})$.

(2) Following M. Hochster [12], we say that a topological space Y is a *spectral space* in the case where Y is homeomorphic to $\mathrm{Spec}(S)$, with the Zariski topology, for some ring S. Spectral spaces have been characterized by Hochster [12, p. 52, Proposition 4] as the topological spaces Y which satisfy the following conditions: (1) Y is a T_0-space; (2) Y is quasi-compact; (3) the quasi-compact open subsets of Y are closed under finite intersections and form a basis of open sets; (4) each irreducible closed subset of Y has a generic point. For examples of modules whose prime spectrum is spectral, see [1, 16].

(3) A Noetherian space is spectral if and only if it is T_0 and every non-empty irreducible closed subspace has a generic point ([12, pp. 57–58]). We recall that if M is a top R-module, then $(\mathrm{Spec}(M), \tau^*)$ is a T_0-space and every irreducible closed subset of $\mathrm{Spec}(M)$ has a generic point (see Remark 1.1(4) and [3, Theorem 3.3]).

2. Main results

We consider the prime spectra of certain torsion modules with different topologies and introduce some conditions under which these are Noetherian or/and spectral spaces. For more information about the modules whose prime spectrums are Noetherian or/and spectral spaces, see [1, 16, 18]. The following lemma is quite useful for our purpose.

Lemma 2.1. *Let M be an R-module and N be a proper submodule of M. Then, the following statements hold.*

1. *Let P be a $\mathfrak{p}$-prime submodule of M for some prime ideal $\mathfrak{p}$ of R. Then, for each ideal J of R such that $J \nsubseteq \mathfrak{p}$, we have $\Gamma_J(M) \subseteq P$.*
2. *If $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where M_λ is an $\mathfrak{m}_\lambda$-torsion submodule of M (where $X := \{m_\lambda \mid \lambda \in \Lambda\}$ is a collection of distinct maximal ideals of R), then $N \in \mathrm{Spec}(M)$ if and only if $(N : M) \in \mathrm{Max}(R)$.*

Proof. (1) Let J be an ideal of R such that $J \nsubseteq \mathfrak{p}$ and let $m \in \Gamma_J(M)$. Then, there is an integer $n \in \mathbb{N}$ such that $J^n m = 0 \in P$. By definition of prime submodule and assumption, we infer that $m \in P$.

(2) ($\Leftarrow$) If N is a proper submodule of M such that $(N : M) \in \mathrm{Max}(R)$, then N is a prime submodule of M by Remark 1.1(1). ($\Rightarrow$) Let N be a $\mathfrak{p}$-prime submodule of M. Then $N \cap M_h \neq M_h$ for some $h \in \Lambda$. By Remark 1.1(2), $N \cap M_h \in \mathrm{Spec}_{\mathfrak{p}}(M_h)$. Therefore, $(N : M) = (N \cap M_h : M_h) = \mathfrak{p}$. It follows from (1) that $(N : M) = \mathfrak{p} = \mathfrak{m}_h \in \mathrm{Max}(R)$. □

There are well-known type of modules that satisfy the assumptions of Lemma 2.1(2). For example, if R is a Noetherian ring and M is a nonzero R-module such that $\mathrm{Ass}(M) \subseteq \mathrm{Max}(R)$, then $M \cong \bigoplus_{\mathfrak{p}\in \mathrm{Ass}(M)} \Gamma_{\mathfrak{p}}(M)$. To see this, we show that $M \to M_p$, where $\mathfrak{p} \in \mathrm{Ass}(M)$ is surjective. By [5, Proposition 3.9], it suffices to prove that for every maximal ideal $\mathfrak{m}$, the induced map $M_{\mathfrak{m}} \to (M_{\mathfrak{p}})_{\mathfrak{m}}$ is surjective. If $\mathfrak{m} = \mathfrak{p}$, then $M_{\mathfrak{m}} \to (M_{\mathfrak{p}})_{\mathfrak{m}}$ is essentially the identity map. On the other hand, if $\mathfrak{m} \neq \mathfrak{p}$, then $\mathfrak{m} \notin \mathrm{Ass}(M_p) = \mathrm{Supp}(M_p)$ so that $(M_{\mathfrak{p}})_{\mathfrak{m}} = 0$ so $M_{\mathfrak{m}} \to (M_{\mathfrak{p}})_{\mathfrak{m}}$ is surely surjective. Likewise this shows that for each maximal ideal $\mathfrak{m}$, the obvious map $M_{\mathfrak{m}} \to (\oplus_{\mathfrak{p}\in \mathrm{Ass}(M)} M_{\mathfrak{p}})_{\mathfrak{m}}$ is an isomorphism (see [23, Theorem 7.37]), showing that $M \to \oplus_{\mathfrak{p}\in \mathrm{Ass}(M)} M_{\mathfrak{p}}$ is an isomorphism. Since R is Noetherian and $\mathrm{Ass}(M) \subseteq \mathrm{Max}(R)$, for each $\mathfrak{q} \in \mathrm{Ass}(M)$ we have $\Gamma_{\mathfrak{q}}(M) \cong \Gamma_{\mathfrak{q}}(\oplus_{\mathfrak{p}\in \mathrm{Ass}(M)} M_{\mathfrak{p}}) = \Gamma_{\mathfrak{q}}(M_{\mathfrak{q}}) = M_{\mathfrak{q}}$. This implies that $M \cong \bigoplus_{\mathfrak{p}\in \mathrm{Ass}(M)} \Gamma_{\mathfrak{p}}(M)$.

Proposition 2.2. *Let $\{\mathfrak{m}_\lambda\}_{\lambda\in\Lambda}$ be a collection of distinct maximal ideals of R. Suppose $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where M_λ is an $\mathfrak{m}_\lambda$-torsion submodule of M. If*

$$\Lambda' = \{\lambda \in \Lambda \mid \mathfrak{m}_\lambda M \neq M\}$$

is a finite set, then the topological space $(\mathrm{Spec}(M), \tau)$ is Noetherian.

Proof. Let $V(N_1) \supseteq V(N_2) \supseteq \cdots$ be a descending chain of closed subsets of $(\mathrm{Spec}(M), \tau)$. Then, we have an ascending chain

$$\Im(V(N_1)) \subseteq \Im(V(N_2)) \subseteq \cdots$$

of radical submodules of M and so we have an ascending chain of radical ideals

$$(\Im(V(N_1)) : M) \subseteq (\Im(V(N_2)) : M) \subseteq \cdots \tag{2.1}$$

of R. By Lemma 2.1(2), each term of Equation (2.1) is $(\cap_\alpha P_\alpha : M) = \cap_\alpha (P_\alpha : M)$ an intersection of maximal ideals of R. Since $\Lambda' = \{\lambda \in \Lambda \mid \mathfrak{m}_\lambda M \neq M\}$ is a finite set, there exists a positive integer k such that

$$(\Im(V(N_k)) : M)M = (\Im(V(N_{k+i})) : M)M$$

for each $i = 1, 2, \ldots$. By Remark 1.1(3),

$$V(\Im(V(N_k))) = V(\Im(V(N_{k+i}))).$$

By Remark 1.3(2), $V(N_k) = V(N_{k+i})$, and so $(\mathrm{Spec}(M), \tau)$ is a Noetherian space. □

The next corollary was proved in [1]. We can use Proposition 2.2 to give a new proof.

Corollary 2.3. *Let M be an Artinian R-module. Then $(\mathrm{Spec}(M), \tau)$ is a Noetherian space.*

Proof. Since M is Artinian, $\mathrm{Ass}(M)$ is finite, and thus there are finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of R such that $M = \Gamma_{\mathfrak{m}_1}(M) \oplus \cdots \oplus \Gamma_{\mathfrak{m}_r}(M)$ (see [25, p.166]). Now the result follows from Proposition 2.2. □

In the next corollary, we investigate the spectralness of the prime spectrum of torsion modules over Dedekind domains.

Proposition 2.4. *Let R be a Dedekind domain which is not field and M be a torsion R-module such that* $(\mathrm{Spec}(M), \tau)$ *is a T_0-space. Then,* $(\mathrm{Spec}(M), \tau)$ *is spectral if and only if it is finite.*

Proof. Obviously, a finite T_0 space is always spectral. On the other hand, we suppose that $\mathrm{Spec}(M)$ is infinite. By assumption we have $\mathrm{Ass}(M) \subseteq \mathrm{Max}(R)$ and $M = \bigoplus_{\mathfrak{p}\in \mathrm{Ass}(M)} \Gamma_{\mathfrak{p}}(M)$.

We claim that $(\mathrm{Spec}(M), \tau)$ is an irreducible space. To see this, let $\mathrm{Spec}(M) = V_1 \cup V_2$, where V_i is a closed subset of $\mathrm{Spec}(M)$. Then there are submodules N and L of M such that $V_1 = V(N)$ and $V_2 = V(L)$. Without loss of generality we can assume that $V_2 = V(L)$ is infinite. This implies that $(L : M)$ is contained in infinitely many prime ideals because $(\mathrm{Spec}(M), \tau)$ is a T_0-space. Since R is a Dedekind domain, $(L : M) = (0)$. Therefore, $\mathrm{Spec}(M) = V(L)$. This shows that $(\mathrm{Spec}(M), \tau)$ is irreducible.

Now, suppose that $(\mathrm{Spec}(M), \tau)$ is a spectral space. By Remark 1.3(3), every irreducible closed subset of $\mathrm{Spec}(M)$ has a generic point. Thus, there exists a prime submodule P of M such that $\mathrm{Spec}(M) = V(P)$. Hence, $(P : M)$ is contained in infinitely many prime ideals. Since R is a Dedekind domain, $(P : M) = (0)$. This is a contradiction, because $M = \bigoplus_{\mathfrak{p}\in \mathrm{Ass}(M)} \Gamma_{\mathfrak{p}}(M)$ and $(P : M)$ is a maximal ideal of R by Lemma 2.1(2). □

Example 2.5. Consider $M = \bigoplus_p (\mathbb{Z}/p\mathbb{Z})$ as a $\mathbb{Z}$-module, where p runs through the set of all prime numbers. It is easy to see that M is a torsion module and $\mathrm{Spec}(M)$ is an infinite set (see [17, Example 1]). By Proposition 2.4, $(\mathrm{Spec}(M), \tau)$ is not a spectral space.

It is well-known that if R is an Artinian ring, then every prime ideal of R is maximal and $\mathrm{Spec}(R)$ is a finite set. Now, we generalize this result to Artinian modules. Note that, for any ring S, $(\mathrm{Spec}(S), \tau) = (\mathrm{Spec}(S), \tau^*)$ is a T_0-space.

Corollary 2.6. *Let M be an Artinian R-module such that* $(\mathrm{Spec}(M), \tau)$ *is a T_0-space. Then, every prime submodule of M is maximal and* $\mathrm{Spec}(M)$ *is a finite set.*

Proof. As we mentioned, there exist finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ of R such that $M = \Gamma_{\mathfrak{m}_1}(M) \oplus \cdots \oplus \Gamma_{\mathfrak{m}_r}(M)$. Thus, M satisfies the assumptions of Lemma 2.1(2). Let $P \in \mathrm{Spec}(M)$. Then, by Lemma 2.1(2), there is a maximal ideal $\mathfrak{m}_\lambda$ of R such that $\mathfrak{m}_\lambda = (P : M)$. Suppose that L is a proper submodule of M such that $P \subseteq L$. Then $\mathfrak{m}_\lambda = (P : M) = (L : M)$. By [13, p. 63, Proposition 4], $\mathfrak{m}_\lambda M$ and L are $\mathfrak{m}_\lambda$-prime submodules of M. Since $(\mathrm{Spec}(M), \tau)$ is a T_0-space, $P = L = \mathfrak{m}_\lambda M$ by Remark 1.2. Consequently, $P = \mathfrak{m}_\lambda M \in \mathrm{Max}(M)$.

Also, this implies that $\mathrm{Spec}(M) = \bigcup_{i=1}^{r} \mathrm{Spec}_{\mathfrak{m}_i}(M)$. Remark 1.2 implies that $\mathrm{Spec}(M)$ must be finite. □

Corollary 2.7. *Let M be an Artinian R-module. Then* $(\mathrm{Spec}(M), \tau)$ *is a T_0-space if and only if* $(\mathrm{Spec}(M), \tau)$ *is a spectral space.*

Proof. If $(\mathrm{Spec}(M), \tau)$ is a T_0-space, then $\mathrm{Spec}(M)$ is a finite set, by Corollary 2.6. Hence, [16, Theorem 6.8] yields that $(\mathrm{Spec}(M), \tau)$ is a spectral space. Conversely, if $(\mathrm{Spec}(M), \tau)$ is a spectral space, then $(\mathrm{Spec}(M), \tau)$ is a T_0-space by Remark 1.3(2). □

Several papers (e.g. [1,3,4,6,7,16,27]) considered the so called top modules. This class of modules was introduced in [21]. Recall that a submodule S of an R-module M is said to be *semiprime* if S is an intersection of prime submodules of M. Also, recall that a prime submodule P of M is called *extraordinary* if whenever N and L are semiprime submodules of M with $N \cap L \subseteq P$, then $N \subseteq P$ or $L \subseteq P$. By [21, Lemma 2.1], an R-module M is top if and only if every prime submodule of M is extraordinary.

Theorem 2.8. *Let I be an ideal of R such that $V(I)$ is a finite subset of* $\mathrm{Max}(R)$ *and let M be an I-torsion R-module.*

(1) *If* $(\mathrm{Spec}(M), \tau)$ *is a T_0-space, then M is top and* $(\mathrm{Spec}(M), \tau^*)$ *is a spectral space.*
(2) $(\mathrm{Spec}(M), \tau)$ *is a Noetherian topological space. Moreover,* $(\mathrm{Spec}(M), \tau)$ *is a spectral space if and only if it is a T_0-space.*

Proof. (1) Let P be a $\mathfrak{p}$-prime submodule of M and also let N and L be two semiprime submodules of M such that $N \cap L \subseteq P$. By Lemma 2.1, $I \subseteq \mathfrak{p}$. We have $\mathfrak{p}M \subseteq P \subsetneq M$ and therefore $\mathfrak{p} = (\mathfrak{p}M : M) = (P : M)$ (since $\mathfrak{p}$ is a maximal ideal of R) and $\mathfrak{p}M$ is a $\mathfrak{p}$-prime submodule of M by Remark 1.1(1). By Remark 1.2, $P = \mathfrak{p}M$ and so

$$\mathrm{Spec}(M) := \{\mathfrak{q}M \mid \mathfrak{q} \in V(I) \text{ and } \mathfrak{q}M \neq M\}$$

is a finite set. Therefore, $(N : M)$ and $(L : M)$ are finite intersections of maximal ideals with $(N : M) \cap (L : M) \subseteq \mathfrak{p}$. This implies that $N \subseteq P$ or $L \subseteq P$. Hence, M is a top module. As a finite space, $(\mathrm{Spec}(M), \tau^*)$ is a spectral space.

(2) Let N be a submodule of M and $P \in V(N)$. Then by Lemma 2.1 and the assumptions, $\{(P : M) \mid P \in V(N)\}$ is a finite set. Let

$$V(N_1) \supseteq V(N_2) \supseteq \cdots$$

be a descending chain of closed subsets of $(\mathrm{Spec}(M), \tau)$. So, we have an ascending chain

$$\Im(V(N_1)) \subseteq \Im(V(N_2)) \subseteq \cdots$$

of radical submodules of M and so we have an ascending chain of radical ideals

$$(\Im(V(N_1)) : M) = \bigcap_{P \in V(N_1)} (P : M) \subseteq (\Im(V(N_2)) : M) \subseteq \cdots .$$

By the above argument, there exists a positive integer k such that

$$(\Im(V(N_k)) : M)M = (\Im(V(N_{k+i})) : M)M$$

for each $i = 1, 2, \ldots$. According to Remark 1.1(3), we have

$$V(\Im(V(N_k))) = V(\Im(V(N_{k+i}))).$$

Now, Remark 1.3(2) implies that $V(N_k) = V(N_{k+i})$, and so $(\mathrm{Spec}(M), \tau)$ is a Noetherian space.

One implication of the last assertion follows from Remark 1.3(3), and so we assume that $(\mathrm{Spec}(M), \tau)$ is a T_0-space. Then, by part (1) and Remark 1.2, $(\mathrm{Spec}(M), \tau)$ is a finite topological space. Hence, it is a spectral space, by [16, Theorem 6.8]. □

Example 2.9. Let R be a one-dimensional Noetherian integral domain and I be a nonzero ideal of R. Let N be an R-module, n be a positive integer and $M = \mathrm{Ext}^n_R(R/I, N)$, where Ext^n_R is n-th right derived functor of Hom (see [24]). Let $x \in I$. Then $xM = \mathrm{Ext}^n_R(x(R/I), N) = \mathrm{Ext}^n_R(0, N) = 0$, since Ext^n_R is an R-linear functor. Therefore, M is I-torsion. If $(\mathrm{Spec}(M), \tau)$ is a T_0-space, then M is a top R-module and $(\mathrm{Spec}(M), \tau^*)$ is a spectral space, by Theorem 2.8.

We recall that a family $\{M_i\}_{i\in I}$ of R-modules is said to be *prime-compatible* if, for all $i \neq j$ in I and every prime ideal $\mathfrak{p}$ of R, at least one between $\mathrm{Spec}_{\mathfrak{p}}(M_i)$ and $\mathrm{Spec}_{\mathfrak{p}}(M_j)$ is empty (see [21]).

Theorem 2.10. *Let $\{M_i\}_{i\in I}$ be R-modules. Then the following statements are equivalent for the R-module $M = \oplus_{i\in I} M_i$.*

(1) *M is a top module.*
(2) *$M_i \oplus M_j$, is a top module for all $i \neq j$ in I.*
(3) *$\{M_i\}_{i\in I}$ are prime-compatible top modules.*

Proof. See [21, Theorem 5.1]. □

Corollary 2.11. *Let I be a nonzero ideal of R.*

(1) *If R is a Dedekind domain and M is an I-torsion R-module such that $(\mathrm{Spec}(M), \tau)$ is a T_0-space, then M is a top module and $(\mathrm{Spec}(M), \tau^*)$ is a spectral space.*
(2) *Let R be a one-dimensional integral domain with Noetherian spectrum and $\{\mathfrak{q}_\lambda\}_{\lambda\in\Lambda}$ be a collection of the comaximal ideals of R. For each $\lambda \in \Lambda$, let M_λ be a $\mathfrak{q}_\lambda$-torsion R-module such that $(\mathrm{Spec}(M_\lambda), \tau)$ is a T_0-space. Then, $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is a top R-module.*

Proof. (1) Since $V(I)$ is a finite subset of $\mathrm{Max}(R)$, the result follows from Theorem 2.8.

(2) By Lemma 2.1, the family $\{M_\lambda\}_{\lambda\in\Lambda}$ is prime-compatible, and by Theorem 2.8 M_λ is top, since by assumption $V(\mathfrak{q}_\lambda)$ is a finite subset of $\mathrm{Max}(R)$. Therefore, $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is a top R-module by [21, Theorem 5.1]. □

Example 2.12. Consider $M = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}$-module, where p runs through the set of all prime numbers. By Corollary 2.11, M is top.

Acknowledgment. The author specially thank the referee for the helpful suggestions and comments.

References

[1] A. Abbasi and D. Hassanzadeh-Lelekaami, *Modules and spectral spaces*, Comm. Algebra **40** (2012), no. 11, 4111–4129.

[2] J. Abuhlail, *A dual Zariski topology for modules*, Topology Appl. **158** (2011), no. 3, 457–467.

[3] H. Ansari-Toroghy and D. Hassanzadeh-Lelekaami, *On the prime spectrum of top modules*, Algebra Discrete Math. **11** (2011), no. 1, 1–16.

[4] H. Ansari-Toroghy and R. Ovlyaee-Sarmazdeh, *On the prime spectrum of X-injective modules*, Comm. Algebra **38** (2010), no. 7, 2606–2621.

[5] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, MA, 1969.

[6] M. Behboodi and M. R. Haddadi, *Classical Zariski topology of modules and spectral spaces. I*, Int. Electron. J. Algebra **4** (2008), 104–130.

[7] ———, *Classical Zariski topology of modules and spectral spaces. II*, Int. Electron. J. Algebra **4** (2008), 131–148.

[8] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge Studies in Advanced Mathematics, **60**, Cambridge University Press, Cambridge, 1998.

[9] T. Duraivel, *Topology on spectrum of modules*, J. Ramanujan Math. Soc. **9** (1994), no. 1, 25–34.

[10] D. Hassanzadeh-Lelekaami and H. Roshan-Shekalgourabi, *Prime submodules and a sheaf on the prime spectra of modules*, Comm. Algebra **42** (2014), no. 7, 3063–3077.

[11] ———, *On regular modules over commutative rings*, Bull. Malays. Math. Sci. Soc. (2017); https://doi.org/10.1007/s40840-017-0501-0.

[12] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969), 43–60.

[13] C.-P. Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Paul. **33** (1984), no. 1, 61–69.

[14] ———, *M-radicals of submodules in modules*, Math. Japon. **34** (1989), no. 2, 211–219.

[15] ———, *Spectra of modules*, Comm. Algebra **23** (1995), no. 10, 3741–3752.

[16] ———, *The Zariski topology on the prime spectrum of a module*, Houston J. Math. **25** (1999), no. 3, 417–432.

[17] ———, *A module whose prime spectrum has the surjective natural map*, Houston J. Math. **33** (2007), no. 1, 125–143.

[18] ———, *Modules with Noetherian spectrum*, Comm. Algebra **38** (2010), no. 3, 807–828.

[19] R. L. McCasland and M. E. Moore, *On radicals of submodules*, Comm. Algebra **19** (1991), no. 5, 1327–1341.

[20] ———, *Prime submodules*, Comm. Algebra **20** (1992), no. 6, 1803–1817.

[21] R. L. McCasland, M. E. Moore, and P. F. Smith, *On the spectrum of a module over a commutative ring*, Comm. Algebra **25** (1997), no. 1, 79–103.

[22] ———, *An introduction to Zariski spaces over Zariski topologies*, Rocky Mountain J. Math. **28** (1998), no. 4, 1357–1369.

[23] C. Peskine, *An Algebraic Introduction to Complex Projective Geometry. 1*, Cambridge Studies in Advanced Mathematics, **47**, Cambridge University Press, Cambridge, 1996.

[24] J. J. Rotman, *An Introduction to Homological Algebra*, Pure and Applied Mathematics, **85**, Academic Press, Inc., New York, 1979.

[25] R. Y. Sharp, *Steps in Commutative Algebra*, second edition, London Mathematical Society Student Texts, **51**, Cambridge University Press, Cambridge, 2000.
[26] Ü. Tekir, *On the sheaf of modules*, Comm. Algebra **33** (2005), no. 8, 2557–2562.
[27] G. Zhang, F. Wang, and W. Tong, *Multiplication modules in which every prime submodule is contained in a unique maximal submodule*, Comm. Algebra **32** (2004), no. 5, 1945–1959.

Hajar Roshan-Shekalgourabi
Department of Basic Sciences
Arak University of Technology
P. O. Box 38135-1177, Arak, Iran
Email address: Hrsmath@gmail.com, roshan@arakut.ac.ir