Commun. Korean Math. Soc. **34** (2019), No. 1, pp. 55–82
https://doi.org/10.4134/CKMS.c170487
pISSN: 1225-1763 / eISSN: 2234-3024

SOLUTIONS AND STABILITY OF TRIGONOMETRIC FUNCTIONAL EQUATIONS ON AN AMENABLE GROUP WITH AN INVOLUTIVE AUTOMORPHISM

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Abstract. Given $\sigma : G \to G$ an involutive automorphism of a semigroup G, we study the solutions and stability of the following functional equations

$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y), \quad x, y \in G,$$
$$f(x\sigma(y)) = f(x)f(y) - g(x)g(y), \quad x, y \in G$$

and

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y), \quad x, y \in G,$$

from the theory of trigonometric functional equations.

(1) We determine the solutions when G is a semigroup generated by its squares.
(2) We obtain the stability results for these equations, when G is an amenable group.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphims. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers's Theorem was generalized by Aoki [4] for additive mappings and by Rassias [20] for linear mappings by considering an unbounded Cauchy difference. The stability problem of several functional equations has been extensively investigated by a number of authors. An account on the further progress and developments in this field can be found in [11], [14], [15] and [18]. We refer also to [16] and [17]. In this paper we investigate the stability of the trigonometric functional equations

$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y),\ x, y \in G, \tag{1.1}$$

$$f(x\sigma(y)) = f(x)f(y) - g(x)g(y),\ x, y \in G, \tag{1.2}$$

Received December 21, 2017; Accepted February 6, 2018.
2010 *Mathematics Subject Classification.* 39B32, 39B72, 39B82.
Key words and phrases. Hyers-Ulam stability, semigroup, group, cosine equation, sine equation, involutive automorphism, multiplicative function, additive function.

and

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y),\ x, y \in G, \tag{1.3}$$

where G is an amenable group, $\sigma : G \to G$ is an involutive automorphism. That is $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$.

The complex-valued solutions of (1.1), (1.2) and (1.3) on groups that need not be abelian are obtained by Poulsen and Stetkær [19]. A particular case of (1.1) and (1.2) is the sine addition law

$$f(xy) = f(x)g(y) + g(x)f(y),\ x, y \in G \tag{1.4}$$

and the cosine addition law

$$f(xy) = f(x)f(y) - g(x)g(y),\ x, y \in G. \tag{1.5}$$

The stability properties of (1.4) and (1.5) have been obtained by Székelyhidi [20] on amenable groups. The stability problems of (1.3) were studied by Chung, Choi and Kim [8] in 2-divisible abelian groups. Chang and chung [7] proved the Hyers-Ulam stability of (1.4) and (1.5) in the spaces of generalized functions.

Recently, Chang et al. [6] studied the stability of equation (1.2) on abelian groups. We refer also to [5] and [16].

The aim of the present paper is

(1) To extend Poulsen and Stetkær's work [15] from groups to the semigroups generated by its squares.
(2) To show how Székelyhidi's results [20] on the stability of equations (1.4) and (1.5) extends to the much wider frame work of (1.1) and (1.2).

Our results encompass not Székelyhidi's in [20], but also those of Chung et al. [8] and Chang et al. [6] about stability of functional equations (1.3) and (1.2).

2. Definitions and preliminaries

Throughout this paper G denotes a semigroup (a set with an associative composition) or a group. That G is generated by its squares means that for all $x \in G$ their exist $x_1, \ldots, x_n \in G$ such that $x = x_1^2 \cdots x_n^2$. We denote by $\mathcal{B}(G)$ the linear space of all bounded complex-valued functions on G. The map $\sigma : G \to G$ denotes an involutive automorphism. That σ is involutive means that $\sigma(\sigma(x)) = x$ for all $x \in G$. We call $a : G \to \mathbb{C}$ additive provided that $a(xy) = a(x) + a(y)$ for all $x, y \in G$ and call $m : G \to \mathbb{C}$ multiplicative provided that $m(xy) = m(x)m(y)$ for all $x, y \in G$. If $m \neq 0$, then $I_m := \{x \in G \mid m(x) = 0\}$ is either empty or a proper subset of G. I_m is a two sided ideal in G if not empty and $G \setminus I_m$ is a subsemigroup of G.

Let $\mathcal{V}$ be a linear space of complex-valued functions on G. We say that the functions $f, g : G \to \mathbb{C}$ are linearly independent modulo $\mathcal{V}$ if $\lambda f + \mu g \in \mathcal{V}$ implies $\lambda = \mu = 0$ for any $\lambda, \mu \in \mathbb{C}$. We say that the linear space $\mathcal{V}$ is two-sided invariant if $f \in \mathcal{V}$ implies that the functions $x \mapsto f(xy)$ and $x \mapsto f(yx)$ belong to $\mathcal{V}$ for any $y \in G$. We say that $\mathcal{V}$ is σ-invariant if $f \in \mathcal{V}$ implies that $f \circ \sigma \in \mathcal{V}$.

The space $\mathcal{B}(G)$ is an obvious example of a linear space of complex-valued functions on G which is two-sided invariant and σ-invariant.

Let $f : G \longrightarrow \mathbb{C}$ be a function. We call $f_e := \frac{f+f\circ\sigma}{2}$ the even part of f and $f_o := \frac{f-f\circ\sigma}{2}$ its odd part.

3. Stability of equation (1.1) on amenable groups

Regular solutions of the functional equation (1.4) were described, on abelian groups, by Aczél [1].

The functional equation (1.4) was solved by Chung et al. [10] on groups. Poulsen and Stetkær [19] determined, on a topological group with continuous involutive automorphism σ, the continuous solutions of the functional equation (1.1). Recently Ajebbar and Elqorachi [3] obtained the solutions of equation (1.1) on a semigroup generated by its squares.

In this section we will extend the result obtained by Székelyhidi [24, Theorem 2.3] to the functional equation (1.1).

Lemma 3.1. *Let G be a semigroup, $f, g : G \to \mathbb{C}$ be functions and let $\mathcal{V}$ be a two-sided invariant linear space of complex-valued functions on G such that $\mathcal{V}$ is σ-invariant. Suppose that f and g are linearly independent modulo $\mathcal{V}$. If the function*

$$x \mapsto f(x\sigma(y)) - f(x)g(y) - g(x)f(y)$$

belongs to $\mathcal{V}$ for all $y \in G$, then

$$f\circ\sigma = f \quad \text{and} \quad g\circ\sigma = g$$

or

$$f\circ\sigma = -f \quad \text{and} \quad g\circ\sigma = g.$$

Proof. We use a similar computation as the one of the proof of [24, Lemma 2.1].

Let ψ be the function defined by

$$\psi(x,y) = f(x\sigma(y)) - f(x)g(y) - g(x)f(y) \tag{3.1}$$

for $x, y \in G$. Since f and g are linearly independent modulo $\mathcal{V}$ we get that $f \neq 0$, then there exists $x_0 \in G$ such that $f(x_0) \neq 0$. Let $\alpha_0 := -f(x_0)^{-1}g(x_0)$ and $\alpha_1 := f(x_0)^{-1} \in \mathbb{C}\setminus\{0\}$. By applying (3.1) to the pair (x, x_0) we derive

$$g(x) = \alpha_0\, f(x) + \alpha_1\, f(x\sigma(x_0)) - \alpha_1\,\psi(x, x_0) \tag{3.2}$$

for all $x \in G$.

Let $x, y, z \in G$. By applying (3.1) to the pair $(x\sigma(y), z)$ and using (3.2) we get that

$$\begin{aligned}
f(x\sigma(y)\sigma(z)) &= f(x\sigma(y))g(z) + g(x\sigma(y))f(z) + \psi(x\sigma(y), z)\\
&= [f(x)g(y) + g(x)f(y) + \psi(x,y)]g(z)\\
&\quad + [\alpha_0\, f(x\sigma(y)) + \alpha_1\, f(x\sigma(yx_0)\\
&\quad - \alpha_1\,\psi(x\sigma(y), x_0)] \times f(z) + \psi(x\sigma(y), z)
\end{aligned}$$

$$
\begin{aligned}
&= f(x)g(y)g(z) + g(x)f(y)g(z) + \psi(x,y)g(z)\\
&\quad + \alpha_0\,[f(x)g(y) + g(x)f(y) + \psi(x,y)]f(z)\\
&\quad + \alpha_1\,[f(x)g(yx_0) + g(x)f(yx_0) + \psi(x,yx_0)]f(z)\\
&\quad - \alpha_1\,\psi(x\sigma(y),x_0)f(z) + \psi(x\sigma(y),z).
\end{aligned}
$$

So that

$$
\begin{aligned}
&f((x\sigma(y))\sigma(z))\\
&= f(x)[g(y)g(z) + \alpha_0\,g(y)f(z) + \alpha_1\,g(yx_0)f(z)]\\
&\quad + g(x)[f(y)g(z) + \alpha_0\,f(y)f(z) + \alpha_1\,f(yx_0)f(z)]\\
&\quad + \psi(x,y)g(z) + [\alpha_0\,\psi(x,y) + \alpha_1\,\psi(x,yx_0) - \alpha_1\,\psi(x\sigma(y),x_0)]f(z)\\
&\quad + \psi(x\sigma(y),z).
\end{aligned}
\tag{3.3}
$$

On the other hand, by applying (3.1) to the pair (x, yz) we get that

$$
f(x\sigma(y)\sigma(z)) = f(x\sigma(yz)) = f(x)g(yz) + g(x)f(yz) + \psi(x,yz). \tag{3.4}
$$

From (3.3) and (3.4) we deduce that

$$
\begin{aligned}
&f(x)[g(y)g(z) + \alpha_0\,g(y)f(z) + \alpha_1\,g(yx_0)f(z) - g(yz)]\\
&\quad + g(x)[f(y)g(z) + \alpha_0\,f(y)f(z) + \alpha_1\,f(yx_0)f(z) - f(yz)]\\
&= -\psi(x,y)g(z) - [\alpha_0\,\psi(x,y) + \alpha_1\,\psi(x,yx_0) - \alpha_1\,\psi(x\sigma(y),x_0)]f(z)\\
&\quad - \psi(x\sigma(y),z) + \psi(x,yz).
\end{aligned}
\tag{3.5}
$$

Now, let $y, z \in G$ be arbitrary. By assumption the functions

$$
\begin{aligned}
&x \mapsto \psi(x,y),\; x \mapsto \psi(x,y)g(z),\; x \mapsto \psi(x,yz),\\
&x \mapsto \psi(x,y)f(z),\; x \mapsto \psi(x,yx_0)f(z)
\end{aligned}
$$

belong to $\mathcal{V}$. Moreover the linear space $\mathcal{V}$ is two-sided invariant, then the functions

$$
x \mapsto \psi(x\sigma(y),x_0),\; x \mapsto \psi(x\sigma(y),z)
$$

belong to $\mathcal{V}$. By using (3.5) it follows that the function

$$
\begin{aligned}
x \mapsto\, &f(x)[g(y)g(z) + \alpha_0\,g(y)f(z) + \alpha_1\,g(yx_0)f(z) - g(yz)]\\
&+ g(x)[f(y)g(z) + \alpha_0\,f(y)f(z) + \alpha_1\,f(yx_0)f(z) - f(yz)]
\end{aligned}
$$

belongs to $\mathcal{V}$. Since f and g are linearly independent modulo $\mathcal{V}$ we get that

$$
f(y)g(z) + \alpha_0\,f(y)f(z) + \alpha_1\,f(yx_0)f(z) - f(yz) = 0.
$$

$y, z \in G$ being arbitrary we deduce that

$$
f(xy) = f(x)g(y) + \alpha_0\,f(y)f(x) + \alpha_1\,f(y)f(xx_0) \tag{3.6}
$$

for all $x, y \in G$.

By applying (3.1) to the pair $(x, \sigma(y))$ we get

$$
f(xy) = f(x)g\circ\sigma(y) + f\circ\sigma(y)g(x) + \psi(x,\sigma(y)). \tag{3.7}
$$

By subtracting (3.7) from (3.6) we get that

$$\psi(x,\sigma(y)) = f(x)(g(y) - g\circ\sigma(y)) + \alpha_0\, f(y)f(x) + \alpha_1\, f(y)f(xx_0) - f\circ\sigma(y)g(x),$$

which can be written

$$\psi(x,\sigma(y)) = 2\, f(x)g_o(y) + \alpha_0\, f(y)f(x) + \alpha_1\, f(y)f(xx_0) - f\circ\sigma(y)g(x). \tag{3.8}$$

By replacing y by $\sigma(y)$ in (3.8) we get

$$\psi(x,y) = -2\, f(x)g_o(y) + \alpha_0\, f\circ\sigma(y)f(x) + \alpha_1\, f\circ\sigma(y)f(xx_0) - f(y)g(x). \tag{3.9}$$

Let

$$\varphi(x) := \alpha_0 f(x) + \alpha_1\, f(xx_0) - g(x)$$

for all $x \in G$.

By adding the identities (3.8) and (3.9) we get that

$$\begin{aligned}\psi(x,\sigma(y)) + \psi(x,y) = \alpha_0\, f(x)[f(y) + f\circ\sigma(y)] + \alpha_1\, f(xx_0)[f(y) + f\circ\sigma(y)] \\ - g(x)[f(y) + f\circ\sigma(y)].\end{aligned}$$

So that

$$\psi(x,\sigma(y)) + \psi(x,y) = 2\, f_e(y)\varphi(x) \tag{3.10}$$

for all $x, y \in G$.

On the other hand, by subtracting (3.9) from (3.8) we get that

$$\begin{aligned}&\psi(x,\sigma(y)) - \psi(x,y) \\ &= 4\, f(x)g_o(y) + 2\,\alpha_0\, f_o(y)f(x) + 2\,\alpha_1\, f_o(y)f(xx_o) + 2\, f_o(y)g(x) \\ &= 4\, f(x)g_o(y) + 2\, f_o(y)[\alpha_0\, f(x) + \alpha_1\, f(xx_o) + g(x)],\end{aligned}$$

which implies

$$\psi(x,\sigma(y)) - \psi(x,y) = 4\, f(x)g_o(y) + 4\, f_o(y)g(x) + 2\, f_o(y)\varphi(x). \tag{3.11}$$

We split the discussion into the cases of $f\circ\sigma = -f$ or $f\circ\sigma \neq -f$.
Case 1: $f\circ\sigma \neq -f$, then $f_e \neq 0$. So, there exists $y_0 \in G$ such that $f_e(y_0) \neq 0$. By replacing y by y_0 in (3.10) and using the fact that the functions $x \mapsto \psi(x, y_0)$ and $x \mapsto \psi(x,\sigma(y_0))$ belong to $\mathcal{V}$ we deduce that

$$\varphi \in \mathcal{V}. \tag{3.12}$$

Let $y \in G$ be arbitrary. As the functions $x \mapsto \psi(x,y)$ and $x \mapsto \psi(x,\sigma(y))$ belong to $\mathcal{V}$, then from (3.11) and (3.12) we deduce that the function $x \mapsto 4\, f(x)g_o(y) + 4\, f_o(y)g(x)$ belongs to $\mathcal{V}$. Since f and g are linearly independent modulo $\mathcal{V}$ we get that $f_o(y) = g_o(y) = 0$. So, $f\circ\sigma(y) = f(y)$ and $g\circ\sigma(y) = g(y)$. So, y being arbitrary, we deduce that $f\circ\sigma = f$ and $g\circ\sigma = g$.
Case 2: $f\circ\sigma = -f$, then we apply (3.1) to the pairs $(x,\sigma(y))$ and $(\sigma(x), y)$, and get respectively

$$\psi(x,\sigma(y)) = f(xy) - f(x)g\circ\sigma(y) - f\circ\sigma(y)g(x) = f(xy) - f(x)g\circ\sigma(y) + f(y)g(x)$$

and

$$\psi(\sigma(x), y) = f\circ\sigma(xy) - f\circ\sigma(x)g(y) - f(y)g\circ\sigma(x)$$

$$= -f(xy) + f(x)g(y) - f(y)g \circ \sigma(x).$$

So,

$$\psi(x, \sigma(y)) + \psi(\sigma(x), y) = 2\, f(x)g_o(y) + 2\, f(y)g_o(x). \tag{3.13}$$

In the following we prove that $g \circ \sigma = g$. Assume that there exists $y_1 \in G$ such that $g_o(y_1) \neq 0$. Let $\phi : G \to \mathbb{C}$ be the function defined by $\phi(x) := \psi(x, \sigma(y_1)) + \psi(\sigma(x), y_1)$.

By replacing y by y_1 in (3.13) we get

$$\phi(x) = 2\, f(x)g_o(y_1) + 2\, f(y_1)g_o(x), \quad x \in G. \tag{3.14}$$

The function $x \mapsto \psi(x, \sigma(y_1))$ belongs to $\mathcal{V}$ by assumption. Furthermore, since $\mathcal{V}$ is σ-invariant then the function $x \mapsto \psi(\sigma(x), y_1)$ belongs to $\mathcal{V}$. Hence,

$$\phi \in \mathcal{V}. \tag{3.15}$$

Taking (3.14), (3.15) and $g_o(y_1) \neq 0$ into account we deduce that there exist $h \in \mathcal{V}$ and a constant $\alpha \in \mathbb{C}$ such that

$$f = \alpha\, g_o + h. \tag{3.16}$$

Substituting (3.16) back into (3.13) we obtain

$$\psi(x, \sigma(y)) + \psi(\sigma(x), y) = 2\,(\alpha g_o(x) + h(x))g_o(y) + 2\,(\alpha g_0(y) + h(y))g_o(x),$$

then

$$\psi(x, \sigma(y)) + \psi(\sigma(x), y) = 2\,(2\,\alpha\, g_o(y) + h(y))g_o(x) + 2\, h(x)g_o(y). \tag{3.17}$$

If there exists $y_0 \in G$ such that $2\,\alpha\, g_o(y_0) + h(y_0) \neq 0$, then $f \in \mathcal{V}$. Indeed, the functions $x \mapsto \psi(x, \sigma(y_0)) + \psi(\sigma(x), y_0)$ and $x \mapsto h(x)g_o(y_0)$ belong to $\mathcal{V}$. So, by replacing y by y_0 in (3.17), we get that the function $x \mapsto (2\,\alpha\, g_o(y_0) + h(y_0))g_o(x)$ belongs to $\mathcal{V}$. Hence, $g_o \in \mathcal{V}$. Taking (3.16) into account we deduce that $f \in \mathcal{V}$.

If $2\,\alpha\, g_o(y) + h(y) = 0$ for all $y \in G$, then $2\,\alpha\, g_o = -h$. Taking (3.16) into account we get that $f \in \mathcal{V}$.

Thus $f \in \mathcal{V}$ in both cases, which contradicts the linear independence modulo $\mathcal{V}$ of f and g. We conclude that $g \circ \sigma = g$. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let G be a semigroup, $f, g : G \to \mathbb{C}$ be functions and let $\mathcal{V}$ be a two-sided invariant linear space of complex-valued functions on G such that $\mathcal{V}$ is σ-invariant. If the function*

$$x \mapsto f(x\sigma(y)) - f(x)g(y) - g(x)f(y)$$

belongs to $\mathcal{V}$ for all $y \in G$, then we have one of the following possibilities:

(1) $f = 0$ *and g is arbitrary.*

(2) $f, g \in \mathcal{V}$.

(3) $g \in \mathcal{V}$ *and g is multiplicative.*

(4) $f = \lambda\, m - \lambda\, \varphi$, $g = \frac{1}{2}\, m + \frac{1}{2}\, \varphi$, *where $\lambda \in \mathbb{C} \backslash \{0\}$ is a constant, $m : G \to \mathbb{C}$ is a multiplicative function and $\varphi \in \mathcal{V}$.*

(5) *$f(x\sigma(y)) = f(x)g(y) + g(x)f(y)$ for all $x, y \in G$.*

(6) *$\psi(x, \sigma(y)) + \psi(y, \sigma(x)) = f(xy) + f(yx)$ for all $x, y \in G$, where ψ is the function defined by* (3.1).

Proof. We split the discussion into the cases of f and g are linearly independent modulo $\mathcal{V}$, or f and g are linearly dependent modulo $\mathcal{V}$.
Case 1: f and g are linearly independent modulo $\mathcal{V}$. Then, according to Lemma 3.1, ($f \circ \sigma = f$ and $g \circ \sigma = g$) or ($f \circ \sigma = -f$ and $g \circ \sigma = g$).

If $f \circ \sigma = f$ and $g \circ \sigma = g$, then, by using similar computations as the ones of the proof of [24, Lemma 2.1], we get that

$$\psi(x, y) = 0$$

for all $x, y \in G$, where ψ is the function defined in (3.1). That is $f(x\sigma(y)) = f(x)g(y) + g(x)f(y)$ for all $x, y \in G$. The result occurs in (5) of Lemma 3.2.

If $f \circ \sigma = -f$ and $g \circ \sigma = g$, then by replacing y by $\sigma(y)$ in (3.1) we get

$$\psi(x, \sigma(y)) = f(xy) - f(x)g(y) + g(x)f(y).$$

Interchanging x and y in the identity above we get

$$\psi(y, \sigma(x)) = f(yx) - f(y)g(x) + g(y)f(x).$$

By adding the two last identities we obtain

$$\psi(x, \sigma(y)) + \psi(y, \sigma(x)) = f(xy) + f(yx)$$

for all $x, y \in G$. The result occurs in (6) of Lemma 3.2.
Case 2: f and g are linearly dependent modulo $\mathcal{V}$. We prove, by a computation adapted to that of the proof of [24, Lemma 2.2], that one of the possibilities (1)-(6) of Lemma 3.2 holds. $\square$

Theorem 3.3. *Let G be an amenable group, $\sigma : G \to G$ be an involutive automorphism and let $f, g : G \to \mathbb{C}$ be functions. The function*

$$(x, y) \mapsto f(x\sigma(y)) - f(x)g(y) - g(x)f(y)$$

is bounded if and only if one of the following assertions holds:

(1) *$f = 0$ and g is arbitrary.*

(2) *$f, g \in \mathcal{B}(G)$.*

(3) *$f = a\,m + b$ and $g = m$, where $a : G \to \mathbb{C}$ is an additive function, $m : G \to \mathbb{C}$ is a bounded multiplicative function and $b : G \to \mathbb{C}$ is a bounded function such that $m \circ \sigma = m$ and $a \circ \sigma = a$.*

(4) *$f = \lambda\, m - \lambda\, b$, $g = \frac{1}{2}\, m + \frac{1}{2}\, b$, where $\lambda \in \mathbb{C} \setminus \{0\}$ is a constant, $b : G \to \mathbb{C}$ is a bounded function and $m : G \to \mathbb{C}$ is a multiplicative function such that $m \circ \sigma = m$ or $m \in \mathcal{B}(G)$.*

(5) *$f(x\sigma(y)) = f(x)g(y) + g(x)f(y)$ for all $x, y \in G$.*

Proof. First we prove the necessity. Let $f, g : G \to \mathbb{C}$ be two functions such that the function ψ defined in (3.1) is bounded. Then the function

$$x \mapsto f(x\sigma(y)) - f(x)g(y) - g(x)f(y)$$

belongs to $\mathcal{B}(G)$ for all $y \in G$. Notice that $\mathcal{B}(G)$ is a two-sided invariant linear space and σ-invariant. According to Lemma 3.2 we have one of the following possibilities:

(1) $f = 0$ and g is arbitrary, which occurs in (1) of Theorem 3.3.

(2) $f, g \in \mathcal{B}(G)$, which occurs in (2) of Theorem 3.3.

(3) $f = \lambda\, m - \lambda\, b$, $g = \frac{1}{2}\, m + \frac{1}{2}\, b$, where $\lambda \in \mathbb{C} \setminus \{0\}$ is a constant, $m : G \to \mathbb{C}$ is a multiplicative function and $b \in \mathcal{B}(G)$. Hence,

$$\begin{aligned}\psi(x, y) &= \lambda\, (m(x\sigma(y)) - b(x\sigma(y))) - \frac{\lambda}{2}\, (m(x) - b(x))(m(y) + b(y)) \\ &\quad - \frac{\lambda}{2}\, (m(y) - b(y))(m(x) + b(x)) \\ &= \lambda\, m(x)\, (m \circ \sigma(y) - m(y)) - \lambda\, (b(x\sigma(y)) - b(x)b(y))\end{aligned}$$

for all $x, y \in G$.

Then the function $(x, y) \mapsto m(x)\, (m\circ\sigma(y) - m(y))$ is bounded. So, $m\circ\sigma = m$ or m is bounded. The result occurs in (4) of Theorem 3.3.

(4) $g = m : G \to \mathbb{C}$ where m is a bounded multiplicative function. Then

$$\psi(x, y) = f(x\sigma(y)) - f(x)m(y) - m(x)f(y) \tag{3.18}$$

for all $x, y \in G$.

If $m = 0$, then $\psi(x, y) = f(x\sigma(y))$ for all $x, y \in G$. Since ψ is bounded by assumption so is f. The result occurs in (2) of Theorem 3.3.

If $m \neq 0$, then by replacing y by $\sigma(y)$ respectively x by $\sigma(x)$ in (3.18) we obtain respectively

$$\psi(x, \sigma(y)) = f(xy) - f(x)m(\sigma(y)) - m(x)f(\sigma(y)), \tag{3.19}$$

$$\psi(\sigma(x), y) = f(\sigma(xy)) - f(\sigma(x))m(y) - m(\sigma(x))f(y). \tag{3.20}$$

We split the discussion into the cases of $m \circ \sigma = m$ or $m \circ \sigma \neq m$.
Case 1: $m \circ \sigma = m$. By adding the identities (3.19) and (3.20) we get that

$$\frac{1}{2}[\psi(x, \sigma(y)) + \psi(\sigma(x), y)] = f_e(xy) - f_e(x)m(y) - m(x)f_e(y).$$

Since m is a nonzero multiplicative function on the group G we get that $m(x) \neq 0$ and $m(x^{-1}) = (m(x))^{-1}$ for all $x \in G$. Hence, $\frac{1}{2}[\psi(x, \sigma(y)) + \psi(\sigma(x), y)]m((xy)^{-1}) = f_e(xy)m((xy)^{-1}) - f_e(x)m(x^{-1}) - f_e(y)m(y^{-1})$ for all $x, y \in G$. Since the function ψ is bounded so is the function $(x, y) \mapsto f_e(xy)m((xy)^{-1}) - f_e(x)m(x^{-1}) - f_e(y)m(y^{-1})$. So, according to Hyers's Theorem [23, Theorem 3.1] there exist an additive function $a : G \to G$ and a bounded function $\varphi : G \to G$ such that $f_e(x)\, m(x^{-1}) - a(x) = \varphi(x)$ for all $x \in G$. Hence, $f_e = (a + \varphi)\, m$.

On the other hand, by subtracting (3.19) and (3.20) we get that $\frac{1}{2}[\psi(x, \sigma(y)) - \psi(\sigma(x), y)] = f_o(xy) - f_o(x)m(y) + m(x)f_o(y)$ for all $x, y \in G$. Hence, the function $(x, y) \mapsto f_o(xy) - f_o(x)m(y) + m(x)f_o(y)$ is bounded. Let e be the identity element of G. By putting $x = e$ we get that $f_o = b_0$ where $b_0 \in \mathcal{B}(G)$. Hence $f = a\,m + b$ with $b := \varphi\, m + b_0 \in \mathcal{B}(G)$. So,

$$\begin{aligned}\psi(x,y) &= a(x\sigma(y))m(x\sigma(y)) + b(x\sigma(y)) - [a(x)m(x) + b(x)]m(y) \\ &\quad - [a(y)m(y) + b(y)]m(x) \\ &= [a\circ\sigma(y) - a(y)]m(xy) - m(x)b(y) - m(y)b(x) + b(x\sigma(y)),\end{aligned}$$

which implies

$$\psi(x,y)m(xy)^{-1} = a\circ\sigma(y) - a(y) - m(y^{-1})b(y) - m(x^{-1})b(x) + b(x\sigma(y))m(xy)^{-1}$$

for all $x, y \in G$. Since the functions ψ, m and b are bounded so is the function $y \mapsto (a\circ\sigma(y) - a(y)$. Since the function $a\circ\sigma - a$ is additive we get, according to [21, Exercise 2.5(a)], that $a\circ\sigma = a$. The result occurs in (3) of Theorem 3.3.

Case 2: $m\circ\sigma \neq m$. Since $m \neq 0$ we have $m(e) = 1$. By putting $x = e$ in (3.18) we obtain $\psi(e,y) = f\circ\sigma(y) - f(e)m(y) - f(y)$. It follows that the function $y \mapsto -2\,f_o(y) - f(e)m(y)$ belongs to $\mathcal{B}(G)$. Since $m \in \mathcal{B}(G)$ then $f_o \in \mathcal{B}(G)$.

On the other hand, by adding (3.19) and (3.20) we get that

$$\begin{aligned}&\psi(x,\sigma(y)) + \psi(\sigma(x),y) \\ &= 2\,f_e(xy) - f(x)m\circ\sigma(y) - m(x)f\circ\sigma(y) - f\circ\sigma(x)m(y) - m\circ\sigma(x)f(y) \\ &= 2\,f_e(xy) - 2\,f_e(x)\,m_e(y) + 2\,f_o(x)\,m_o(y) - 2\,m_e(x)\,f_e(y) + 2\,m_o(x)\,f_o(y),\end{aligned}$$

hence

$$\begin{aligned}f_e(xy) - f_e(x)m_e(y) = \frac{1}{2}[\psi(x,\sigma(y)) + \psi(\sigma(x),y)] - f_o(x)m_o(y) - m_o(x)f_o(y) \\ + m_e(x)f_e(y)\end{aligned}$$

for all $x, y \in G$.

Since the functions

$$x \mapsto \psi(x,y), \quad x \mapsto f_o(x)m_o(y), \quad x \mapsto m_o(x)f_o(y) \text{ and } x \mapsto m_e(x)f_e(y)$$

belong to $\mathcal{B}(G)$ for all $y \in G$, $\mathcal{B}(G)$ is a two-sided invariant and σ-invariant linear space of complex-valued functions on G, and seeing that m_e is not multiplicative because $m\circ\sigma \neq m$, we deduce by applying [22, Theorem] that $f_e \in \mathcal{B}(G)$, so $f \in \mathcal{B}(G)$. It follows that (2) of Theorem 3.3 holds.

(5) $f(x\sigma(y)) = f(x)g(y) + g(x)f(y)$ for all $x, y \in G$. The result occurs in (5) of Theorem 3.3.

(6) $\psi(x,\sigma(y)) + \psi(y,\sigma(x)) = f(xy) + f(yx)$ for all $x, y \in G$. If $f = 0$, then the functional equation (1.1) is satisfied, which corresponds to (5) of Theorem 3.3.

In what follows we assume that $f \neq 0$. By putting $y = e$ in the identity above we get

$$\psi(x,e) + \psi(e,\sigma(x)) = 2\,f(x)$$

for all $x \in G$. Since the functions $x \mapsto \psi(x,e)$ and $x \mapsto \psi(e,x)$ belong to $\mathcal{B}(G)$ and $\mathcal{B}(G)$ is σ-invariant, we get that $f \in \mathcal{B}(G)$. So, we get from (3.1) that $g \in \mathcal{B}(G)$, which implies (2) of Theorem 3.3.

Conversely, we check by elementary computations that if one of the assertions (1)-(5) in Theorem 3.3 is satisfied, then the function ψ defined in (3.1) is bounded. This completes the proof of Theorem 3.3. □

By taking $\sigma(x) = x$ for all $x \in G$ in Theorem 3.3 we obtain the following corollary.

Corollary 3.4 ([24, Theorem 2.3])**.** *Let G be an amenable group and let $f, g : G \to \mathbb{C}$ be functions. The function*

$$(x, y) \mapsto f(xy) - f(x)g(y) - g(x)f(y)$$

is bounded if and only if one of the following assertions holds:

(1) *$f = 0$ and g is arbitrary.*

(2) *$f, g \in \mathcal{B}(G)$.*

(3) *$f = a\,m + b$ and $g = m$, where $a : G \to \mathbb{C}$ is an additive function, $m : G \to \mathbb{C}$ is a bounded multiplicative function and $b : G \to \mathbb{C}$ is a bounded function.*

(4) *$f = \lambda\, m - \lambda\, b$, $g = \frac{1}{2}\, m + \frac{1}{2}\, b$, where $\lambda \in \mathbb{C} \setminus \{0\}$ is a constant, $b : G \to \mathbb{C}$ is a bounded function and $m : G \to \mathbb{C}$ is a multiplicative function.*

(5) *$f(xy) = f(x)g(y) + g(x)f(y)$ for all $x, y \in G$.*

4. Solutions and stability of equation (1.2)

4.1. Solutions of equation (1.2) on semigroup generated by its squares

Regular solutions of the functional equation (1.5) were described, on abelian groups, by Aczél [1]. Poulsen and Stetkær [19] determined, on a topological group with continuous involutive automorphism σ, the continuous solutions of the functional equation (1.2).

In this subsection we will solve the functional equation (1.2) on a semigroup G generated by its squares, and so, extend the results obtain by Poulsen and Stetkær [19].

Lemma 4.1. *Let G be a semigroup generated by its squares. The solutions $f, g : G \to \mathbb{C}$ of the functional equation*

$$f(xy) = f(x)f(y) - g(x)g(y), \quad x, y \in G \tag{4.1}$$

can be listed as follows:

(1) *$f = 0$ and $g = 0$.*

(2) *$f = \frac{1}{1-\lambda^2}\, m$ and $g = \frac{\lambda}{1-\lambda^2}\, m$, where $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ is a constant and $m : G \to \mathbb{C}$ is a nonzero multiplicative function.*

(3) *$f = \frac{\lambda\, M + \frac{1}{\lambda}\, m}{\lambda + \frac{1}{\lambda}}$ and $g = \frac{M - m}{(\lambda + \frac{1}{\lambda})i}$, where $\lambda \in \mathbb{C} \setminus \{0, -i, i\}$ is a constant and $m, M : G \to \mathbb{C}$ are two multiplicative functions such that $m \neq M$.*

(4) $\begin{cases} f = m\,(1+a) \quad \textit{and} \quad g = m\,a \quad \textit{on} \quad G \setminus I_m, \\ f = g = 0 \qquad\qquad\qquad\qquad\quad \textit{on} \quad I_m, \end{cases}$

where $m : G \to \mathbb{C}$ *is a nonzero multiplicative function and* $a : G \setminus I_m \to \mathbb{C}$ *is a nonzero additive function.*

(5) $\begin{cases} f = m\,(1+a) \quad \textit{and} \quad g = -m\,a \quad \textit{on} \quad G \setminus I_m, \\ f = g = 0 \qquad\qquad\qquad\qquad\quad \textit{on} \quad I_m, \end{cases}$

where $m : G \to \mathbb{C}$ *is a nonzero multiplicative function and* $a : G \setminus I_m \to \mathbb{C}$ *is a nonzero additive function.*

Proof. Let $f, g : G \to \mathbb{C}$ satisfy the functional equation (4.1).

If $f = 0$, then $g(x)g(y) = 0$ for all $x, y \in G$, hence $g = 0$. This is case (1) of Lemma 4.1. Assume that $f \neq 0$. We split the discussion into the cases of f and g are linearly dependent or f and g are linearly independent.

Case 1: f and g are linearly dependent. Since $f \neq 0$ there exists a constant $c \in \mathbb{C}$ such that $g = c\,f$. By substituting this in Eq. (4.1) we obtain $f(xy) = (1-c^2)f(x)f(y)$ for all $x, y \in G$. Since $f \neq 0$ and G is generated by its squares we get that $c \neq 1$ and $c \neq -1$. From the last equation we obtain $(1-c^2)f(xy) = (1-c^2)^2 f(x)f(y)$ for all $x, y \in G$, then there exists a nonzero multiplicative function $m : G \to \mathbb{C}$ such that $(1-c^2)f = m$. So that $f = \frac{1}{1-c^2}\,m$ and $g = \frac{c}{1-c^2}\,m$. The result occurs in (2) of the list of Lemma 4.1.

Case 2: f and g are linearly independent. By similar computations as the ones of the proof of [21, Theorem 4.15], we get that

$$g(xy) = g(x)f(y) + g(y)f(x) + \alpha\, g(x)g(y) \tag{4.2}$$

for all $x, y \in G$, where $\alpha, \lambda_1, \lambda_2 \in \mathbb{C}$ are constants such that λ_1 and λ_2 are the roots of the polynomial $z^2 + \alpha\, z + 1$, and the functions $m : f - \lambda_1 g$ and $M := f - \lambda_2 g$ are multiplicative. Notice that $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 = -\alpha$.

If $\lambda_1 \neq \lambda_2$, then

$$f = \frac{\lambda\, M + \frac{1}{\lambda}\, m}{\lambda + \frac{1}{\lambda}} \quad \text{and} \quad g = \frac{M - m}{(\lambda + \frac{1}{\lambda})i},$$

where $\lambda = -i\,\lambda_1 = \frac{-i}{\lambda_2}$. Notice that $\lambda \in \mathbb{C} \setminus \{0, -i, i\}$. The result occurs in (3) of the list of Lemma 4.1.

If $\lambda_1 = \lambda_2$, then $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$.

If $\lambda_1 = \lambda_2 = 1$, then the functional equation (4.1) becomes

$$g(xy) = g(x)m(y) + g(y)m(x) \tag{4.3}$$

for all $x, y \in G$. As $g \neq 0$, because f and g are linearly independent, and G is generated by its squares we get from Eq. (4.3) that $m \neq 0$. By similar computations as the ones in the proof of [21, Lemma 3.4] we deduce from (4.3) that there exists a nonzero additive function $a : G \setminus I_m \to \mathbb{C}$ such that $g = m\,a$ on $G \setminus I_m$ and $g = 0$ on I_m, then $f = m\,(1+a)$ on $G \setminus I_m$ and $f = 0$. The result occurs in (4) of the list of Lemma 4.1.

If $\lambda_1 = \lambda_2 = -1$, then, by similar arguments as above, we obtain a solution of the form (5) of the list of Lemma 4.1.

Conversely, we check by elementary computations that the pairs (f, g) described in Lemma 4.1 are solutions of equation (4.1). This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let G be a semigroup generated by its squares. Let $f, g : G \to \mathbb{C}$ a solution of the functional equation* (1.2)*. Then*

(1) *$f \circ \sigma = f$, i.e., f is even with respect to σ, and f is central.*

(2) *$g \circ \sigma = g$ or $g \circ \sigma = -g$.*

Proof. (1) Let $x, y, z \in S$ be arbitrary. By interchanging x and y in (1.2) we get that $f(x\sigma(y)) = f(y\sigma(x))$. By replacing x by $\sigma(x)$ in the last identity we obtain $f \circ \sigma(xy) = f(\sigma(x)\sigma(y)) = f(yx)$. So, $f \circ \sigma(xyz) = f \circ \sigma(x(yz)) = f(yzx) = f \circ \sigma(zxy) = f(xyz)$. Since G is generated by its squares there exist $x_1, \ldots, x_n \in G$ such that $x = x_1^2 \cdots x_n^2$. So, we have $f \circ \sigma(x) = f \circ \sigma(x_1^2 \cdots x_n^2)$.

If $n = 1$ we obtain $f \circ \sigma(x) = f \circ \sigma(x_1^2) = f(x_1^2) = f(x)$.

If $n \geq 2$ we get that $f \circ \sigma(x) = f \circ \sigma(x_1 x_1 (x_2^2 \cdots x_n^2)) = f(x_1 x_1 (x_2^2 \cdots x_n^2)) = f(x)$. In both cases we get that $f \circ \sigma(x) = f(x)$. Since x is arbitrary, we deduce that $f \circ \sigma = f$. Moreover, since $f \circ \sigma(xy) = f(yx)$ for all $x, y \in G$, we get that $f(xy) = f(yx)$ for all $x, y \in G$. Hence, f is central. This is the result (1) of Lemma 4.2.

(2) By applying Eq. (1.2) to the pairs $(x, \sigma(y))$ and $(\sigma(x), y)$, and taking into account that $f \circ \sigma = f$, we get respectively

$$f(xy) = f(x)f(y) - g(x)g \circ \sigma(y)$$

and

$$f(xy) = f(x)f(y) - g \circ \sigma(x)g(y).$$

Hence, $g(x)g \circ \sigma(y) = g \circ \sigma(x)g(y)$ for all $x, y \in G$, which implies that the two functions g and $g \circ \sigma$ are linearly dependent. Since $\sigma \circ \sigma(x) = x$ for all $x \in G$, we get $g \circ \sigma = g$ or $g \circ \sigma = -g$, which is the result (2) of Lemma 4.2 and completes the proof of Lemma 4.2. □

Theorem 4.3. *Let G be a semigroup generated by its squares. The solutions $f, g : G \to \mathbb{C}$ of the functional equation* (1.2) *can be listed as follows*:

(1) *$f = 0$ and $g = 0$.*

(2) *$f = \frac{1}{1-\lambda^2}\, m$ and $g = \frac{\lambda}{1-\lambda^2}\, m$, where $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ is a constant and $m : G \to \mathbb{C}$ is a nonzero multiplicative function such that $m \circ \sigma = m$.*

(3) *$f = \frac{\lambda M + \frac{1}{\lambda} m}{\lambda + \frac{1}{\lambda}}$ and $g = \frac{M - m}{(\lambda + \frac{1}{\lambda})i}$, where $\lambda \in \mathbb{C} \setminus \{0, -i, i\}$ is a constant and $m, M : G \to \mathbb{C}$ are two different multiplicative functions such that $m \circ \sigma = m$ and $M \circ \sigma = M$.*

(4) $$\begin{cases} f = m\,(1 + a) \quad \text{and} \quad g = m\,a & \text{on} \quad G \setminus I_m, \\ f = g = 0 & \text{on} \quad I_m, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero multiplicative function and $a : G \setminus I_m \to \mathbb{C}$ is an nonzero additive function such that $m \circ \sigma = m$ and $a \circ \sigma = a$.

$$(5)\ \begin{cases} f = m\,(1+a) \quad \textit{and} \quad g = -m\,a & \textit{on} \quad G \setminus I_m, \\ f = g = 0 & \textit{on} \quad I_m, \end{cases}$$

where $m : G \to \mathbb{C}$ *is a nonzero multiplicative function and* $a : G \setminus I_m :\to \mathbb{C}$ *is a nonzero additive function such that* $m \circ \sigma = m$ *and* $a \circ \sigma = a$.

(6) $f = \frac{m+m\circ\sigma}{2}$ *and* $g = \frac{m-m\circ\sigma}{2}$, *where* $m : G \to \mathbb{C}$ *is a multiplicative function such that* $m \circ \sigma \neq m$.

Proof. Let $f, g : G \to \mathbb{C}$ satisfy the functional equation (1.2). According to Lemma 4.2(2) we have two cases: $g \circ \sigma = g$ or $g \circ \sigma = -g$.

Case 1: $g \circ \sigma = g$. By applying (1.2) to the pair $(x, \sigma(y))$ we get, according to Lemma 4.2(1), that

$$f(xy) = f(x)f(y) - g(x)g(y)$$

for all $x, y \in G$. According to Lemma 4.1 we get that one of the following possibilities holds:

(1) $f = 0$ and $g = 0$, which is (1) of Theorem 4.3.

(2) $f = \frac{1}{1-\lambda^2}m$ and $g = \frac{\lambda}{1-\lambda^2}m$, where $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ is a constant and $m : G \to \mathbb{C}$ is a nonzero multiplicative function. Since $f \circ \sigma = f$ we get that $m \circ \sigma = m$. So, we obtain a solution of the form (2) in Theorem 4.3.

(3) $f = \frac{\lambda M + \frac{1}{\lambda} m}{\lambda + \frac{1}{\lambda}}$ and $g = \frac{M-m}{(\lambda+\frac{1}{\lambda})i}$, where $\lambda \in \mathbb{C} \setminus \{0, -i, i\}$ is a constant and $m, M : G \to \mathbb{C}$ are two multiplicative functions such that $m \neq M$. Since $f \circ \sigma = f$, $g \circ \sigma = g$ and $\lambda \neq 0$, we get that $m \circ \sigma + \lambda^2 M \circ \sigma = m + \lambda^2 M$ and $M \circ \sigma - m \circ \sigma = M - m$, which implies $(1+\lambda^2)(M - M \circ \sigma) = 0$. As $1 + \lambda^2 \neq 0$ we get that $M \circ \sigma = M$, and then $m \circ \sigma = m$. The solution occurs in (3) of the list of Theorem 4.3.

$$(4)\ \begin{cases} f = m\,(1+a) \quad \text{and} \quad g = m\,a & \text{on} \quad G \setminus I_m, \\ f = g = 0 & \text{on} \quad I_m, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero multiplicative function and $a : G \setminus I_m \to \mathbb{C}$ is a nonzero additive function. Since $f \circ \sigma = f$, $g \circ \sigma = g$ we get that $m \circ \sigma\, a \circ \sigma = ma$ and $m \circ \sigma + m \circ \sigma\, a \circ \sigma = m + ma$ on $G \setminus I_m$, which implies $m \circ \sigma = m$ on $G \setminus I_m$ and $a \circ \sigma = a$. Moreover, $\sigma(I_m) \subseteq I_m$. Indeed, if there exists $x \in I_m$ such that $\sigma(x) \in G \setminus I_m$, then $f(\sigma(x)) = m(\sigma(x)) + m(\sigma(x))\,a(\sigma(x))$, $f(x) = 0$, $g(\sigma(x)) = m(\sigma(x))\,a(\sigma(x))$ and $g(x) = 0$. We infer from $f \circ \sigma = f$ and $g \circ \sigma = g$ that $m(\sigma(x)) = 0$, which is a contradiction. Hence, $\sigma(I_m) \subseteq I_m$. We deduce that $m \circ \sigma(x) = m(\sigma(x)) = 0$, and then $m \circ \sigma(x) = m(x)$ for all $x \in I_m$. Hence, $m \circ \sigma = m$. The solution occurs in (4) of Theorem 4.3.

$$(5)\ \begin{cases} f = m\,(1+a) \quad \text{and} \quad g = -m\,a & \text{on} \quad G \setminus I_m, \\ f = g = 0 & \text{on} \quad I_m, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero multiplicative function and $a : G \setminus I_m \to \mathbb{C}$ is a nonzero additive function. As in the case (4) we prove that $m \circ \sigma = m$ and $a \circ \sigma = a$. The solution occurs in (5) of Theorem 4.3.

Case 2: $g \circ \sigma = -g$. By applying (1.2) to the pair $(x, \sigma(y))$ we get, according to Lemma 4.2(1), that

$$f(xy) = f(x)f(y) + g(x)g(y)$$

for all $x, y \in G$. By writing ig instead of g we go back to the functional equation (4.1). So, as in case 1, we have the following possibilities:

(1) $f = 0$ and $g = 0$, which is (1) of Theorem 4.3.

(2) $f = \frac{1}{1-\lambda^2} m$ and $ig = \frac{\lambda}{1-\lambda^2} m$, where $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ is a constant and $m : G \to \mathbb{C}$ is a nonzero multiplicative function. Since $f \circ \sigma = f$ and $g \circ \sigma = -g$ we get that $\lambda = 0$ and $m \circ \sigma = m$. It follows that $f = m$ and $g = 0$ with $m \circ \sigma = m$ which (2) of Theorem 4.3.

(3) $f = \frac{\lambda M + \frac{1}{\lambda} m}{\lambda + \frac{1}{\lambda}}$ and $ig = \frac{M - m}{(\lambda + \frac{1}{\lambda}) i}$, where $\lambda \in \mathbb{C} \setminus \{0, -i, i\}$ is a constant and $m, M : G \to \mathbb{C}$ are two multiplicative functions such that $m \neq M$. Hence, $f = \frac{m + \lambda^2 M}{1 + \lambda^2}$ and $g = \frac{\lambda}{1+\lambda^2}(m - M)$. Since $g \circ \sigma = -g$ we get $m \circ \sigma + m = M \circ \sigma + M$. According to [21, Corollary 3.19] and taking into account that $m \neq M$, we get that $M = m \circ \sigma$, and then $m \circ \sigma \neq m$. So, $f = \frac{m + \lambda^2 m \circ \sigma}{1 + \lambda^2}$. Since $f \circ \sigma = f$ we deduce that $(\lambda^2 - 1)(m - m \circ \sigma) = 0$. Hence $\lambda^2 = 1$.

If $\lambda = 1$, then $f = \frac{m + m \circ \sigma}{2}$ and $g = \frac{m - m \circ \sigma}{2}$. The solution occurs in (6) of Theorem 4.3.

If $\lambda = -1$, then $f = \frac{m + m \circ \sigma}{2}$ and $g = \frac{m \circ \sigma - m}{2}$. By writing $m \circ \sigma$ instead of m, we obtain a solution of the form (6) of Theorem 4.3.

(4) $\begin{cases} f = m\,(1 + a) \quad \text{and} \quad g = -i\, m\, a & \text{on} \quad G \setminus I_m, \\ f = g = 0 & \text{on} \quad I_m, \end{cases}$

where $m : G \to \mathbb{C}$ is a nonzero multiplicative function and $a : G \setminus I_m \to \mathbb{C}$ is a nonzero additive function. As in (4) of case 1, we check that $\sigma(I_m) \subseteq I_m$. So, σ being an involution, we obtain $\sigma(G \setminus I_m) = G \setminus I_m$. Since $f \circ \sigma = f$ and $g \circ \sigma = -g$ we get that

$$m(\sigma(x))(1 + a(x)) = m(x)(1 + a(x))$$

and

$$m(\sigma(x))a(\sigma(x)) = -m(x)a(x)$$

for all $x \in G \setminus I_m$.

By adding the two last identity, we obtain $m \circ \sigma(x) + 2\, m \circ \sigma(x)\, a \circ \sigma(x) = m(x)$ for all $x \in G \setminus I_m$. So that $m(x) - m \circ \sigma(x) = 2\, m \circ \sigma(x)\, a \circ \sigma(x)$ for all $x \in G \setminus I_m$. Since $G \setminus I_m$ is a semigroup, then, according to [3, Lemma 4.4] (due to Stetkær) and using that $m \circ \sigma(x) \neq 0$ for all $x \in G \setminus I_m$, we get that $a(\sigma(x)) = 0$, and then $m \circ \sigma(x) = m(x)$ for all $x \in G \setminus I_m$. So, σ being an involution and $\sigma(G \setminus I_m) = G \setminus I_m$, we obtain $a(x) = 0$ for all $x \in G \setminus I_m$, which contradicts the fact that a a nonzero function on $G \setminus I_m$. Hence, the functional equation (1.2) has no solution in this case.

(5) $\begin{cases} f = m\,(1 + a) \quad \text{and} \quad g = m\, a & \text{on} \quad G \setminus I_m, \\ f = g = 0 & \text{on} \quad I_m, \end{cases}$

where $m : G \to \mathbb{C}$ is a nonzero multiplicative function and $a : G \setminus I_m \to \mathbb{C}$ is a nonzero additive function. Proceeding as above we prove that the functional equation (1.2) has no solution in this case.

Conversely, we check by elementary computations that the pairs (f, g) described in Theorem 4.3 are solutions of equation (1.2). This completes the proof of Theorem 4.3. $\square$

4.2. Stability of equation (1.2) on amenable groups

The stability of the functional equation (1.5) was established by Székelyhidi [24, Theorem 3.3] on an amenable group. In this subsection we will study the stability of the functional equation (1.2) on an amenable group. The results obtained are generalizations of those in [24, Theorem 3.3].

By using similar computations to the ones of the proofs of [24, Lemma 3.1 and Lemma 3.2] we get the following lemma.

Lemma 4.4. *Let G be a semigroup, $f, g : G \to \mathbb{C}$ be functions and let $\mathcal{V}$ be a two-sided invariant linear space of complex-valued functions on G. If the functions*

$$x \mapsto f(x\sigma(y)) - f(x)f(y) + g(x)g(y)$$

and

$$x \mapsto f(x\sigma(y)) - f(y\sigma(x))$$

belong to $\mathcal{V}$ for all $y \in G$, then we have one of the following possibilities:

(1) *$f, g \in \mathcal{V}$.*

(2) *f is multiplicative and $g \in \mathcal{V}$.*

(3) *$f + g$ or $f - g$ is multiplicative in $\mathcal{V}$.*

(4) *$f = \frac{\lambda^2}{\lambda^2-1} m - \frac{1}{\lambda^2-1} \varphi$ and $g = \frac{\lambda}{\lambda^2-1} m - \frac{\lambda}{\lambda^2-1} \varphi$, where $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ is a constant, $m : G \to \mathbb{C}$ is multiplicative and $\varphi \in \mathcal{V}$.*

(5) *$f(x\sigma(y)) = f(x)f(y) - g(x)g(y)$ for all $x, y \in G$.*

Theorem 4.5. *Let G be an amenable group, $\sigma : G \to G$ be an involutive automorphism and let $f, g : G \to \mathbb{C}$ be functions. The function*

$$(x, y) \mapsto f(x\sigma(y)) - f(x)f(y) + g(x)g(y)$$

is bounded if and only if one of the following assertions holds:

(1) *$f, g \in \mathcal{B}(G)$.*

(2) *f is multiplicative and $g \in \mathcal{B}(G)$.*

(3) *$f = (1 + a)\, m + b$ and $g = a\, m + b$, or $f = a\, m + b$ and $g = (1 - a)\, m - b$, where $a : G \to \mathbb{C}$ is additive, $m : G \to \mathbb{C}$ is a bounded multiplicative function and $b \in \mathcal{B}(G)$ such that $m \circ \sigma = m$ and $a \circ \sigma = a$.*

(4) *$f = \frac{\lambda^2}{\lambda^2-1} m - \frac{1}{\lambda^2-1} b$ and $g = \frac{\lambda}{\lambda^2-1} m - \frac{\lambda}{\lambda^2-1} b$, where $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ is a constant, $m : G \to \mathbb{C}$ is multiplicative and $b : G \to \mathbb{C}$ is a bounded function.*

(5) *$f(x\sigma(y)) = f(x)f(y) - g(x)g(y)$ for all $x, y \in G$.*

Proof. First we prove the necessity. We define the function

$$F(x, y) = f(x\sigma(y)) - f(x)f(y) + g(x)g(y) \tag{4.4}$$

for $x, y \in G$. Since F is bounded then the functions

$$x \mapsto f(x\sigma(y)) - f(x)f(y) + g(x)g(y)$$

and

$$x \mapsto f(x\sigma(y)) - f(y\sigma(x))$$

belong to $\mathcal{B}(G)$ for all $y \in G$. Since $\mathcal{B}(G)$ is a two-sided invariant linear space we have, according to Lemma 4.4, one of the following possibilities:

(1) $f, g \in \mathcal{B}(G)$, which occurs in (1) of Theorem 4.5.

(2) f is multiplicative and $g \in \mathcal{B}(G)$, which is (2) of Theorem 4.5.

(3) $f+g$ or $f-g$ is multiplicative in $\mathcal{B}(G)$. We will study the case $f-g = m$ with m multiplicative in $\mathcal{B}(G)$. The case $f+g$ multiplicative in $\mathcal{B}(G)$ go back to the first one by writing $-g$ instead of g.

If $m = 0$, then $F(x, y) = f(x\sigma(y))$ for all $x, y \in G$, and consequently $f, g \in \mathcal{B}(G)$. The result occurs in (1) of Theorem 4.5.

If $m \neq 0$, then

$$\begin{aligned} F(x, y) &= f(x\sigma(y)) - f(x)f(y) + [f(x) - m(x)][f(y) - m(y)] \\ &= f(x\sigma(y)) - f(x)m(y) - m(x)f(y) + m(x)m(y) \end{aligned}$$

for all $x, y \in G$. So,

$$F(x, \sigma(y)) = f(xy) - f(x)m(\sigma(y)) - m(x)f(\sigma(y)) + m(x)m(\sigma(y))$$

and

$$F(\sigma(x), y) = f \circ \sigma(xy) - f(\sigma(x)m(y) - m(\sigma(x))f(y) + m(\sigma(x))m(y).$$

We split the discussion into the cases of $m \circ \sigma = m$ or $m \circ \sigma \neq m$.

Case A: $m \circ \sigma = m$. Let $x, y \in G$. The identities above become

$$F(x, \sigma(y)) = f(xy) - f(x)m(y) - m(x)f(\sigma(y)) + m(x)m(y) \tag{4.5}$$

and

$$F(\sigma(x), y) = f \circ \sigma(xy) - f(\sigma(x))m(y) - m(x)f(y) + m(x)m(y) \tag{4.6}$$

for all $x, y \in G$.

By adding the identities (4.5) and (4.6), and using the fact that m is multiplicative, we obtain

$$F(x, \sigma(y)) + F(\sigma(x), y) = 2\, f_e(xy) - 2\, f_e(x)m(y) - 2m(x)f_e(y) + 2m(xy).$$

As m is a nonzero multiplicative function on the group G we get that $m(x) \neq 0$ and $m(x^{-1}) = (m(x))^{-1}$ for all $x \in G$. So,

$$\begin{aligned} &\frac{1}{2}[F(x, \sigma(y)) + F(\sigma(x), y)]m((xy))^{-1}) \\ &= f_e(xy)(m(xy))^{-1} - f_e(x)(m(x))^{-1} - f_e(y)(m(y))^{-1} + 1 \\ &= [f_e(xy)(m(xy))^{-1} - 1] - [f_e(x)(m(x))^{-1} - 1] - [f_e(y)(m(y))^{-1} - 1]. \end{aligned}$$

On the other hand, by subtracting (4.6) from (4.5), we get similarly that

$$\begin{aligned}&\frac{1}{2}[F(x,\sigma(y))-F(\sigma(x),y)]m((xy)^{-1})\\ &= f_o(xy)(m(xy))^{-1})-f_o(x)(m(x))^{-1}-f_o(y)(m(y))^{-1}.\end{aligned}$$

Since the functions F and m are bounded, and $\mathcal{B}(G)$ is a two-sided invariant and σ-invariant linear space of complex-valued functions on G, we get that the right hand side of the identity above is bounded as a function in (x, y). Moreover, G being an amenable group, we get, according to Hyers's Theorem [23, Theorem 3.1], that there exist two additive functions a_1, $a_2 : G \to \mathbb{C}$ and two functions $b_1, b_2 \in \mathcal{B}(G)$ such that $f_e = (1 + a_1)\, m + b_1$ and $f_o = a_2\, m + b_2$. Hence, $f = (1 + a)\, m + b$ and $g = a\, m + b$, where $a := a_1 + a_2$ is additive and $b := b_1 + b_2$ is a bounded function on G. Substituting this into (4.4), and taking into account that a is additive and m is multiplicative such that $m \circ \sigma = m$, we get that

$$\begin{aligned}F(x,y) &= (1+a(x\sigma(y)))m(x\sigma(y))+b(x\sigma(y))-[(1+a(x))m(x)+b(x)]\\ &\quad\times[(1+a(y))m(y)+b(y)]+[a(x)m(x)+b(x)][a(y))m(y)+b(y)]\\ &= (a\circ\sigma(y)-a(y))m(xy)-m(x)b(y)-m(y)b(x)+b(x\sigma(y))\end{aligned}$$

for all $x, y \in G$.

As $m(x) \neq 0$ and $m(x^{-1}) = (m(x))^{-1}$ for all $x \in G$, we get that

$$\begin{aligned}F(x,y)(m(xy)^{-1}) &= [a\circ\sigma(y)-a(y)]m(x)-m(y)b(y)-m(x)b(x)\\ &\quad+b(x\sigma(y))(m(xy)^{-1})\end{aligned}$$

for all $x, y \in G$. Since the functions F, m and b are bounded so is the function $(x, y) \mapsto (a\circ\sigma(y)-a(y))m(x)$. Since $m \neq 0$, we deduce that the function $a\circ\sigma-a$ is bounded. Since $a\circ\sigma-a$ is additive we get, according to [21, Exercise 2.5(a)], that $a \circ \sigma = a$. The result obtained in this case occurs in (3) of Theorem 4.5.

Case B: $m\circ\sigma \neq m$. By similar computations to the ones in Case 2 of the proof of Theorem 3.3 we prove that $f \in \mathcal{B}(G)$ and then $g \in \mathcal{B}(G)$, which occurs in (1) of Theorem 4.5.

(4) $f = \frac{\lambda^2}{\lambda^2-1}\, m - \frac{1}{\lambda^2-1}\, b$ and $g = \frac{\lambda}{\lambda^2-1}\, m - \frac{\lambda}{\lambda^2-1}\, b$, where $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ is a constant, $m : G \to \mathbb{C}$ is multiplicative and $b \in \mathcal{B}(G)$. The result occurs in (4) of Theorem 4.5.

(5) $f(x\sigma(y)) = f(x)f(y) - g(x)g(y)$ for all $x, y \in G$. The result occurs in (5) of Theorem 4.5.

Conversely, we check by elementary computations that if one of assertions (1)-(4) in Theorem 4.5 is satisfied, then the function F is bounded. This completes the proof of Theorem 4.5. $\square$

By taking $\sigma(x) = x$ for all $x \in G$ in Theorem 4.5 we obtain the following corollary.

Corollary 4.6 ([24, Theorem 3.3]). *Let G be an amenable group and let $f, g : G \to \mathbb{C}$ be functions. The function*

$$(x, y) \mapsto f(xy) - f(x)f(y) + g(x)g(y)$$

is bounded if and only if one of the following assertions holds:

(1) $f, g \in \mathcal{B}(G)$.

(2) f *is multiplicative and* $g \in \mathcal{B}(G)$.

(3) $f = (1+a)\,m + b$ *and* $g = a\,m + b$, *or* $f = a\,m + b$ *and* $g = (1-a)\,m - b$, *where* $a : G \to \mathbb{C}$ *is additive,* $m : G \to \mathbb{C}$ *is a bounded multiplicative function and* $b \in \mathcal{B}(G)$.

(4) $f = \frac{\lambda^2}{\lambda^2-1}\,m - \frac{1}{\lambda^2-1}\,b$ *and* $g = \frac{\lambda}{\lambda^2-1}\,m - \frac{\lambda}{\lambda^2-1}\,b$, *where* $\lambda \in \mathbb{C} \setminus \{-1, 1\}$ *is a constant,* $m : G \to \mathbb{C}$ *is multiplicative and* $b : G \to \mathbb{C}$ *is a bounded function.*

(5) $f(xy) = f(x)f(y) - g(x)g(y)$ *for all* $x, y \in G$.

5. Solutions and stability of equation (1.3)

The general solution of the functional equation $f(x - y) = f(x)g(y) - g(x)f(y)$ is given by Aczél and Dhombres in [2, p. 217, Theorem 11] on abelian group. Stetkær determined in [21, Theorem 4.12] the continuous solutions of the functional equation (1.3) on a topological group with σ a continuous involutive automorphism of G.

Chung et al. [9, Theorem 2] proved the Hyers-Ulam stability of (1.3) on an abelian 2-divisible group [9, Theorem 9]. In [7, Theorem 2.3] Chang and Chung proved the Hyers-Ulam stability of the functional equation $f(x - y) = f(x)g(y) - g(x)f(y)$ on an abelian 2-divisible group. They proved the Hyers-Ulam stability of the same equation on an abelian group [8, Theorem 2.5].

In this section we generalize the cited results by solving the functional equation (1.3) on a semigroup generated by its squares, and proving the Hyers-Ulam stability of (1.3) on an amenable group.

5.1. Solutions of equation (1.3) on semigroup generated by its squares

In this subsection we assume that G is a semigroup generated by its squares.

By using similar computations used in the proof of Lemma 4.2 we get the following result.

Lemma 5.1. *Let G be a semigroup generated by its squares. Let $f, g : G \to \mathbb{C}$ be a solution of the functional equation* (1.3). *Then*

(1) $f \circ \sigma = -f$, *i.e.,* f *is odd with respect to* σ.

(2) $f(xy) = f(yx)$ *for all* $x, y \in S$, *i.e.,* f *is central.*

Theorem 5.2. *The solutions $f, g : G \to \mathbb{C}$ of the functional equation* (1.3) *can be listed as follows*:

(A) $f = 0$ *and* g *is arbitrary.*

(B) $f = \frac{m - m\circ\sigma}{2\,\alpha}$ *and* $g = \frac{m + m\circ\sigma}{2} + \frac{\rho\,(m - m\circ\sigma)}{2}$, *where* $\alpha, \rho \in \mathbb{C}$ *are two constants with* $\alpha \neq 0$, *and* $m : G \to \mathbb{C}$ *is a multiplicative function such that* $m \circ \sigma \neq m\cdot$

(C) $\begin{cases} f = ma \quad and \quad g = m(1+\beta\, a) & on \quad G \setminus I_m, \\ f = g = 0 & on \quad I_m, \end{cases}$

where $\beta \in G$ is a constant, $m : G \to \mathbb{C}$ is a nonzero multiplicative function and $a : G \setminus I_m \to \mathbb{C}$ is a nonzero additive function such that $m \circ \sigma = m$ and $a \circ \sigma = -a$.

Proof. Let $f, g : G \to \mathbb{C}$ be a solution of the functional equation (1.3). If $f = 0$, then g is arbitrary, and the solution occurs in (A) of Theorem 5.2. So, in what follows we assume that $f \neq 0$. Since f is central and odd with respect to σ, then by using similar computations to that of the proof of [21, Theorem 4.12] we get that there exists a constant $\beta \in \mathbb{C}$ such

$$g_o = \beta f \tag{5.1}$$

and that

$$f(xy) = f(x)g_e(y) + g_e(x)f(y),\ x, y \in G. \tag{5.2}$$

According to [12, Lemma 3.4] there exist two multiplicative functions $m_1, m_2 : G \to \mathbb{C}$ such $g_e = \frac{m_1+m_2}{2}$.

If $m_1 \neq m_2$, then there exists a constant $\alpha \in \mathbb{C} \setminus \{0\}$ such that $f = \frac{m_1-m_2}{2\,\alpha}$. Since $g_e \circ \sigma = g_e$ and $f \circ \sigma = -f$ we get that $m_1 \circ \sigma - m_2 \circ \sigma = m_2 - m_1$ and $m_1 \circ \sigma + m_2 \circ \sigma = m_1 + m_2$. It follows that $m_2 = m_1 \circ \sigma$. So, $f = \frac{m - m\circ\sigma}{2\,\alpha}$ and $g_e = \frac{m+m\circ\sigma}{2}$, where $m := m_1$. Taking (5.2) into account we get that $g_o = \frac{\beta\,(m-m\circ\sigma)}{2\,\alpha} = \frac{\rho\,(m-m\circ\sigma)}{2}$, where $\rho := \frac{\beta}{\alpha}$. As $g = g_e + g_o$ we obtain $g = \frac{m+m\circ\sigma}{2} + \frac{\rho\,(m-m\circ\sigma)}{2}$. The solution occurs in (B) of the list of Theorem 5.2.

If $m_1 = m_2$, then letting $m := m_1$ we get $g_e = m$. Since $f \neq 0$ and G is generated by its squares we deduce, from (5.2), that $m \neq 0$ and there exists a nonzero additive function $a : G \setminus I_m \to \mathbb{C}$ such that $f = m\,a$ on $G \setminus I_m$ and $f = 0$ on I_m. Hence, we get from (5.1) that $g_o = \beta\, m\, a$ on $G \setminus I_m$ and $g_o = 0$ on I_m. It follows that $g = m\,(1+\beta\, a)$ on $G \setminus I_m$ and $g = 0$ on I_m. Moreover $m \circ \sigma = g_e \circ \sigma = g_e = m$, then $\sigma(G \setminus I_m) = G \setminus I_m$. Let $x \in G \setminus I_m$ be arbitrary. Since $f \circ \sigma = -f$ we get that $m(\sigma(x))\, a(\sigma(x)) = -m(x)\, a(x)$, which implies $m(x)\, a \circ \sigma(x) = -a(x)$. As $m(x) \neq 0$ we obtain $a \circ \sigma(x) = -a(x)$. We deduce that $a \circ \sigma = -a$. The solution occurs in (C) of the list of Theorem 5.2.

Conversely, if f and g are of the forms (A)-(C) in Theorem 5.2, we check by elementary computations that f and g satisfy the functional equation (1.3). This completes the proof of Theorem 5.2. ☐

5.2. Stability of equation (1.3) on amenable groups

Lemma 5.3. *Let G be a semigroup, $f, g : G \to \mathbb{C}$ be functions and let $\mathcal{V}$ be a two-sided invariant linear space of complex-valued functions on G such that $\mathcal{V}$ is σ-invariant. Suppose that f and g are linearly independent modulo $\mathcal{V}$. If the function*

$$x \mapsto f(x\sigma(y)) - f(x)g(y) + g(x)f(y)$$

belongs to $\mathcal{V}$ *for all* $y \in G$*, then*

$$f \circ \sigma = f \quad \textit{and} \quad g \circ \sigma = g$$

or

$$f \circ \sigma = -f \quad \textit{and} \quad g_o = \gamma f,$$

where $\gamma \in \mathbb{C}$ *is a constant.*

Proof. Let F be the function defined by

$$F(x,y) = f(x\sigma(y)) - f(x)g(y) + g(x)f(y) \tag{5.3}$$

for $x, y \in G$. Using similar computations as the ones of the proof of Lemma 3.1 we prove that there exist $y_0 \in G$ and λ_0, $\lambda_1 \in \mathbb{C}$ such that the function φ_1 defined by $\varphi_1(x) = -\lambda_0 f(x) + \lambda_1 f(xy_0) + g(x)$ for $x \in G$, satisfies the following functional equations

$$F(x,y) + F(x,\sigma(y)) = 2\, f_e(y)\, \varphi_1(x) \tag{5.4}$$

and

$$F(x,\sigma(y)) - F(x,y) = 4\, f(x)\, g_o(y) + 4\, g(x)\, f_o(y) + 2 f_o(y)\, \varphi_1(x) \tag{5.5}$$

for all $x, y \in G$.

If $f \circ \sigma \neq -f$, then $f_e \neq 0$. So, there exists $y_1 \in G$ such that $f_e(y_1) \neq 0$. By replacing y by y_1 in (5.4) and using the fact that the function $x \mapsto F(x, y_1) + F(x, \sigma(y_1))$ belongs to $\mathcal{V}$ we get that

$$\varphi_1 \in \mathcal{V}. \tag{5.6}$$

Let $y \in G$ be arbitrary. Equations (5.5) and (5.6) implies that the function $x \mapsto f(x)\, g_o(y) + g(x)\, f_o(y)$ belongs to $\mathcal{V}$. As f and g are linearly independent modulo $\mathcal{V}$ we get that $g_o(y) = f_o(y) = 0$. So, y being arbitrary, we deduce that $f \circ \sigma = f$ and $g \circ \sigma = g$.

If $f \circ \sigma = -f$, then

$$F(x,\sigma(y)) = f(xy) - f(x)\, g \circ \sigma(y) - g(x)\, f(y)$$

and

$$F(\sigma(x),y) = -f(xy) + f(x)\, g(y) + g \circ \sigma(x)\, f(y)$$

for all $x, y \in G$. By adding the identities above we get that

$$F(x,\sigma(y)) + F(\sigma(x),y) = 2\, f(x)\, g_o(y) - 2\, g_o(x)\, f(y). \tag{5.7}$$

On the other hand $f \neq 0$, because f and g are linearly independent modulo $\mathcal{V}$, so there exists $z_0 \in G$ such that $f(z_0) \neq 0$. Moreover, the functions $x \mapsto F(x, \sigma(z_0))$ and $x \mapsto F(x, z_0)$ belong to $\mathcal{V}$. Since $\mathcal{V}$ is σ-invariant the function $x \mapsto F(\sigma(x), z_0)$ belongs to $\mathcal{V}$, so does the function $x \mapsto F(x, \sigma(z_0)) + F(\sigma(x), z_0)$. By replacing y by z_0 in (5.7) and dividing by $f(z_0)$ we get that there exist a constant $\gamma \in \mathbb{C}$ and a function $h \in \mathcal{V}$ such that

$$g_o = \gamma f + h. \tag{5.8}$$

Let $y \in G$ be arbitrary. Substituting (5.8) back into (5.7) we obtain

$$\begin{aligned} F(x,\sigma(y)) + F(\sigma(x),y) &= 2\,f(x)(\gamma f(y) + h(y)) - 2\,(\gamma f(x) + h(x))f(y) \\ &= 2\,f(x)h(y) - 2h(x)f(y) \end{aligned}$$

for all $x \in G$.

Since the functions $x \mapsto F(x,\sigma(y)) + F(\sigma(x),y)$ and $x \mapsto h(x)$ belong to $\mathcal{V}$, we deduce from the identity above that the function $x \mapsto f(x)h(y)$ belongs to $\mathcal{V}$. As $f \notin \mathcal{V}$ we infer that $h(y) = 0$. So, y being arbitrary, we deduce that $h = 0$. Hence (5.8) becomes $g_o = \gamma f$. This completes the proof of Lemma 5.3. □

Lemma 5.4. *Let G be a semigroup, $f, g : G \to \mathbb{C}$ be functions and let $\mathcal{V}$ be a two-sided invariant linear space of complex-valued functions on G such that $\mathcal{V}$ is σ-invariant. If the function*

$$x \mapsto f(x\sigma(y)) - f(x)g(y) + g(x)f(y)$$

belongs to $\mathcal{V}$ for all $y \in G$, then we have one of the following possibilities:

(1) *$f = 0$ and g is arbitrary.*

(2) *$f, g \in \mathcal{V}$.*

(3) *$f \notin \mathcal{V}$, $g \in \mathcal{V}$ and g is multiplicative.*

(4) *$f \notin \mathcal{V}$, $g \notin \mathcal{V}$ and $g = \delta\, f + m$, where $\delta \in \mathbb{C}\backslash\{0\}$ is a constant and $m \in \mathcal{V}$ is a multiplicative function.*

(5) *$f(x\sigma(y)) = f(x)g(y) - g(x)f(y)$ for all $x, y \in G$.*

(6) *$F(x,\sigma(y)) + F(y,\sigma(x)) = f(xy) + f(yx)$ for all $x, y \in G$, where F is the function defined in* (5.3).

Proof. We use a similar computation as the one of the proof of [24, Lemma 2.1]. Let F be the function defined in (5.3). We split the discussion into the cases of f and g are linearly independent modulo $\mathcal{V}$, or f and g are linearly dependent modulo $\mathcal{V}$.

<u>Case 1</u>: f and g are linearly independent modulo $\mathcal{V}$. Then, according to Lemma 5.3, we have one of the following subcases:

<u>Subcase 1.1</u>: $f \circ \sigma = -f$ and $g_o = \gamma\, f$, where $\gamma \in \mathbb{C}$ is a constant.

Since f and g are linearly independent modulo $\mathcal{V}$, then $f \neq 0$. So, there exists $y_0 \in G$ such that $f(y_0) \neq 0$. Let $x, y, z \in G$. By similar computation as the one of the proof of equations (3.2) and (3.6) (See the proof of Lemma 3.1) we prove that there exist two constants $\lambda_0, \lambda_1 \in \mathbb{C}$ such that

$$g(x) = \lambda_0\, f(x) - \lambda_1\, f(x\sigma(y_0)) + \lambda_1\, F(x, y_0), \tag{5.9}$$

$$f(yz) = f(y)g(z) - \lambda_0\, f(y)\, f(z) + \lambda_1\, f(yy_0)\, f(z) \tag{5.10}$$

and

$$g(yz) = g(y)g(z) - \lambda_0\, g(y)f(z) + \lambda_1\, g(yy_0)f(z). \tag{5.11}$$

By replacing x by $\sigma(y)$ in (5.9) and using that $f \circ \sigma = -f$ we get that

$$g(\sigma(y)) = -\lambda_0\, f(y) + \lambda_1\, f(yy_0) + \lambda_1\, F(\sigma(y), y_0)$$

so that

$$g(\sigma(y)) - \lambda_1 F(\sigma(y), y_0) = -\lambda_0 f(y) + \lambda_1 f(yy_0).$$

Substituting this back into (5.10) we obtain

$$f(yz) = f(y)g(z) + g(\sigma(y))f(z) - \lambda_1 F(\sigma(y), y_0)f(z). \tag{5.12}$$

Moreover, from (5.3) we have

$$f(y\sigma(z)) = f(y)g(z) - g(y)f(z) + F(y, z).$$

By replacing y by $\sigma(y)$ in the identity above and using that $f \circ \sigma = -f$ we get

$$f(yz) = f(y)g(z) + g(\sigma(y))f(z) - F(\sigma(y), z). \tag{5.13}$$

From (5.12) and (5.13) we deduce that

$$F(y, z) = \lambda_1 F(y, y_0)f(z) \tag{5.14}$$

for all $y, z \in G$.

By applying (5.10) to the pair $(\sigma(x), y)$ and using that $f \circ \sigma = -f$ we get that

$$\begin{aligned} f(x\sigma(y)) &= f(x)g(y) - \lambda_0 f(x)f(y) + \lambda_1 f(x\sigma(y_0))f(y) \\ &= f(x)g(y) - f(y)[\lambda_0 f(x) - \lambda_1 f(x\sigma(y_0))]. \end{aligned}$$

Moreover, the identity (5.9) implies $\lambda_0 f(x) - \lambda_1 f(x\sigma(y_0)) = g(x) - \lambda_1 F(x, y_0)$. Hence,

$$f(x\sigma(y)) = f(x)g(y) - f(y)[g(x) - \lambda_1 F(x, y_0)]. \tag{5.15}$$

Computing $f(x\sigma(y)\sigma(z))$ first as $f((x\sigma(y))\sigma(z))$ and then as $f(x\sigma(yz))$, by using (5.15) and a computation adapted to that of the proof of [24, Lemma 2.1], we derive

$$\begin{aligned} &- \psi(x)f(y) + \lambda_1 F(x\sigma(y), y_0) \\ &= \lambda_1 F(\sigma(y), y_0)g(x) + \lambda_1 F(x, y_0)g(\sigma(y)) - \lambda_1^2 F(x, y_0) F(y, y_0) \\ &\quad - f(x)\psi(y), \end{aligned} \tag{5.16}$$

where

$$\psi(x) := \lambda_0 g(x) - \lambda_1 g(xy_0)$$

for all $x \in G$.

By interchanging x and y in (5.16) we get

$$\begin{aligned} &- \psi(y)f(x) + \lambda_1 F(y\sigma(x), y_0) \\ &= \lambda_1 F(\sigma(x), y_0)g(y) + \lambda_1 F(y, y_0)g(\sigma(x)) - \lambda_1^2 F(x, y_0)F(y, y_0) \\ &\quad - f(y)\psi(x). \end{aligned} \tag{5.17}$$

By adding (5.16) and (5.17) we obtain

$$\begin{aligned} &\lambda_1 [F(x\sigma(y), y_0) + F(y\sigma(x), y_0)] \\ &= \lambda_1 F(\sigma(y), y_0)g(x) + \lambda_1 F(y, y_0)g(\sigma(x)) + \lambda_1 F(x, y_0)g(\sigma(y)) \\ &\quad + \lambda_1 F(\sigma(x), y_0)g(y) - 2\lambda_1^2 F(x, y_0)F(y, y_0) \end{aligned} \tag{5.18}$$

Let $y \in G$ be arbitrary. Since $\mathcal{V}$ a two-sided invariant and σ-invariant linear space of complex-valued functions on G, and the function $x \mapsto F(x, y_0)$ belongs to $\mathcal{V}$ by assumption, we derive that the functions $x \mapsto F(x\sigma(y), y_0)$, $x \mapsto F(y\sigma(x), y_0)$ and $x \mapsto F(\sigma(x), y_0)$ belong to $\mathcal{V}$. So, taking (5.18) into account, the function $x \mapsto \lambda_1 F(\sigma(y), y_0)g(x) + \lambda_1 F(y, y_0)g \circ \sigma(x)$ belongs to $\mathcal{V}$. As $g_o = \gamma f$ we get that $g \circ \sigma = g - 2\gamma f$. It follows that the function $x \mapsto \lambda_1 [F(\sigma(y), y_0) + F(y, y_0)]g(x) + 2\gamma \lambda_1 F(y, y_0)f(x)$ belongs to $\mathcal{V}$. Since f and g are linearly independent modulo $\mathcal{V}$ and y being arbitrary, we obtain

$$\gamma \lambda_1 F(y, y_0) = 0 \tag{5.19}$$

for all $y \in G$.

If $\gamma \neq 0$, then we get, from (5.19), that $\lambda_1 F(y, y_0) = 0$. It follows, from (5.14), that $F(y, z) = 0$ for all $y, z \in G$. Hence, $f(x\sigma(y)) = f(x)g(y) - g(x)f(y)$ for all $x, y \in G$. The result occurs in (5) of Lemma 5.4.

If $\gamma = 0$, then $g_0 = 0$, which implies that $g \circ \sigma = g$. So, $F(x, \sigma(y)) = f(xy) - f(x)g(y) - g(x)f(y)$ for all $x, y \in G$. Hence, the function $x \mapsto f(xy) - f(x)g(y) - g(x)f(y)$ belongs to $\mathcal{V}$ for each fixed y in G. According to [24, Lemma 2.1] we get that

$$f(xy) = f(x)g(y) + g(x)f(y)$$

for all $x, y \in G$. By applying this functional equation to the pair $(x, \sigma(y))$ we get that

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y)$$

for all $x, y \in G$. The result occurs in (5) of Lemma 5.4.

Subcase 1.2: $f \circ \sigma = f$ and $g \circ \sigma = g$. Let $x, y \in G$. By applying (5.3) to the pairs $(y, \sigma(x))$ and $(\sigma(x), y)$ we obtain respectively

$$F(y, \sigma(x)) = f(yx) - f(y)g(x) + g(y)f(x)$$

and

$$F(\sigma(x), y) = f(xy) - f(x)g(y) + g(x)f(y).$$

By adding the last identities we get that

$$F(\sigma(x), y) + F(y, \sigma(x)) = f(xy) + f(yx)$$

for all $x, y \in G$, which occurs in (6) of Lemma 5.4.

Case 2: f and g are linearly dependent modulo $\mathcal{V}$. Then, there exist two constants μ, $\nu \in \mathbb{C}$, not both zero, and a function $h \in \mathcal{V}$ such that $\mu f + \nu g = h$.

If $f = 0$, then g is arbitrary. This is (1) of Lemma 5.4.

If $f \notin \mathcal{V}$ and $g \notin \mathcal{V}$, then $\mu \neq 0$ and $\nu \neq 0$. So $g = \delta f + l$, where $\delta := -\frac{\mu}{\nu} \in \mathbb{C} \setminus \{0\}$ is a constant and $l := -\frac{1}{\nu} h \in \mathcal{V}$. Hence,

$$\begin{aligned} F(x, \sigma(y)) &= f(xy) - f(x)[\delta f(\sigma(y)) + l(\sigma(y))] + [\delta f(x) + l(x)]f(\sigma(y)) \\ &= f(xy) - f(x)l \circ \sigma(y) + l(x)f(\sigma(y)) \end{aligned}$$

for all $x, y \in G$. Let $y \in G$ be arbitrary. Since the functions $x \mapsto F(x, \sigma(y))$ and $x \mapsto l(x)f(\sigma(y))$ belong to $\mathcal{V}$ so does the function $x \mapsto f(xy) - f(x)l \circ \sigma(y)$. As $f \notin \mathcal{V}$ we derive, according to [22, Theorem], that $l \circ \sigma$ is a multiplicative function, so is l. Hence $g = \delta f + m$, where $m \in \mathcal{V}$ is multiplicative. The result occurs in (4) of Lemma 5.4.

If $f \in \mathcal{V}$ and $f \neq 0$, then the function $x \mapsto g(x)f(y)$ belongs to $\mathcal{V}$ for all $y \in G$, because the functions $x \mapsto F(x, y)$ and $x \mapsto f(x)g(y)$ belong also to $\mathcal{V}$. As $f \neq 0$ we derive that $g \in \mathcal{V}$. So we obtain the result (2) of Lemma 5.4.

If $g \in \mathcal{V}$ and $f \notin \mathcal{V}$. Let y be arbitrary. We have $f(xy) - f(x)g \circ \sigma(y) = F(x, \sigma(y)) - g(x)f(\sigma(y))$ for all $x \in G$. Since the functions $x \mapsto F(x, \sigma(y))$ and $x \mapsto g(x)f(\sigma(y))$ belong to $\mathcal{V}$ so does the function $x \mapsto f(xy) - f(x)g \circ \sigma(y)$. As $f \notin \mathcal{V}$ we get, according to [22, Theorem], that g is a multiplicative function. The result occurs in (3) of Lemma 5.4. This completes the proof of Lemma 5.4. □

Theorem 5.5. *Let G be an amenable group, $\sigma : G \to G$ be an involutive automorphism and let $f, g : G \to \mathbb{C}$ be functions. The function*

$$(x, y) \mapsto f(x\sigma(y)) - f(x)g(y) + g(x)f(y)$$

is bounded if and only if one of the following assertions holds:

(1) *$f = 0$ and g is arbitrary.*

(2) *$f, g \in \mathcal{B}(G)$.*

(3) *$f \notin \mathcal{B}(G)$, $g \notin \mathcal{B}(G)$, and $f = a\,m + b$ and $g = (1 + \delta\, a)m + \delta\, b$, where $\delta \in \mathbb{C}$ is a constant, $b : G \to \mathbb{C}$ is a bounded function, $a : G \to \mathbb{C}$ is nonzero additive function and $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function such that $m \circ \sigma = m$ and $a \circ \sigma = -a$.*

(4) *$f(x\sigma(y)) = f(x)g(y) - g(x)f(y)$ for all $x, y \in G$.*

Proof. First we prove the necessity. Let F be the function defined in (5.3). Since F is bounded, then the function

$$x \mapsto f(x\sigma(y)) - f(x)g(y) + g(x)f(y)$$

belongs to $\mathcal{B}(G)$ for every $y \in G$. Notice that $\mathcal{B}(G)$ is a two-sided invariant and σ-invariant linear space of complex-valued functions on G. According to Lemma 5.4 we have one of the following possibilities:

(1) $f = 0$ and g is arbitrary, which occurs in (1) of Theorem 5.5.

(2) $f, g \in \mathcal{B}(G)$, the result occurs in (2) of Theorem 5.5.

(3) $g \in \mathcal{B}(G)$ and g is multiplicative. Let $m := g$. If $m = 0$, then $F(x, y) = f(x\sigma(y))$ for all $x, y \in G$. Since F is bounded so is f. The result occurs in (2) of Theorem 5.5.

Suppose that $m \neq 0$. Let $x, y \in G$, we have

$$F(x, \sigma(y)) = f(xy) - f(x)m \circ \sigma(y) + m(x)f \circ \sigma(y) \tag{5.20}$$

and

$$F(\sigma(x), y) = f \circ \sigma(xy) - f \circ \sigma(x)m(y) + m \circ \sigma(x)f(y). \tag{5.21}$$

We discuss two cases: $m \circ \sigma = m$, and $m \circ \sigma \neq m$.

Case 1: $m \circ \sigma = m$, then by adding the equations (5.20) and (5.21) we get

$$H(x, y) = f_e(xy) - f_e(x)m(y) + m(x)f_e(y),$$

where $H(x,y) := \frac{1}{2}[F(\sigma(x),y) + F(x,\sigma(y))]$. So,

$$H(x,y) + H(y,x) = f_e(xy) + f_e(yx).$$

Let e be the identity element of the group G. By putting $y = e$ in the identity above we get that

$$f_e(x) = \frac{1}{2}[H(x,e) + H(e,x)] \tag{5.22}$$

for all $x \in G$.

Since the function F is bounded so is the function $x \mapsto \frac{1}{2}[H(x,e) + H(e,x)]$. Hence, we deduce from (5.22) that

$$f_e \in \mathcal{B}(G). \tag{5.23}$$

On the other hand, by subtracting (5.20) from (5.21) and taking into account that $m \circ \sigma = m$, we get that

$$F(x,\sigma(y)) - F(\sigma(x),y) = 2\, f_o(xy) - f_o(x)m(y) - m(x)f_o(y).$$

Since m is a nonzero multiplicative function on the group G we get that $m(x) \neq 0$ and $m(x^{-1}) = (m(x))^{-1}$ for all $x \in G$. Hence, multiplying the last equation by $\frac{1}{2}m((xy)^{-1})$, we get that

$$\begin{aligned}&\frac{1}{2}[F(x,\sigma(y)) - F(\sigma(x),y)]m((xy)^{-1})\\ &= f_o(xy)(m(xy))^{-1} - f_o(x)(m(x))^{-1} - f_o(y)(m(y))^{-1}.\end{aligned}$$

Since the functions F and m are bounded so are the right hand sides of the identity above as a function in (x,y). As G is an amenable group we get, according to Hyers's theorem [23, Theorem 3.1], that there exist an additive function $a : G \to \mathbb{C}$ and a function $b_1 \in \mathcal{B}(G)$ such that $f_o(x)m(x)^{-1} - a(x) = b_1(x)$ for all $x \in G$. Hence,

$$f_o = (a + b_1)m. \tag{5.24}$$

We derive from (5.23) and (5.24) that $f = am + b$, where $b := f_e + b_1 m$ is a bounded function.

On the other hand, using that $f = am + b$, $g = m$, a is additive, m is multiplicative and $m \circ \sigma = m$, we obtain from (5.3) that

$$\begin{aligned}F(x,y) &= a(x\sigma(y))m(x\sigma(y)) + b(x\sigma(y)) - [a(x)m(x) + b(x)]m(y)\\ &\quad + [a(y)m(y) + b(y)]m(x)\\ &= [a(x) + a\circ\sigma(y)]m(xy) - a(x)m(xy) + a(y)m(xy) - b(x)m(y)\\ &\quad + b(y)m(x) + b(x\sigma(y))\\ &= [a\circ\sigma(y) + a(y)]m(xy) - b(x)m(y) + b(y)m(x) + b(x\sigma(y))\end{aligned}$$

for all $x, y \in G$. As m is a nonzero multiplicative function on the group G we get that $m(x) \neq 0$ and $m(x^{-1}) = (m(x))^{-1}$ for all $x \in G$. So,

$$F(x,y)m((xy)^{-1}) = a\circ\sigma(y) + a(y) - b(x)m(x^{-1}) + b(y)m(y^{-1})$$

$$+ b(x\sigma(y))m((xy)^{-1}).$$

Let $x \in G$ be fixed. Since the functions $y \mapsto F(x,y)$, $y \mapsto m((xy)^{-1})$ and b belong to $\mathcal{B}(G)$ so does the function $y \mapsto a \circ \sigma(y) + a(y)$. Since $a \circ \sigma + a$ is additive we get, according to [21, Exercise 2.5(a)], that $a \circ \sigma = -a$. The result occurs in (3) of Theorem 5.5.

Case 2: $m \circ \sigma \neq m$. Since $m \neq 0$ we have $m(e) = 1$. Hence,

$$F(e,y) = f \circ \sigma(y) - f(e)m(y) + f(y) = 2\, f_e(y) - f(e)m(y)$$

for all $y \in G$. Since the functions $y \mapsto F(e,y)$, $y \mapsto f(e)m(y)$ belong to $\mathcal{B}(G)$ we get that

$$f_e \in \mathcal{B}(G). \tag{5.25}$$

On the other hand, by subtracting (5.20) from (5.21) we obtain

$$\begin{aligned} &F(x,\sigma(y)) - F(\sigma(x),y) \\ &= 2\, f_o(xy) - f(x)m \circ \sigma(y) + m(x)f \circ \sigma(y) + f \circ \sigma(x)m(y) - m \circ \sigma(x)f(y) \end{aligned}$$

for all $x, y \in G$. Notice that $f = f_e + f_o$, $f \circ \sigma = f_e - f_o$, $m = m_e + m_o$ and $m \circ \sigma = m_e - m_o$. So, we have

$$\begin{aligned} &\frac{1}{2}[F(x,\sigma(y)) - F(\sigma(x),y)] \\ &= f_o(xy) - f_o(x)m_e(y) + m_o(x)f_e(y) - m_e(x)f_o(y) + f_e(x)m_e(y) \end{aligned}$$

for all $x, y \in G$. Let $y \in G$ be arbitrary. Since the functions $x \mapsto F(x,\sigma(y))$, $x \mapsto F(\sigma(x),y)$, $x \mapsto m_o(x)f_e(y)$, $x \mapsto m_e(x)f_o(y)$ and $x \mapsto f_e(x)m_e(y)$ belong to $\mathcal{B}(G)$ so does the function $x \mapsto f_o(xy) - f_o(x)m_e(y)$. As $\mathcal{B}(G)$ is a two-sided invariant linear space of complex-valued functions on G and m_e is not multiplicative, because $m \circ \sigma \neq m$, we deduce, according to [22, Theorem], that

$$f_o \in \mathcal{B}(G). \tag{5.26}$$

We deduce from (5.23) and (5.24) that $f \in \mathcal{B}(G)$. The result occurs in (2) of Theorem 5.5.

(4) $f \notin \mathcal{B}(G)$, $g \notin \mathcal{B}(G)$ and $g = \delta\, f + m$, where $\delta \in \mathbb{C}\backslash\{0\}$ is a constant and $m \in \mathcal{B}(G)$ is a multiplicative function. Then

$$\begin{aligned} F(x,y) &= f(x\sigma(y)) - f(x)[\delta\, f(y) + m(y)] + f(y)[\delta\, f(x) + m(x)] \\ &= f(x\sigma(y)) - f(x)m(y) + m(x)f(y) \end{aligned}$$

for all $x, y \in G$. So we go back to (3) (see page 21).

If $m = 0$, then $g = \delta\, f$. Hence, $F(x,y) = f(x\sigma(y))$ for all $x, y \in G$. Sine the function $(x,y) \mapsto F(x,y)$ is bounded so are f and g, which contradicts that $f \notin \mathcal{B}(G)$ and $g \notin \mathcal{B}(G)$.

If $m \neq 0$, then, proceeding exactly as in (3) (see page 21), and seeing that $f \notin \mathcal{B}(G)$ and $g \notin \mathcal{B}(G)$ we prove that there exist an additive function $a : G \to \mathbb{C}$ and a function $b \in \mathcal{B}(G)$ such that $f = am + b$, $g = (1 + \delta\, a)m + \delta\, b$,

$m \circ \sigma = m$ and $a \circ \sigma = -a$. Moreover a is nonzero because $f \notin \mathcal{B}(G)$. The result occurs in (3) of Theorem 5.5.

(5) $f(x\sigma(y)) = f(x)g(y) - g(x)f(y)$ for all $x, y \in G$, which is the assertion (4) of Theorem 5.5.

(6) $F(x, \sigma(y)) + F(y, \sigma(x)) = f(xy) + f(yx)$ for all $x, y \in G$. If $f = 0$, then the functional equation (1.3) is satisfied, which corresponds to (4) of Theorem 5.5. In what follows we assume that $f \neq 0$. By putting $y = e$ in the identity above we get

$$F(x, e) + F(e, \sigma(x)) = 2\, f(x)$$

for all $x \in G$. Since the functions $x \mapsto F(x, e)$ and $x \mapsto F(e, x)$ belong to $\mathcal{B}(G)$ and $\mathcal{B}(G)$ is σ-invariant, we get that $f \in \mathcal{B}(G)$. So, we get from the identity

$$F(x, y) = f(x\sigma(y)) - f(x)g(y) + g(x)f(y),$$

that $g \in \mathcal{B}(G)$. The result occurs in (2) of Theorem 5.5.

Conversely, we check by elementary computations that if one of the assertions (1)-(4) in Theorem 5.5 is satisfied, then the function

$$(x, y) \mapsto f(x\sigma(y)) - f(x)g(y) + g(x)f(y)$$

is bounded. This completes the proof of Theorem 5.5. $\square$

Acknowledgments. The authors would like to thank the referee for his/her valuable comments and suggestions which improve the presentation of the paper.

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