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SYMMETRIC PROPERTY OF RINGS WITH RESPECT TO THE JACOBSON RADICAL

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Abstract. Let R be a ring with identity and $J(R)$ denote the Jacobson radical of R, i.e., the intersection of all maximal left ideals of R. A ring R is called *J-symmetric* if for any $a, b, c \in R$, $abc = 0$ implies $bac \in J(R)$. We prove that some results of symmetric rings can be extended to the J-symmetric rings for this general setting. We give many characterizations of such rings. We show that the class of J-symmetric rings lies strictly between the class of symmetric rings and the class of directly finite rings.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. A ring is *reduced* if it has no nonzero nilpotent elements. A weaker condition than "reduced" is defined by Lambek in [6]. A ring R is *symmetric* if for any a, $b, c \in R$, $abc = 0$ implies $acb = 0$ if and only if $abc = 0$ implies $bac = 0$. An equivalent condition on a ring is that whenever a product of any number of elements is zero, any permutation of the factors still yields product zero. Clearly, symmetric rings are J-symmetric, but the converse is not true in general. We investigate characterizations of J-symmetric rings, and that many families of J-symmetric rings are presented.

In what follows, $\mathbb{Z}$ denotes the ring of integers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. We write $M_n(R)$ for the ring of all $n \times n$ matrices and $T_n(R)$ for the ring of all $n \times n$ upper triangular matrices over R. Also we write $R[x]$, $R[[x]]$, $U(R)$, $I(R)$ for the polynomial ring, the power series ring over a ring R, the set of all invertible elements and the set of all idempotent elements of R, respectively.

2. J-symmetric rings

In this section we introduce a class of rings, so-called J-symmetric rings, which is a generalization of symmetric rings. We investigate which properties of symmetric rings hold for the J-symmetric case. It is clear that symmetric

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rings are J-symmetric. We supply an example (see Example 2.4) to show that all J-symmetric rings need not be symmetric. Then, we prove that every J-symmetric ring is directly finite and we give an example to illustrate there are directly finite rings which are not J-symmetric. Therefore, the class of J-symmetric rings lies strictly between classes of symmetric rings and directly finite rings. It is shown that the class of J-symmetric rings is closed under finite direct sums. We have an example which shows that the homomorphic image of a J-symmetric ring is not J-symmetric. Then, we determine under what conditions a homomorphic image of a ring is J-symmetric.

We now give our main definition.

Definition 2.1. A ring R is called *J-symmetric* if for any $a, b, c \in R$, $abc = 0$ implies $bac \in J(R)$.

Note that $R/J(R)$ is J-symmetric if and only if $R/J(R)$ is symmetric, since $J(R/J(R)) = 0$. Then, we have the following result.

Lemma 2.2. *If $R/J(R)$ is a symmetric ring, then R is J-symmetric.*

Proof. Assume that $abc = 0$ for any a,$b, c \in R$. Then, $\overline{abc} = \overline{0} \in R/J(R)$. Since $R/J(R)$ is symmetric, $\overline{bac} = \overline{0}$ and so $bac \in J(R)$, as asserted. $\square$

All commutative rings, reduced rings, symmetric rings are J-symmetric. Recall that a ring R is called *local* if it has only one maximal left ideal (equivalently, maximal right ideal). It is well known that a ring R is local if and only if $a + b = 1$ in R implies that either a or b is invertible if and only if $R/J(R)$ is a division ring. In this direction, we prove that local rings are J-symmetric.

Lemma 2.3. *Every local ring is J-symmetric.*

Proof. Let R be a local ring, $a, b, c \in R$ with $abc = 0$. If $a \in J(R)$ or $b \in J(R)$ or $c \in J(R)$, then clearly $bac \in J(R)$. Let $a, b, c \in U(R)$. This contradicts with $abc = 0$. This completes the proof. $\square$

One may suspect that J-symmetric rings are symmetric. But the following example erases the possibility.

Example 2.4. (1) Consider the ring $S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in R \right\}$ where R is a J-symmetric ring. It is easy to show that S is J-symmetric. But it is not symmetric as for $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $ABI_3 = 0$ and $BAI_3 \neq 0$, where I_3 is the 3×3 identity matrix.

(2) Let K be field and $F = K\langle x, y, z \rangle$ a free algebra and let

$$I = (FxF)^2 + (FyF)^2 + (FzF)^2 + (FxyzF)^2 + (FyzxF)^2 + (FzxyF)^2 \subseteq F$$

be an ideal of F, and consider the ring $R = F/I$. In [7, Example 5], it is proved that R is not symmetric. It is easy to show that R is local and by Lemma 2.3, R is J-symmetric.

The following theorem is very useful to determine whether a ring is J-symmetric.

Theorem 2.5. *For a ring R, the following are equivalent.*

(1) *R is a J-symmetric ring.*
(2) *$abc \in I(R)$ implies $b(1-cab)ac \in J(R)$.*
(3) *$abc \in I(R)$ implies $ba(1-cab)c \in J(R)$.*
(4) *$abc \in I(R)$ implies $ba(1-bca)c \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $abc \in I(R)$. Then, $ab(1-cab)c = 0$. Since R is J-symmetric, $b(1-cab)ac \in J(R)$.

(2) $\Rightarrow$ (1) Firstly, we show that $ab = 0$ implies $baR \subseteq J(R)$. Let $ab = 0$, then $abr \in I(R)$ for all $r \in R$. By hypothesis, $b(1-rab)ar = bar - brabar = bar \in J(R)$, as asserted. To see R is J-symmetric, let $abc = 0$. Then, $bac - bcabac \in J(R)$. Hence, $bac \in J(R)$, since $bcabac \in J(R)$. Thus, R is J-symmetric.

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) It can be proved similar to the proof of (1) $\Leftrightarrow$ (2). $\square$

Note that the direct product of J-symmetric rings is again J-symmetric by Proposition 2.13 to follow. But the homomorphic image of a J-symmetric ring need not be J-symmetric. Consider the following example.

Example 2.6. Let D be a division ring. $R = D[x, y]$ and $I = \langle xy \rangle$ where $xy \neq yx$. R is J-symmetric since R is a domain. On the other hand, $(x+I)(y+I)(1+I) = 0$, but $(y+I)(x+I)(1+I) \notin J(R/I)$. Hence, R/I is not J-symmetric.

Lemma 2.7. *Let R be a ring and I a nil ideal of R. If R/I is a J-symmetric ring, then so is R.*

Proof. Let $a, b, c \in R$ and $abc = 0$. Then, $abc + I = 0 + I$. Hence, $bac + I \in J(R/I)$. Thus, for any $x \in R$, $1-bacx+I \in U(R/I)$. So there exists $y+I \in R/I$ such that $(1-bacx)y + I = 1 + I$. Then, $1-(1-bacx)y \in I$. Since I is nil, $(1-bacx)y \in U(R)$. Therefore, $1-bacx$ is right invertible. Similarly, it can be shown that $1-bacx$ is left invertible. So we have $1-bacx \in U(R)$ which completes the proof. $\square$

Theorem 2.8. *Let I be an ideal of R where R is a J-symmetric ring and S a subring of R with $I \subseteq S$. If S/I is J-symmetric, then S is J-symmetric.*

Proof. Let $abc = 0$ for $a, b, c \in S$. This implies that $bac \in J(R)$. Then, for every $x \in R$, $1 - bacx \in U(R)$. Hence, there exists $y \in R$ such that $y(1-bacx) = 1$. By hypothesis $\overline{bac} \in J(S/I)$. So $\overline{1} - \overline{bacx} \in U(S/I)$. Thus, there exists $\overline{s} \in S/I$ such that $(\overline{1} - \overline{bacx})\overline{s} = \overline{1}$. Therefore, $1-(1-bacx)s \in I$. So $y(1-(1-bacx)s) = y - s \in I$. It is clear that $y \in S$. Hence, $1-bacx$ is left invertible in S. Similarly, it can be shown that $1-bacx$ is right invertible in S. Hence, we have $1-bacx \in U(S)$ and so $bac \in J(S)$, as asserted. $\square$

Corollary 2.9. *Let R be a J-symmetric ring and I an ideal of R. If S is a J-symmetric subring of R, then $I + S$ is J-symmetric.*

Proof. We have $I \subseteq I + S \subseteq R$. Also, it is clear that $I + S/I$ is J-symmetric. Hence, $I + S$ is J-symmetric by Theorem 2.8. $\square$

Proposition 2.10. *Every subdirect product of J-symmetric rings is J-symmetric.*

Proof. Let R be a ring with R/I and R/J are J-symmetric where I, J are ideals of R and $I \cap J = 0$. To show that R is J-symmetric consider the map $f : R \to R/I \oplus R/J$ which is defined by $f(r) = (r + I, r + J)$. Then, $R \cong \mathrm{Im}(f)$ since $I \cap J = 0$. Hence, $R/I \oplus R/J$ and $\mathrm{Im}(f)/f(I) \cong R/I$ are J-symmetric by hypothesis. Since $f(I) \subseteq \mathrm{Im}(f) \subseteq R/I \oplus R/J$, R is J-symmetric by Theorem 2.8. $\square$

Lemma 2.11. *Let I and J be an ideals of a ring R. For J-symmetric rings R/I and R/J, $R/(IJ)$ is J-symmetric.*

Proof. Let $f : R/(I \cap J) \to R/I$ and $g : R/(I \cap J) \to R/J$ which are defined by $f(r + (I \cap J)) = r + I$ and $g(r + (I \cap J)) = r + J$. Then, $\mathrm{Ker} f \cap \mathrm{Ker} g = 0$, f and g are epimorphisms. Hence, $R/(I \cap J)$ is the subdirect product of R/I and R/J. Thus, $R/(I \cap J)$ is J-symmetric by Proposition 2.10. It is easy to check that $IJ \subseteq I \cap J$. Also, we have $R/(I \cap J) \cong (R/(IJ))/((I \cap J)/(IJ))$ and $((I \cap J)/(IJ))^2 = 0$. Consequently, $R/(IJ)$ is J-symmetric by Lemma 2.7. $\square$

Theorem 2.12. *The following are equivalent for a ring R.*

(1) *R is J-symmetric.*
(2) *$S = \{(x, y) \in R \times R : x - y \in J(R)\}$ is J-symmetric.*

Proof. (1) $\Rightarrow$ (2) It is clear that S is a subring of R. Consider the ideals $I = 0 \times J(R)$ and $J = J(R) \times 0$ of S. Then, $I \cap J = 0$ and $S/I \cong R \cong S/J$. Hence, S is a subdirect product of R and so the proof is completed by Proposition 2.10.

(2) $\Rightarrow$ (1) Let $a, b, c \in R$ with $abc = 0$. Then, $(a, a)(b, b)(c, c) = (0, 0)$. Hence, $(b, b)(a, a)(c, c) = (bac, bac) \in J(S)$, by hypothesis. Thus, for any $x \in R$, $(1, 1) - (bac, bac)(x, x) \in U(S)$. Therefore, $(1 - bacx, 1 - bacx) \in U(S)$. Consequently, $1 - bacx \in U(R)$. This implies that $bac \in J(R)$, as asserted. $\square$

Proposition 2.13. *Let $\{R_i\}_{i\in I}$ be a class of rings for an index set I. Then, $\prod_{i\in I} R_i$ is J-symmetric if and only if for each $i \in I$, R_i is J-symmetric.*

Proof. Let R_i be J-symmetric for all $i \in I$ and $(a_i)_{i\in I}, (b_i)_{i\in I}, (c_i)_{i\in I} \in \prod_{i\in I} R_i$ with $(a_i)(b_i)(c_i) = 0$. Then, $a_i b_i c_i = 0$ and by hypothesis $b_i a_i c_i \in J(R_i)$ for all $i \in I$. Hence, $(b_i)(a_i)(c_i) \in \prod_{i\in I} J(R_i) = J(\prod_{i\in I} R_i)$. Therefore, $\prod_{i\in I} R_i$ is J-symmetric. The sufficiency is clear. $\square$

The following result is a direct consequence of Proposition 2.13.

Corollary 2.14. *Let R be a ring. Then, eR and $(1 - e)R$ are J-symmetric for some central idempotent element e of R if and only if R is J-symmetric.*

Lemma 2.15. *A ring R is J-symmetric if and only if so is eRe for all idempotent $e \in R$.*

Proof. Let $(eae)(ebe)(ece) = 0$ for $a, b, c \in R$ and $e^2 = e \in R$. Since R is J-symmetric, $(ebe)(eae)(ece) \in J(R)$ and so $(ebe)(eae)(ece) \in eJ(R)e = J(eRe)$. Therefore, eRe is J-symmetric. The converse is trivial. $\square$

We will see in Example 3.7 that J-symmetric rings need not be abelian. The following example shows that the ring R being J-symmetric does not imply $R/J(R)$ is abelian.

Example 2.16. Let $R = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z}_{(3)}\}$ be the ring of quaternions over $\mathbb{Z}_{(3)}$ the localization of $\mathbb{Z}$ at $3\mathbb{Z}$. Then, R is J-symmetric. Consider the ring $R/J(R) = R/3R$. Then, $2 + 2i + j + J(R)$ is an idempotent in $R/J(R)$. But $2 + 2i + j + J(R)$ is not central in $R/J(R)$. So, $R/J(R)$ is not abelian.

In Proposition 2.17, we show that if R is J-symmetric and idempotents lift modulo $J(R)$, then $R/J(R)$ is abelian.

Proposition 2.17. *Let R be a ring whose idempotents lift modulo $J(R)$. If R is J-symmetric, then $R/J(R)$ is abelian.*

Proof. Let $\overline{e} = \overline{e}^2 \in R/J(R)$. Then, $e^2 = e \in R$ by hypothesis. Then, for every $x \in R$, $xe(1-e) = 0 = x(1-e)e$. Since R is J-symmetric, for every $x \in R$, $ex(1-e)$, $(1-e)xe \in J(R)$. Hence, for every $x \in R$, $ex - xe \in J(R)$, and so $R/J(R)$ is abelian. $\square$

In [8], a ring R is called *clean* if every element of R is the sum of a unit and an idempotent. Clean rings are exchange, but the converse is not true in general. It is well known that abelian exchange rings are clean. Hence, we have the following theorem.

Theorem 2.18. *Let R be a J-symmetric ring. Then, R is clean if and only if R is exchange.*

Proof. Clean rings are always exchange. For the converse let R be an exchange ring. Then, idempotents lift modulo $J(R)$. By Proposition 2.17, $R/J(R)$ is abelian. Then, $R/J(R)$ is a clean ring by [8]. Hence, R is clean by [1, Proposition 7]. $\square$

In [10] a ring R is said to *has stable range* 1 if for any a, $b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. It is clear that R has stable range 1 if and only if $R/J(R)$ has stable range 1. It is known from [13] that exchange rings in which every idempotent is central have stable range 1. So, we have the following.

Theorem 2.19. *J-symmetric exchange rings have stable range* 1.

Proof. Let R be a J-symmetric exchange ring. Then, $R/J(R)$ is exchange and idempotents lift modulo $J(R)$, since R is exchange. Hence, $R/J(R)$ is abelian by Proposition 2.17. Thus, $R/J(R)$ has the stable range 1 by [13, Theorem 6]. Therefore, R has the stable range 1. □

Similar to the definition of strongly J-clean rings [2], one can define J-clean rings. A ring R is called *J-clean*, for every $x \in R$, there exist an idempotent $e \in R$ and $j \in J(R)$ such that $x = e+j$. In this direction we have the following.

Proposition 2.20. *Let R be an abelian ring. Then, we have the following.*

(1) *If R is J-clean, then it is J-symmetric.*
(2) *If R is J-quasipolar, then it is J-symmetric.*

Proof. (1) Let a, b, $c \in R$ with $abc = 0$. By J-cleanness, there are $e^2 = e$ $f^2 = f$, $g^2 = g$ and r, s, $t \in J(R)$ such that $a = e + r$, $b = f + s$ and $c = g + t$. Then, $abc = 0$ implies $efg + x = 0$ where $x \in J(R)$. Since R is abelian, efg is an idempotent. Hence, $efg = 0$. So $bac = feg + y$ where $y \in J(R)$. Since $feg = 0$, $bac \in J(R)$.

(2) It is similar to the proof of (1). □

Recall that an element r of a ring R is called *left minimal* if Rr is minimal left ideal of R, an idempotent $e \in R$ is called *left minimal idempotent* if e is a left minimal element. Also, $ME_l(R)$ denotes the set of all left minimal idempotents of R. A ring R is left minimal abelian if every element of $ME_l(R)$ is left semicentral in R (see [11]).

Theorem 2.21. *Every J-symmetric ring is left minimal abelian ring.*

Proof. Let $e \in ME_l(R)$, $a \in R$ and $h = ae - eae$. If $h \neq 0$, then $eh1_R = 0$, $he = h$ and $Rh = Re$. Thus, $he = h \in J(R)$ since R is J-symmetric. There exists $r \in R$ such that $e = rh$ since $Rh = Re$. Hence, $e \in J(R)$ and so $e = 0$ which is a contradiction. So we have $h = 0$. Therefore, $ae = eae$ as asserted. □

In [12], a ring R is defined to be *quasi-normal* if $ae = 0$ implies $eaRe = 0$ for nilpotent a and idempotent e in R. It is proved that R is quasi-normal if and only if $eR(1-e)Re = 0$ for each idempotent e and R is said to be *weakly quasi-normal* if $eR(1-e)Re \subseteq J(R)$ for each $e^2 = e \in R$.

Proposition 2.22. *Every J-symmetric ring is weakly quasi-normal.*

Proof. Let $e^2 = e \in R$. Then, $r(1-e)e = 0$ and hypothesis imply $(1-e)re \in J(R)$ for every $r \in R$. Hence, $(1-e)Re \subseteq J(R)$. Since $J(R)$ is an ideal, $eR(1-e)Re \subseteq J(R)$. □

Recall that a ring R is called *directly finite* whenever $a, b \in R$, $ab = 1$ implies $ba = 1$. Then, we have the following.

Proposition 2.23. *Every J-symmetric ring is directly finite.*

Proof. Let R be a J-symmetric ring and assume that $ab = 1$ for $a, b \in R$. Then, $a(1 - ba) = 0$. Hence, $ba(1 - ba) = 0$. Since R is J-symmetric, $ab(1 - ba) = 1 - ba \in J(R)$. Therefore, $1 - ba = 0$ and so $ba = 1$. $\square$

It is well known that for any positive integer n, the $n \times n$ full matrix rings $M_n(\mathbb{R})$ over a real number field $\mathbb{R}$ are directly finite. But, by Remark 3.11, we know that $M_n(\mathbb{R})$ are not J-symmetric for $n \geq 2$. Hence, the converse of Proposition 2.23 is not true in general.

3. Extensions of J-symmetric rings

In [9], Rege and Chhawchharia introduced the notion of an Armendariz ring. A ring R is called *Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for all i and j. The name of the ring was given due to Armendariz who proved that reduced rings (i.e., rings without nonzero nilpotent elements) satisfied this condition. The symmetric ring property does not go up to polynomial rings by [4, Example 3.1]. We have a similar situation for J-symmetric rings.

Proposition 3.1. *Let R be a ring. If $R[x]$ is J-symmetric, then R is J-symmetric. The converse holds if R is Armendariz.*

Proof. Assume that $R[x]$ is J-symmetric. Let $a, b, c \in R$ with $abc = 0$. Since $R[x]$ is J-symmetric, $bac \in J(R[x])$. Then, $1 - (bac)r$ is invertible in $R[x]$ for all $r \in R$ and so $bac \in J(R)$. Therefore, R is J-symmetric. Conversely, suppose that R is Armendariz. Let $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j, h(x) = \sum_{k=0}^{t} c_k x^k \in R[x]$ with $f(x)g(x)h(x) = 0$. By hypothesis, we have $a_i b_j c_k = 0$ for all i, j and k. Since R is J-symmetric, $b_j a_i c_k \in J(R)$ for all i, j and k. By Amitsur Theorem, $J(R[x]) = (J(R[x]) \cap R)[x]$ implies $J(R)[x] \subseteq J(R[x])$ and so $g(x)f(x)h(x) \in J(R[x])$. This completes the proof. $\square$

Proposition 3.2. *Let R be a ring. Then, the ring of formal power series $R[[x]]$ is J-symmetric if and only if R is J-symmetric.*

Proof. It can be easily seen by the fact that $J(R[[x]]) = J(R) + \langle x \rangle$. $\square$

Let S and T be any rings, M an S-T-bimodule and R the formal triangular matrix ring $\left[\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right]$. It is well known that $J(R) = \left[\begin{smallmatrix} J(S) & M \\ 0 & J(T) \end{smallmatrix}\right]$.

Proposition 3.3. *Let $R = \left[\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right]$. Then, R is J-symmetric if and only if S and T are J-symmetric.*

Proof. The necessity is obvious by Lemma 2.15. For the other inclusion, assume that S and T are J-symmetric and

$$\begin{bmatrix} s_1 & m_1 \\ 0 & t_1 \end{bmatrix} \begin{bmatrix} s_2 & m_2 \\ 0 & t_2 \end{bmatrix} \begin{bmatrix} s_3 & m_3 \\ 0 & t_3 \end{bmatrix} = 0.$$

Then, $\left[\begin{smallmatrix} s_1s_2s_3 & * \\ 0 & t_1t_2t_3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Since S and T are J-symmetric, $s_2s_1s_3 \in J(S)$ and $t_2t_1t_3 \in J(T)$. Therefore,

$$\begin{bmatrix} s_2 & m_2 \\ 0 & t_2 \end{bmatrix} \begin{bmatrix} s_1 & m_1 \\ 0 & t_1 \end{bmatrix} \begin{bmatrix} s_3 & m_3 \\ 0 & t_3 \end{bmatrix} \in J(R)$$

and so R is J-symmetric. $\square$

The following result directly follows from Theorem 3.3.

Corollary 3.4. *Let R be a ring. Then, R is J-symmetric if and only if $T_n(R)$ is J-symmetric for every positive integer n.*

Proposition 3.5. *The following are equivalent for a ring R.*

(1) *R is J-symmetric.*
(2) $S = \left\{ \begin{pmatrix} r & r_{12} & \cdots & r_{1n} \\ 0 & r & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix} : r, r_{ij} \in R (i < j) \right\}$ *is J-symmetric.*

Proof. (1) $\Rightarrow$ (2) Consider the ideal $I = \begin{pmatrix} 0 & w_{12} & \cdots & w_{1n} \\ 0 & 0 & \cdots & w_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ of S. Hence, $R \cong S/I$ is J-symmetric. Thus, S is J-symmetric, as $I^n = 0$ and by Lemma 2.7.

(2) $\Rightarrow$ (1) If S is J-symmetric, then obviously R is J-symmetric by Lemma 2.15. $\square$

Corollary 3.6. *Let R be a ring. Then, the following are equivalent.*

(1) *R is J-symmetric.*
(2) *$R[x]/(x^n)$ is J-symmetric for all $n \geq 2$.*

Proof. Since

$$R[x]/(x^n) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \ddots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{bmatrix} \mid a_i \in R \right\},$$

it is clear by Proposition 3.5. $\square$

Example 3.7. Let $R = \mathbb{Z}_2$. Then, for every positive integer n, $T_n(R)$ is J-symmetric by Corollary 3.4. But $T_n(R)$ is not abelian.

Let A be an algebra over a commutative ring R. In [3], the *Dorroh extension* of R by A is the abelian group $R \oplus A$ with multiplication given by $(r_1, a_1)(r_2, a_2) = (r_1r_2, r_1a_2 + a_1r_2 + a_1a_2)$ for $r_i \in R$ and $a_i \in A$. We use $I(R; A) = R \oplus A$ to denote the Dorroh extension of R by A.

Proposition 3.8. *Suppose that for any $a \in A$ there exists $b \in A$ such that $a + b + ab = 0$. Then, the following are equivalent for a ring R.*

(1) *R is J-symmetric.*
(2) *$S = I(R; A)$ is J-symmetric.*

Proof. (1) $\Rightarrow$ (2) Let $(r, a),\ (s, b),\ (t, c) \in S$ with $(r, a)(s, b)(t, c) = (0, 0)$. Then, $(rst, d) = (0, 0)$ where $d \in A$. Since R is J-symmetric, $srt \in J(R)$. For any $x \in A$, $(srt, x) = (srt, 0) + (0, x)$. Since $(0, A) \subseteq J(S)$, it is enough to see $(srt, 0) \in J(S)$ to complete the proof. To see that let $(m, y) \in S$. Then, $(1, 0) - (srt, 0)(m, y) = (1, 0) - (srtm, srty) = (1 - srtm, -srty) \in U(S)$, since $(1 - srtm, -srty) = (1 - srtm, 0)(1, (1 - srtm)^{-1}(-srty))$ and $1 - srtm \in U(R)$, $(1, (1 - srtm)^{-1}(-srty)) = (1, 0) + (0, (1 - srtm)^{-1}(-srty) \in U(S)$ by $(0, A) \subseteq J(S)$. Hence, $(srt, 0) \in J(S)$.

(2) $\Rightarrow$ (1) Assume that S is J-symmetric and $a, b, c \in R$ with $abc = 0$. Then, $(a, 0)(b, 0)(c, 0) = (0, 0)$ and so $(b, 0)(a, 0)(c, 0) \in J(S)$ by hypothesis. Hence, for any $x \in R$, $(1, 0) - (bac, 0)(x, 0) = (1 - bacx, 0) \in U(S)$. Thus, $1 - bacx \in U(R)$ as asserted. □

Recall that $R[[x, \sigma]]$ denotes the ring of skew formal power series over a ring R where $\sigma : R \to R$ is a ring homomorphism. That is, $R[[x, \sigma]]$ is the set of all formal power series in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$ for every $r \in R$. It is clear that $R[[x, 1_R]] = R[[x]]$ is the formal power series ring over R. Also it is well-known that $J(R[[x, \sigma]]) = J(R) + \langle x \rangle$. The following is the direct consequence of Proposition 3.8 by the fact that $R[[x, \sigma]] \cong I(R; \langle x \rangle)$ where $\langle x \rangle$ is the ideal generated by x.

Corollary 3.9. *Let R be a ring and $\sigma : R \to R$ a ring homomorphism. Then, the following are equivalent.*

(1) *R is J-symmetric.*
(2) *$R[[x, \sigma]]$ is J-symmetric.*

Let A be a ring and B a subring of A and consider the set

$$R[A, B] = \{a_1, a_2, \ldots, a_n, b, b, \ldots : a_i \in A,\ b \in B,\ 1 \leqslant i \leqslant n\}.$$

Hence, $R[A, B]$ is a ring with componentwise addition and multiplication. Note that $J(R[A, B]) = R[J(A), J(A) \cap J(B)]$.

Proposition 3.10. *Let A be a ring and with a subring B. Then, the following are equivalent.*

(1) *A and B are J-symmetric.*
(2) *$R[A, B]$ is J-symmetric.*

Proof. (1) $\Rightarrow$ (2) Let $(a_{i1}, \ldots, a_{in}, b_i, b_i, \ldots) \in R[A, B]$ for every $1 \le i \le 3$,

$$(a_{11}, \ldots, a_{1n}, b_1, b_1, \ldots)(a_{21}, \ldots, a_{2n}, b_2, b_2, \ldots)(a_{31}, \ldots, a_{3n}, b_3, b_3, \ldots) = 0.$$

Then, $a_{1i}a_{2i}a_{3i} = 0 = b_1b_2b_3$ for every $1 \le i \le 3$. Since A and B are J-symmetric, for every $1 \le i \le 3$, $a_{2i}a_{1i}a_{3i}, b_2b_1b_3 \in J(A)$ and $b_2b_1b_3 \in J(B)$.

So,

$$\begin{aligned}&(a_{21},\dots,a_{2n},b_2,b_2,\dots)(a_{11},\dots,a_{1n},b_1,b_1,\dots)(a_{31},\dots,a_{3n},b_3,b_3,\dots)\\ &\in R[J(A),J(A)\cap J(B)]=J(R[A,B]),\end{aligned}$$

as desired.

(2) $\Rightarrow$ (1) Let $a_1,a_2,a_3\in A$ and $a_1a_2a_3=0$. Then,

$$(a_1,\dots,0,0,0,\dots)(a_2,\dots,0,0,0,\dots)(a_3,\dots,0,0,0,\dots)=0.$$

Since $R[A,B]$ is J-symmetric, $a_2a_1a_3\in J(A)$ and so A is J-symmetric. Similarly, it can be shown that B is a J-symmetric ring. □

Remark 3.11. Let R be a ring with identity and $A=\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$, $B=\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$, $C=\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\in M_2(R)$. Then, $ABC=0$ but $BAC\notin J(M_2(R))$ since $J(M_2(R))=M_2(J(R))$. Therefore, $M_n(R)$ is not J-symmetric.

Let R be a ring and $s\in R$ a central element. In the ring $K_s(R)$, matrices are multiplied according to the following relation:

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}e&f\\g&h\end{pmatrix}=\begin{pmatrix}ae+sbg&af+bh\\ce+dg&scf+dh\end{pmatrix}.$$

Note that $J(K_s(R))=\left(\begin{smallmatrix}J(R)&(s:J(R))\\(s:J(R))&J(R)\end{smallmatrix}\right)$ where $(s:J(R))=\{r\in R: rs\in J(R)\}$ (see [5]). By Lemma 2.15, if $K_s(R)$ is J-symmetric, then the ring R is J-symmetric. But as the following shows that the converse is not true, in general.

Example 3.12. Let $R=\mathbb{Z}_4$ and $s=3$. Then, R is J-symmetric. For $A=\left(\begin{smallmatrix}\bar1&\bar1\\\bar3&\bar1\end{smallmatrix}\right)$, $B=\left(\begin{smallmatrix}\bar1&\bar0\\\bar1&\bar0\end{smallmatrix}\right)$ and $C=\left(\begin{smallmatrix}\bar1&\bar0\\\bar0&\bar1\end{smallmatrix}\right)$, $ABC=\left(\begin{smallmatrix}\bar0&\bar0\\\bar0&\bar0\end{smallmatrix}\right)$ but $BAC=\left(\begin{smallmatrix}\bar1&\bar1\\\bar1&\bar3\end{smallmatrix}\right)\notin J(K_s(R))$. Hence, $K_s(R)$ is not J-symmetric.

Proposition 3.13. *For a ring R, the following are equivalent*:

(1) *R is J-symmetric.*
(2) *$K_0(R)$ is J-symmetric.*

Proof. (1) $\Rightarrow$ (2) Let R be a J-symmetric ring. Assume that

$$\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}\begin{pmatrix}b_1&b_2\\b_3&b_4\end{pmatrix}\begin{pmatrix}c_1&c_2\\c_3&c_4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

Then

$$\begin{pmatrix}a_1b_1c_1&*\\*&a_4b_4c_4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

Hence, $a_1b_1c_1=0$ and $a_4b_4c_4=0$. Since R is J-symmetric, $b_1a_1c_1\in J(R)$ and $b_4a_4c_4\in J(R)$. Therefore,

$$\begin{pmatrix}b_1&b_2\\b_3&b_4\end{pmatrix}\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}\begin{pmatrix}c_1&c_2\\c_3&c_4\end{pmatrix}=\begin{pmatrix}b_1a_1c_1&*\\*&b_4a_4c_4\end{pmatrix}\in J(K_0(R))$$

as asserted.

(2) ⇒ (1) Assume that $K_0(R)$ is J-symmetric and $abc = 0$. Then

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and so

$$\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \in J(K_0(R))$$

which implies that $bac \in J(R)$. Hence, R is J-symmetric. □

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