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STRUCTURE OF 3-PRIME NEAR-RINGS SATISFYING SOME IDENTITIES

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Abstract. In this paper, we investigate commutativity of 3-prime near-rings $\mathcal{N}$ in which $(1,\alpha)$-derivations satisfy certain algebraic identities. Some well-known results characterizing commutativity of 3-prime near-rings have been generalized. Furthermore, we give some examples show that the restriction imposed on the hypothesis is not superfluous.

1. Introduction

In the present paper, $\mathcal{N}$ will denote a left near-ring with center $Z(\mathcal{N})$. A near-ring $\mathcal{N}$ is called zero-symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that left distributivity yields $x0 = 0$). $\mathcal{N}$ is 3-prime, that is, for $a, b \in \mathcal{N}$, $a\mathcal{N}b = \{0\}$ implies $a = 0$ or $b = 0$. A non empty subset U of $\mathcal{N}$ is said to be a semigroup left (resp. right) ideal of $\mathcal{N}$ if $\mathcal{N}U \subseteq U$ (resp. $U\mathcal{N} \subseteq U$) and if U is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$. As usual for all x, y in $\mathcal{N}$, the symbol $[x, y]$ stands for Lie product (commutator) $xy - yx$ and $x \circ y$ stands for Jordan product (anticommutator) $xy+yx$. We note that for a left near-ring, $-(x+y) = -y-x$ and $-xy = x(-y)$. For terminologies concerning near-rings we refer to G. Pilz [9].

An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [11], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called a semiderivation if there exists a map $g : \mathcal{N} \to \mathcal{N}$ such that $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$ hold for all $x, y \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called a two sided α-derivation if there exists a map $\alpha : \mathcal{N} \to \mathcal{N}$ such that $d(xy) = d(x)y + \alpha(x)d(y)$ and $d(xy) = d(x)\alpha(y) + xd(y)$ hold for all $x, y \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called $(1, \alpha)$-derivation if there exists a map $\alpha : \mathcal{N} \to \mathcal{N}$ such that $d(xy) = d(x)y + \alpha(x)d(y)$ holds for all $x, y \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called $(\alpha, 1)$-derivation if there exists a map $\alpha : \mathcal{N} \to \mathcal{N}$ such that $d(xy) = d(x)\alpha(y) + xd(y)$ holds for all

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$x, y \in \mathcal{N}$. Obviously, a two sided α-derivation is both a $(1, \alpha)$-derivation as well as an $(\alpha, 1)$-derivation. Also, any derivation on $\mathcal{N}$ is a $(1, \alpha)$-derivation, but the converse is not true in general (see [6]). The study of commutativity of 3-prime near-rings by using derivations was initiated by H. E. Bell and G. Mason in 1987 (see [5]). In [8] A. A. M. Kamal generalizes some results of Bell and Mason by studying the commutativity of 3-prime near-rings using σ-derivations instead of the usual derivations, where σ is an automorphism on the near-ring. M. Ashraf, A. Ali and Shakir Ali in [1] and N. Aydin and Ö. Gölbasi in [7] generalize Kamal's work by using a (s, t)-derivation instead of a s-derivation, where s and t are automorphisms. Recently many authors (see [2], [4], [5]) for reference where further references can be found) have studied commutativity of 3-prime near-rings satisfying certain identities involving derivations, semiderivations, two sided α-derivations. Now our aim in this paper is to study the commutativity behavior of 3-prime near-ring which admits $(1, \alpha)$-derivations satisfying certain properties. In fact, our results generalize, extend and complement several results obtained earlier in [2], [6], [10] on derivations, semiderivations and two sided α-derivations for 3-prime near-rings.

2. Some preliminaries

In this section, we give some well-known results and we add some new lemmas which will be used throughout the next sections of the paper.

Lemma 2.1 ([4, Theorem 2.9]). *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{U}$ is a nonzero semigroup ideal of $\mathcal{N}$, then the following assertions are equivalent*:

(i) $[x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{U}$;

(ii) $\mathcal{N}$ *is a commutative ring.*

Lemma 2.2 ([4, Theorem 2.10]). *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{U}$ be a nonzero semigroup ideal. If $u \circ v \in Z(\mathcal{N})$ for all $u, v \in \mathcal{U}$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.3. *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{U}$ be a nonzero semigroup ideal of $\mathcal{N}$.*

(i) [3, Lemma 1.5] *If $\mathcal{U} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

(ii) [3, Lemma 1.4(i)] *If $x, y \in \mathcal{N}$ and $x\mathcal{U}y = \{0\}$, then $x = 0$ or $y = 0$.*

(iii) [3, Lemma 1.3(i)] *If x is an element of $\mathcal{N}$ such that $\mathcal{U}x = \{0\}$ (resp. $x\mathcal{U} = \{0\}$), then $x = 0$.*

(iv) *If z centralizes a nonzero semigroup ideal $\mathcal{U}$, then $z \in Z(\mathcal{N})$.*

Lemma 2.4. *Let $\mathcal{N}$ be a near-ring and d an additive map.*

(i) *If d is a $(1, \alpha)$-derivation associated with a map α, then $\mathcal{N}$ satisfies the following property*:

$$\Big(d(x)y + \alpha(x)d(y)\Big)z = d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - \alpha(xy)d(z)$$

for all $x, y, z \in \mathcal{N}$.

(ii) *If d is an $(\alpha,1)$-derivation, then $\mathcal{N}$ satisfies the following relation*:

$$\Big(d(x)\alpha(y)+xd(y)\Big)\alpha(t)=d(x)\alpha(yt)+xd(y)\alpha(t) \text{ for all } t,x,y\in\mathcal{N}.$$

Proof. (i) We have

$$\begin{aligned} d((xy)z)&=d(xy)z+\alpha(xy)d(z)\\ &=\Big(d(x)y+\alpha(x)d(y)\Big)z+\alpha(xy)d(z) \text{ for all } x,y,z\in\mathcal{N}. \end{aligned}$$

Also

$$\begin{aligned} d(x(yz))&=d(x)yz+\alpha(x)d(yz)\\ &=d(x)yz+\alpha(x)d(y)z+\alpha(x)\alpha(y)d(z) \text{ for all } x,y,z\in\mathcal{N}. \end{aligned}$$

Combining the above two equalities, we find that

$$\Big(d(x)y+\alpha(x)d(y)\Big)z+\alpha(xy)d(z)=d(x)yz+\alpha(x)d(y)z+\alpha(x)\alpha(y)d(z)$$

for all $x,y,z\in\mathcal{N}$, which gives the required result.

(ii) Using the same proof of [10, Lemma 2.1], we find the required result. □

Lemma 2.5. *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{U}$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero $(1,\alpha)$-derivation d associated with a map α, the following properties are satisfied*:

(i) *If $d(\mathcal{U})=\{0\}$, then $d=0$.*
(ii) *If $ad(\mathcal{U})=\{0\}$, $a\in\mathcal{N}$ and $\alpha(\mathcal{U})=\mathcal{U}$, then $a=0$.*
(iii) *If $d(\mathcal{U})a=\{0\}$, $a\in\mathcal{N}$ and $\alpha(xy)=\alpha(x)\alpha(y)$ for all $x,y\in\mathcal{U}$, then $a=0$.*

Proof. (i) Suppose that $d(\mathcal{U})=\{0\}$. Then

$$\begin{aligned} 0&=d(mnu)\\ &=d(m)nu+\alpha(m)d(nu)\\ &=d(m)nu \quad \text{for all } u\in\mathcal{U},\ m\in\mathcal{N}, \end{aligned}$$

which implies that $d(m)\mathcal{N}u=\{0\}$ for all $u\in\mathcal{U}$, $m\in\mathcal{N}$. But $\mathcal{N}$ is 3-prime and $\mathcal{U}\neq\{0\}$, then $d=0$.

(ii) If $ad(\mathcal{U})=\{0\}$ and $a\in\mathcal{N}$, then $ad(xy)=0$ for all $x,y\in\mathcal{U}$. This implies that $ad(x)y+a\alpha(x)d(y)=0$ for all $x,y\in\mathcal{U}$, and hence $a\alpha(x)d(y)=0$ for all $x,y\in\mathcal{U}$. But $\alpha(\mathcal{U})=\mathcal{U}$, then $a\mathcal{U}d(y)=\{0\}$ for all $y\in\mathcal{U}$. Using (i) and Lemma 2.3(ii), we obtain $a=0$.

(iii) If $d(\mathcal{U})a=\{0\}$, then $d(xy)a=0$ for all $x,y\in\mathcal{U}$. By Lemma 2.4, we get $d(x)ya+\alpha(x)d(y)a+\alpha(x)\alpha(y)d(a)-\alpha(xy)d(a)=0$ for all $x,y\in\mathcal{U}$. Using the given hypothesis, we find that $d(x)ya=0$ for all $x,y\in\mathcal{U}$, i.e., $d(x)\mathcal{U}a=\{0\}$ for all $x\in\mathcal{U}$. Since $d\neq 0$, we arrive at $a=0$. □

3. Main results

In [5], H. E. Bell and G. Mason proved that a 3-prime near-ring $\mathcal{N}$ must be commutative if it admits a derivation d such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$. This result was generalized by the authors in [6], [10]. They replaced the derivation by a semiderivation or a two sided α-derivation where α is an homomorphism. Our objective in the following theorem is to generalize this result by treating the cases of $(1, \alpha)$-derivations, $(\alpha, 1)$-derivations and two sided α-derivations where α is an additive map.

Theorem 3.1. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero map d such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring if d has one of the following properties*:

(i) *d is a $(1, \alpha)$-derivation associated with an additive map α.*
(ii) *d is a $(\alpha, 1)$-derivation associated with an additive map α.*
(iii) *d is a two sided α-derivation associated with an additive map α.*

Proof. (i) Using our assumptions, we have $zd(xy) = d(xy)z$ and $d(z)d(xy) = d(xy)d(z)$ for all $x, y, z \in \mathcal{N}$. By Lemma 2.4, we obtain

$$zd(x)y + z\alpha(x)d(y) = d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - \alpha(xy)d(z) \tag{3.1}$$

and

$$\begin{aligned} &d(z)d(x)y + d(z)\alpha(x)d(y) \\ &= d(x)yd(z) + \alpha(x)d(y)d(z) + \alpha(x)\alpha(y)d^2(z) - \alpha(xy)d^2(z). \end{aligned} \tag{3.2}$$

Since $d^2(z) = d(d(z)) \in Z(\mathcal{N})$, (3.2) becomes

$$d^2(z)\mathcal{N}\big(\alpha(xy) - \alpha(x)\alpha(y)\big) = \{0\} \quad \text{for all } x, y, z \in \mathcal{N}. \tag{3.3}$$

Since $\mathcal{N}$ is 3-prime, we have

$$d^2(z) = 0 \text{ or } \alpha(xy) = \alpha(x)\alpha(y) \quad \text{for all } x, y, z \in \mathcal{N}. \tag{3.4}$$

Assume that $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in \mathcal{N}$. For $z = y$, (3.1) implies $d(y)\mathcal{N}[\alpha(x), y] = \{0\}$ for all $x, y \in \mathcal{N}$. Using $\mathcal{N}$ is 3-prime, we obtain $d(y) = 0$ or $y\alpha(x) = \alpha(x)y$ for all $x, y \in \mathcal{N}$. The last two cases give the following equation

$$d(x)\mathcal{N}[y, z] = \{0\} \quad \text{for all } x, y, z \in \mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, for each $y \in \mathcal{N}$, either y centralizes $\mathcal{N}$ or $d(\mathcal{N}) = \{0\}$; and by Lemma 2.3(iv) together with Lemma 2.5(i), we conclude that $\mathcal{N}$ is a commutative ring.

Suppose that $d^2(z) = 0$ for all $z \in \mathcal{N}$. We have

$$\begin{aligned} 0 &= d^2(xy) \\ &= d\big(d(x)y + \alpha(x)d(y)\big) \\ &= \alpha\big(d(x)) + d(\alpha(x)\big)d(y) \quad \text{for all } x, y \in \mathcal{N}, \end{aligned}$$

this implies that $\big(\alpha(d(x)) + d(\alpha(x))\big)d(y) = 0$ for all $x, y \in \mathcal{N}$. Replacing y by yt in the preceding equation and using it again, we arrive at

$$d(y)\mathcal{N}\big(\alpha(d(x)) + d(\alpha(x))\big) = \{0\} \quad \text{for all } x, y \in \mathcal{N}.$$

By 3-primeness of $\mathcal{N}$ and $d \neq 0$, we obtain

$$\alpha(d(x)) + d(\alpha(x)) = 0 \quad \text{for all } x \in \mathcal{N}. \tag{3.5}$$

Using our hypothesis, we have $d(xd(y)) = d(d(y)x)$ for all $x, y \in \mathcal{N}$. By the definition of d, we get

$$d(x)d(y) + \alpha(x)d^2(y) = d^2(y)x + \alpha(d(y))d(x) \quad \text{for all } x, y \in \mathcal{N},$$

which can be rewritten as

$$(\alpha(d(y)) - d(y))d(x) = 0 \quad \text{for all } x, y \in \mathcal{N}.$$

From the above, one can easily see that

$$\alpha(d(y)) = d(y) \quad \text{for all } y \in \mathcal{N}. \tag{3.6}$$

Using (3.5) and (3.6), we obtain

$$d(x) + d(\alpha(x)) = 0 \quad \text{for all } x \in \mathcal{N}. \tag{3.7}$$

Taking $d(u)$ instead of y in (3.1), we find that

$$\Bigg(\alpha(xd(u)) - \alpha(x)d(u)\Bigg)d(z) = 0 \quad \text{for all } x, u, z \in \mathcal{N}.$$

Which implies that

$$\alpha(xd(u)) = \alpha(x)d(u) \quad \text{for all } x, u \in \mathcal{N}. \tag{3.8}$$

Using (3.6) with the definition of d, we get

$$\alpha\big(d(x)y + \alpha(x)d(y)\big) = d(x)y + \alpha(x)d(y) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.9}$$

Since α is an additive map, using the fact that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$ and (3.8), we arrive at

$$d(x)\alpha(y) + \alpha^2(x)d(y) = d(x)y + \alpha(x)d(y) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.10}$$

Setting $x = y$ in (3.10) and using our hypothesis, we obviously get

$$(\alpha^2(x) - x)\mathcal{N}d(x) = \{0\} \quad \text{for all } x \in \mathcal{N}. \tag{3.11}$$

Since $\mathcal{N}$ is 3-prime, by (3.11) we can easily find that

$$\alpha^2 = id_{\mathcal{N}} \text{ or } d(x) = 0 \quad \text{for all } x \in \mathcal{N}. \tag{3.12}$$

Suppose there exists an element $u \in \mathcal{N}$ such that $d(u) = 0$ and writing u instead of y in (3.10), we find that $d(x)\alpha(u) = d(x)u$ for all $x \in \mathcal{N}$ which forces that

$$d(\mathcal{N})\mathcal{N}(\alpha(u) - u) = \{0\}. \tag{3.13}$$

Since $d \neq 0$ and $\mathcal{N}$ is 3-prime, we conclude that $\alpha(u) = u$ so that $\alpha^2(u) = \alpha(u) = u$. In this case, (3.12) forces that $\alpha^2 = id_{\mathcal{N}}$. Using Lemma 2.4, we get

$$\alpha(z)d(x)\alpha(y)+\alpha(z)xd(y) = d(x)\alpha(yz)+xd(y)\alpha(z) \quad \text{for all } x, y, z \in \mathcal{N}. \tag{3.14}$$

For $z = \alpha(x)$, (3.14) becomes

$$d(x)\mathcal{N}\big(\alpha(y\alpha(x)) - x\alpha(y)\big) = \{0\} \quad \text{for all } x, y \in \mathcal{N}. \tag{3.15}$$

Since $\mathcal{N}$ is 3-prime, either $d(x) = 0$ or $\alpha(y\alpha(x)) = x\alpha(y)$ for all $x, y \in \mathcal{N}$.

Suppose there exists $x_0 \in \mathcal{N}$ such that $d(x_0) = 0$. By (3.14), we arrive at $d(y)\mathcal{N}[x_0, \alpha(z)] = \{0\}$ for all $y, z \in \mathcal{N}$. Replacing z by $\alpha(u)$ and using the 3-primeness of $\mathcal{N}$ and $d \neq 0$, we get $x_0u = ux_0$ for all $u \in \mathcal{N}$.

If there exists $x_0 \in \mathcal{N}$ such that $\alpha(y\alpha(x_0)) = x_0\alpha(y)$ for all $y \in \mathcal{N}$. Replace z by $\alpha(x_0)$ in (3.14), we get $d(y)\mathcal{N}(x_0x - xx_0) = \{0\}$ for all $x, y \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime and $d \neq 0$, we obtain $x_0x = xx_0$ for all $x \in \mathcal{N}$. In both cases, we conclude that x centralizes $\mathcal{N}$ which forces that $\mathcal{N}$ is a commutative ring by Lemma 2.3(iv).

(ii) Assume that $d(xy) = d(x)\alpha(y) + xd(y)$ for all $x, y \in \mathcal{N}$. By hypothesis, we have $d(xy) = \alpha(y)d(x) + d(y)x$ for all $x, y \in \mathcal{N}$. Calculating $d(x(y + y))$ in two different ways, we obtain

$$\begin{aligned} d(x)\alpha(y) + xd(y) &= xd(y) + d(x)\alpha(y) \\ &= d(y)x + \alpha(y)d(x) \quad \text{for all } x, y \in \mathcal{N}. \end{aligned}$$

From the last expression, we remark that d plays a role of a $(1, \alpha)$-derivation, in this case, using the same proof of (i), we find that $\mathcal{N}$ is a commutative ring.

(iii) It is clear that if d is a two sided α-derivation, then d is both a $(1, \alpha)$-derivation and an $(\alpha, 1)$-derivation, which proves that $\mathcal{N}$ is a commutative ring by (i) and (ii). □

In the following, we study the commutativity of a near-ring $\mathcal{N}$ admitting nonzero two sided α-derivations ($(1, \alpha)$-derivations) d satisfying the condition $d(xy) = d(yx)$ ($d(xy) = -d(yx)$) for all $x, y \in \mathcal{N}$. These results have been demonstrated by several authors in cases the derivations, semiderivations and two sided α-derivations on 3-prime near-rings for more details see the following references [2], [3], [5], [6] and [10]. Our goal in the next part is to generalize these results in the case of $(1, \alpha)$-derivations and two sided α-derivations where α is an additive map instead of a homomorphism.

Theorem 3.2. *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{U}$ be a nonzero semigroup of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero map d such that $d([x, y]) = 0$ for all $x, y \in \mathcal{U}$, then $\mathcal{N}$ is a commutative ring if d has one of the following properties*:

(i) *d is a $(1, \alpha)$-derivation associated with a map α.*
(ii) *d is a two sided α-derivation associated with a map α.*

Proof. (i) By our assumptions, we have $d([x, y]) = 0$ for all $x, y \in \mathcal{U}$. Replacing y by xy, then

$$\begin{aligned} 0 &= d([x, xy]) \\ &= d(x)[x, y] + \alpha(x)d([x, y]) \\ &= d(x)[x, y] \quad \text{for all } x, y \in \mathcal{U} \end{aligned}$$

which implies that $d(x)xy = d(x)yx$ for all $x, y \in \mathcal{U}$. Taking yz instead of y where $z \in \mathcal{N}$, we obtain $d(x)\mathcal{U}[x, z] = \{0\}$ for all $x \in \mathcal{U}$, $z \in \mathcal{N}$. Invoking Lemma 2.3(ii), we get

$$d(x) = 0 \text{ or } x \in Z(\mathcal{N}) \quad \text{for all } x \in \mathcal{U}. \tag{3.16}$$

Suppose there is an element $x_0 \in \mathcal{U}$ such that $d(x_0) = 0$. Using the fact that $d(x_0y) = d(yx_0)$ for all $y \in \mathcal{U}$, we obtain $\alpha(x_0)d(y) = d(y)x_0$ for all $y \in \mathcal{U}$. Putting yt instead of y and using Lemma 2.4, we get

$$\begin{aligned} \alpha(x_0)d(y)t + \alpha(x_0)\alpha(y)d(t) &= d(yt)x_0 \\ &= d(y)tx_0 + \alpha(y)d(t)x_0 \quad \text{for all } y \in \mathcal{U}, t \in \mathcal{N} \end{aligned}$$

which can be rewritten as

$$d(y)x_0t + \alpha(x_0)\alpha(y)d(t) = d(y)tx_0 + \alpha(y)d(t)x_0 \quad \text{for all } y \in \mathcal{U}, t \in \mathcal{N}.$$

Taking $t = [u, v]$ in last equation, we obviously get $d(y)(x_0[u, v] - [u, v]x_0) = 0$ for all $y, u, v \in \mathcal{U}$. Calculating the expression $d\big(y\big(x_0[u, v] - [u, v]x_0\big)\big)$, one can easily find that $d\big(y\big(x_0[u, v] - [u, v]x_0\big)\big) = d(y)(x_0[u, v] - [u, v]x_0) = 0$ for all $y, u, v \in \mathcal{U}$. Substituting yt for y, where $t \in \mathcal{U}$ in the above equation, we arrive at $d(y)\mathcal{U}\big(x_0[u, v]-[u, v]x_0\big) = \{0\}$ for all $y, u, v \in \mathcal{U}$. Lemma 2.5(ii) and Lemma 2.3(ii) force that $x_0[u, v] = [u, v]x_0$ for all $u, v \in \mathcal{U}$, in this case, (3.16) forces that $x[u, v] = [u, v]x$ for all $x, u, v \in \mathcal{U}$. Replacing x by xt where $t \in \mathcal{N}$ in the preceding equation and using it again, we arrive at $\mathcal{U}(\big[[u, v], t\big]) = \{0\}$ for all $u, v \in \mathcal{U}$, $t \in \mathcal{N}$. By virtu of Lemma 2.3(iii), we obtain $[u, v] \in Z(\mathcal{N})$ for all $u, v \in \mathcal{U}$ which together with Lemma 2.1, yields that $\mathcal{N}$ is a commutative ring.

(ii) It is clear that if d is a two sided α-derivation, then d is a $(1, \alpha)$-derivation, which proves that $\mathcal{N}$ is a commutative ring by (i). □

As an application of Theorem 3.1, we obtain the following corollaries.

Corollary 3.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and d a nonzero derivation.*

(i) [5, Theorem 2] *If $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

(ii) [2, Theorem 4.1] *If $d[x, y] = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Corollary 3.2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and d a nonzero semi-derivation.*

(i) [6, Theorem 1] *If $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

(ii) [6, Theorem 2] *If $d([x, y]) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Corollary 3.3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and d a nonzero two sided α-derivation.*

(i) [10, Theorem 1] *If $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

(ii) [10, Theorem 2] *If $d([x,y]) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

The following example shows the necessity of the 3-primeness of $\mathcal{N}$ in the previous theorems.

Example 3.1. Let S be a 2-torsion free near-ring. Let us define $\mathcal{N}$ and $d, \alpha : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y \in S \right\},$$

$$d\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \alpha\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that $\mathcal{N}$ is not a 3-prime near-ring and d is a nonzero two sided α-derivation associated with an additive map α satisfying the following properties:

(i) $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, (ii) $d([A,B]) = 0$ for all $A, B \in \mathcal{N}$,

but, since the addition in $\mathcal{N}$ is not commutative, $\mathcal{N}$ cannot be a commutative ring.

The conclusion of Theorem 3.2 no remains valid if we replace the product $[x,y]$ by $x \circ y$, provided that $\mathcal{N}$ is 2-torsion free. In fact, we obtain the following result.

Theorem 3.3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{U}$ be a nonzero semigroup ideal of $\mathcal{N}$. Then there exists no nonzero map d on $\mathcal{N}$ such that $d(x \circ y) = 0$ for all $x, y \in \mathcal{U}$ in the following cases:*

(i) *d is a $(1,\alpha)$-derivation associated with a map α.*

(ii) *d is a two sided α-derivation associated with a map α.*

Proof. (i) Suppose that d is a $(1,\alpha)$-derivation associated with an additive map such that $d(x \circ y) = 0$ for all $x, y \in \mathcal{U}$. Replacing y by xy, then

$$\begin{aligned} 0 &= d(x \circ xy) \\ &= d(x)(x \circ y) + \alpha(x)d(x \circ y) \\ &= d(x)(x \circ y) \quad \text{for all } x, y \in \mathcal{U} \end{aligned}$$

which implies that $d(x)xy = -d(x)yx$ for all $x, y \in \mathcal{U}$. Taking yz instead of y where $z \in \mathcal{N}$, we obtain $d(x)\mathcal{U}\big(-z(-x) + (-x)z\big) = \{0\}$ for all $x \in \mathcal{U}$, $z \in \mathcal{N}$. Using Lemma 2.3(ii), we get

$$d(x) = 0 \text{ or } -x \in Z(\mathcal{N}) \quad \text{for all } x \in \mathcal{U}. \tag{3.17}$$

Suppose there exists an element $x_0 \in \mathcal{U}$ such that $-x_0 \in Z(\mathcal{N})$. We have

$$\begin{aligned} 0 &= -d(x_0 \circ x_0) \\ &= d(-x_0 \circ x_0) \\ &= d(2(-x_0)x_0). \end{aligned}$$

By the 2-torsion freeness of $\mathcal{N}$, we get $d((-x_0)x_0) = d(x_0(-x_0)) = 0$. On the other hand, we have

$$\begin{aligned} 0 &= d(x_0 \circ x_0(-x_0)) \\ &= d((-x_0)(x_0^2 + x_0^2)) \\ &= 2d((-x_0)x_0^2) \\ &= d((-x_0)x_0^2) \\ &= d(x_0x_0(-x_0)) \\ &= d(x_0)x_0(-x_0) + \alpha(x_0)d(x_0(-x_0)) \\ &= d(x_0)x_0(-x_0) \end{aligned}$$

which implies that $d(x_0)x_0\mathcal{N}(-x_0) = \{0\}$. In light of 3-primeness of $\mathcal{N}$, we conclude that $d(x_0)x_0 = 0 = d(x_0)(-x_0)$ and $d(x_0)\mathcal{N}(-x_0) = \{0\}$. By the 3-primeness of $\mathcal{N}$, we obtain $d(x_0) = 0$. In all cases (3.17) becomes $d(x) = 0$ for all $x \in \mathcal{U}$ which is a contradiction with Lemma 2.5(i).

(ii) It is clear that if d is a two sided α-derivation, then d is a $(1, \alpha)$-derivation, which proves that $\mathcal{N}$ is a commutative ring by (i). □

The following example shows the necessity of the 3-primeness of $\mathcal{N}$ in the previous theorems.

Example 3.2. Let S be a 2-torsion free near-ring. Let us define $\mathcal{N}$, d and $\alpha : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; x, y \in S \right\}$$

$$d\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \alpha\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that $\mathcal{N}$ is a non 3-prime near-ring and d is a nonzero two sided α-derivation such that $d(A \circ B) = 0$ for all $A, B \in \mathcal{N}$, but $\mathcal{N}$ is not a commutative ring because the addition is not commutative.

Example 3.3. Let $\mathcal{N} = M_2(\mathbb{Z}_3)$ the noncommutative prime ring and d the nonzero map on $\mathcal{N}$ such that $d\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} c & d-a \\ 0 & -c \end{smallmatrix}\right)$. Taking $x = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $d(x \circ y) = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 2 \end{smallmatrix}\right) \neq 0$, which shows that the condition "$d(x \circ y) = 0$ for all $x, y \in \mathcal{N}$" is necessary.

Example 3.4. Let $\mathcal{N} = \mathbb{Z}_2[x]$. Then $\mathcal{N}$ is an integral domain which means that $\mathcal{N}$ is a commutative prime ring. Also, we have $2\mathcal{N} = \{0\}$. If we take d to be the identical application on N and $\alpha = 0$. Then d is a nonzero $(1, \alpha)$-derivation and also is a nonzero two sided α-derivation on $\mathcal{N}$ and $d(p \circ q) = 2d(pq) = 0$ for all $p, q \in \mathcal{N}$. But $\mathcal{N}$ is not 2-torsion free.

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