Commun. Korean Math. Soc. **34** (2019), No. 1, pp. 1–16
https://doi.org/10.4134/CKMS.c170261
pISSN: 1225-1763 / eISSN: 2234-3024

SOME NEW IDENTITIES CONCERNING THE HORADAM SEQUENCE AND ITS COMPANION SEQUENCE

Refik Keskin and Zafer Şiar

Abstract. Let a, b, P, and Q be real numbers with $PQ \neq 0$ and $(a, b) \neq (0, 0)$. The Horadam sequence $\{W_n\}$ is defined by $W_0 = a$, $W_1 = b$ and $W_n = PW_{n-1} + QW_{n-2}$ for $n \geq 2$. Let the sequence $\{X_n\}$ be defined by $X_n = W_{n+1} + QW_{n-1}$. In this study, we obtain some new identities between the Horadam sequence $\{W_n\}$ and the sequence $\{X_n\}$. By the help of these identities, we show that Diophantine equations such as

$$x^2 - Pxy - y^2 = \pm(b^2 - Pab - a^2)(P^2 + 4),$$
$$x^2 - Pxy + y^2 = -(b^2 - Pab + a^2)(P^2 - 4),$$
$$x^2 - (P^2 + 4)y^2 = \pm 4(b^2 - Pab - a^2),$$

and

$$x^2 - (P^2 - 4)y^2 = 4(b^2 - Pab + a^2)$$

have infinitely many integer solutions x and y, where a, b, and P are integers. Lastly, we make an application of the sequences $\{W_n\}$ and $\{X_n\}$ to trigonometric functions and get some new angle addition formulas such as

$$\sin r\theta \sin(m + n + r)\theta = \sin(m + r)\theta \sin(n + r)\theta - \sin m\theta \sin n\theta,$$
$$\cos r\theta \cos(m + n + r)\theta = \cos(m + r)\theta \cos(n + r)\theta - \sin m\theta \sin n\theta,$$

and

$$\cos r\theta \sin(m + n)\theta = \cos(n + r)\theta \sin m\theta + \cos(m - r)\theta \sin n\theta.$$

1. Introduction

Many number sequences can be defined, characterized, evaluated, and classified by linear recurrence relations with certain orders. In this paper, we consider the sequences defined by linear recurrence relations with second order. The best known of these sequences is called the Horadam sequence, which was introduced in 1965 by Horadam [3]. The Horadam sequence $\{W_n\} = \{W_n(a, b; P, Q)\}$ is defined by

$$W_0 = a,\ W_1 = b \ \text{ and } \ W_n = PW_{n-1} + QW_{n-2} \ \text{ for } n \geq 2,$$

Received June 21, 2017; Revised July 2, 2018; Accepted December 4, 2018.
2010 *Mathematics Subject Classification.* 11B37, 11B39.
Key words and phrases. Horadam sequence, second-order recurring sequences.

where a, b, P, and Q are real numbers with $PQ \neq 0$ and $(a, b) \neq (0, 0)$. Particular cases of $\{W_n\}$ are the Lucas sequence of the first kind $\{U_n(P, Q)\} = \{W_n(0, 1; P, Q)\}$ and the Lucas sequence of the second kind $\{V_n(P, Q)\} = \{W_n(2, P; P, Q)\}$. Instead of $U_n(P, Q)$ and $V_n(P, Q)$, we write U_n and V_n, respectively. If we define the sequence $\{X_n\} = \{X_n(a, b; P, Q)\}$ by

$$X_0 = 2b - aP,\ X_1 = bP + 2aQ \quad \text{and} \quad X_n = PX_{n-1} + QX_{n-2} \quad \text{for } n \geq 2,$$

then it is convenient to consider it to be a companion sequence of $\{W_n\}$, in the same way that $\{V_n\}$ is the companion of $\{U_n\}$. Let α and β be the roots of the equation $x^2 - Px - Q = 0$. Then $\alpha = (P + \sqrt{P^2 + 4Q})/2$ and $\beta = (P - \sqrt{P^2 + 4Q})/2$. Clearly $\alpha + \beta = P$, $\alpha - \beta = \sqrt{P^2 + 4Q}$, and $\alpha\beta = -Q$. We will assume from now on that $P^2 + 4Q \neq 0$. In [3], Binet formula express the number W_n in terms of α and β by

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1}$$

where $A = b - a\beta$, $B = b - a\alpha$. Clearly, $AB = b^2 - abP - a^2Q$.

We obtain some identities concerning the Horadam sequence and its companion sequence with the help of the matrices given in the next section. Some of these identities are well known and some are new. But, since we prove these identities by matrix method not used in the literature, we also give the proof of the well known identities. Moreover, we show that some Diophantine equations such as

$$x^2 - Pxy - y^2 = \pm(b^2 - Pab - a^2)(P^2 + 4),$$
$$x^2 - Pxy + y^2 = -(b^2 - Pab + a^2)(P^2 - 4),$$
$$x^2 - (P^2 + 4)y^2 = \pm 4(b^2 - Pab - a^2),$$

and

$$x^2 - (P^2 - 4)y^2 = 4(b^2 - Pab + a^2)$$

have infinitely many integer solutions x and y. Lastly, we make an application of the sequences $\{W_n\}$ and $\{X_n\}$ to trigonometric functions and get some new angle addition formulas such as

$$\sin r\theta \sin(m + n + r)\theta = \sin(m + r)\theta \sin(n + r)\theta - \sin m\theta \sin n\theta,$$

$$\cos r\theta \cos(m + n + r)\theta = \cos(m + r)\theta \cos(n + r)\theta - \sin m\theta \sin n\theta,$$

and

$$\cos r\theta \sin(m + n)\theta = \cos(n + r)\theta \sin m\theta + \cos(m - r)\theta \sin n\theta.$$

2. Preliminaries

In this section, we will give some close relations between the sequences $\{W_n\}$, $\{X_n\}$, $\{U_n\}$, and $\{V_n\}$ and some lemmas, which will be used in the next sections.

$$X_n = W_{n+1} + QW_{n-1} = PW_n + 2QW_{n-1}, \tag{2}$$

(3) $$(P^2+4Q)W_n = X_{n+1} + QX_{n-1},$$

(4) $$W_n = bU_n + aQU_{n-1},$$

and

(5) $$X_n = bV_n + aQV_{n-1}$$

for $n \geq 1$. From (2), it can be seen that Binet formula of $\{X_n\}$ is given by

(6) $$X_n = A\alpha^n + B\beta^n.$$

It is well known that the numbers U_n and V_n for negative subscript are defined as

(7) $$U_{-n} = \frac{-U_n}{(-Q)^n} \text{ and } V_{-n} = \frac{V_n}{(-Q)^n}$$

for $n \geq 1$. By using (1) together with (6), it is convenient to extend the numbers W_n and X_n for negative subscript by

$$W_{-n} = \frac{A\alpha^{-n} - B\beta^{-n}}{\alpha - \beta} \text{ and } X_{-n} = A\alpha^{-n} + B\beta^{-n}.$$

Then it follows that

(8) $$W_{-n} = \frac{-bU_n + aU_{n+1}}{(-Q)^n} \text{ and } X_{-n} = \frac{bV_n - aV_{n+1}}{(-Q)^n}$$

and therefore

$$W_{-n} = bU_{-n} + aQU_{-n-1} \text{ and } X_{-n} = bV_{-n} + aQV_{-n-1}.$$

Thus it is seen that the identities (1)–(7) are valid for all integers n. For more information about the Horadam sequence one can consult [1, 3, 7–10]. Many identities concerning the terms of the Lucas sequence of the first and second kind can be proved by using Binet formulae, induction and matrices. In the literature, the matrices

$$\begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} P/2 & (P^2+4Q)/2 \\ 1/2 & P/2 \end{bmatrix}$$

are used to produce identities (see [2,6,11]). The n-th powers of these matrices, which will be used in the next section, are

(9) $$\begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} U_{n+1} & QU_n \\ U_n & QU_{n-1} \end{bmatrix}$$

and

(10) $$\begin{bmatrix} P/2 & (P^2+4Q)/2 \\ 1/2 & P/2 \end{bmatrix}^n = \begin{bmatrix} V_n/2 & (P^2+4Q)U_n/2 \\ U_n/2 & V_n/2 \end{bmatrix}.$$

The following two lemmas are given in [11].

Lemma 1. *If X is a square matrix satisfying the relation $X^2 = PX + QI$, then $X^n = U_nX + QU_{n-1}I$ for every $n \in \mathbb{Z}$.*

Lemma 2. *Let X be an arbitrary 2×2 matrix. Then $X^2 = PX + QI$ if and only if X is of the form*

$$X = \begin{bmatrix} x & y \\ z & P-x \end{bmatrix} \text{ with } \det X = -Q$$

or $X = \lambda I$, where $\lambda \in \{\alpha,\beta\}$, $\alpha = (P+\sqrt{P^2+4Q})/2$, and $\beta = (P-\sqrt{P^2+4Q})/2$.

3. Main theorems

Theorem 3. *If X is a square matrix satisfying the relation $X^2 = PX + QI$, then $(bX + aQI)X^n = W_{n+1}X + QW_nI$ for every $n \in \mathbb{Z}$.*

Proof. By Lemma 1 and the identity (4), it follows that

$$\begin{aligned}(bX + aQI)X^n &= bX^{n+1} + aQX^n \\ &= b(U_{n+1}X + QU_nI) + aQ(U_nX + QU_{n-1}I) \\ &= (bU_{n+1} + aQU_n)X + Q(bU_n + aQU_{n-1})I \\ &= W_{n+1}X + QW_nI.\end{aligned}$$

□

The following corollary is given in [5]. Now we give a simple proof of it.

Corollary 4. *$(b\alpha+aQ)\alpha^n = \alpha W_{n+1}+QW_n$ and $(b\beta+aQ)\beta^n = \beta W_{n+1}+QW_n$ for every $n \in \mathbb{Z}$.*

Proof. Let $X = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right]$. Then $\det X = \alpha\beta = -Q$ and this implies that

$$\begin{bmatrix} (b\alpha + aQ)\alpha^n & 0 \\ 0 & (b\beta + aQ)\beta^n \end{bmatrix} = \begin{bmatrix} \alpha W_{n+1} + QW_n & 0 \\ 0 & \beta W_{n+1} + QW_n \end{bmatrix}$$

by Theorem 3. Therefore, $(b\alpha + aQ)\alpha^n = \alpha W_{n+1} + QW_n$ and $(b\beta + aQ)\beta^n = \beta W_{n+1} + QW_n$ for every $n \in \mathbb{Z}$. □

Corollary 5. *$W_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta}$ and $X_n = A\alpha^n + B\beta^n$ for every $n \in \mathbb{Z}$.*

Proof. The result follows from (2) and Corollary 4. □

In this section, we will obtain some identities concerning the sequences mentioned above by using Theorem 3. Some of these identities, particularly given in Theorems 11, 14, 20, 23, 27, 29 and Corollaries 15, 16, 17, 22, 24, 30 are new, and others are well known. And we will use these identities to obtain some new formulas related to trigonometric functions. Also, we will use these identities in order to obtain integer solutions of some Diophantine equations.

Theorem 6. *$W_{m+n} = W_{m+1}U_n + QW_mU_{n-1}$ and $(-Q)^nW_{m-n} = W_mU_{n+1} - W_{m+1}U_n$ for every $m, n \in \mathbb{Z}$.*

Proof. Let $X = \left[\begin{smallmatrix} P & Q \\ 1 & 0 \end{smallmatrix}\right]$. Then $X^2 = PX + QI$ and therefore $DX^{m+n} = W_{m+n+1}X + QW_{m+n}I$ by Theorem 3, where $D = bX + aQI$. Hence,

$$DX^{m+n} = W_{m+n+1}X + QW_{m+n}I = \begin{bmatrix} W_{m+n+2} & QW_{m+n+1} \\ W_{m+n+1} & QW_{m+n} \end{bmatrix}$$

and

$$DX^{m-n} = W_{m-n+1}X + QW_{m-n}I = \begin{bmatrix} W_{m-n+2} & QW_{m-n+1} \\ W_{m-n+1} & QW_{m-n} \end{bmatrix}.$$

Also,

$$DX^{m+n} = (DX^m)X^n = \begin{bmatrix} W_{m+2} & QW_{m+1} \\ W_{m+1} & QW_m \end{bmatrix} \begin{bmatrix} U_{n+1} & QU_n \\ U_n & QU_{n-1} \end{bmatrix}$$

and

$$DX^{m-n} = (DX^m)X^{-n} = \begin{bmatrix} W_{m+2} & QW_{m+1} \\ W_{m+1} & QW_m \end{bmatrix} \begin{bmatrix} U_{-n+1} & QU_{-n} \\ U_{-n} & QU_{-n-1} \end{bmatrix}$$

by (9). Using (7), the proof follows. □

Corollary 7. $W_{m+1}^2 - W_mW_{m+2} = (-Q)^m AB$ *for every* $m \in \mathbb{Z}$.

Proof. By using the equality $\det(DX^m) = \det(D)\,(\det(X))^m$, we get $W_{m+1}^2 - W_mW_{m+2} = (-Q)^m AB$ by the proof of Theorem 6, where $\det(D) = -Q(b^2 - abP - a^2Q) = -QAB$. □

By using the above corollary, the following corollary can be given.

Corollary 8. $W_m^2 - PW_mW_{m-1} - QW_{m-1}^2 = (-Q)^{m-1}AB$ *for every* $m \in \mathbb{Z}$.

Since $AB = b^2 - Pab - a^2Q$, taking $Q = 1$ and $Q = -1$, respectively, we have:

Corollary 9. *Let* a, b, *and* P *be integers. Then the Diophantine equations* $x^2 - Pxy - y^2 = b^2 - Pab - a^2$ *and* $x^2 - Pxy - y^2 = -(b^2 - Pab - a^2)$ *have infinitely many integer solutions given by* $(x, y) = (W_{2n+1}, W_{2n})$ *and* $(x, y) = (W_{2n}, W_{2n-1})$ *with* $n \in \mathbb{Z}$, *respectively, where* $W_n = W_n(a, b; P, 1)$.

Corollary 10. *Let* a, b, *and* $P \geq 3$ *be integers. Then the Diophantine equation* $x^2 - Pxy + y^2 = b^2 - Pab + a^2$ *has infinitely many integer solutions given by* $(x, y) = (W_n, W_{n-1})$ *with* $n \in \mathbb{Z}$, *where* $W_n = W_n(a, b; P, -1)$.

Theorem 11.
i) $QX_n^2 - (P^2 + 4Q)X_nW_{n+1} + (P^2 + 4Q)W_{n+1}^2 = -(-Q)^n P^2 AB$,
ii) $X_{n+1}^2 - (P^2 + 4Q)X_{n+1}W_n + Q(P^2 + 4Q)W_n^2 = (-Q)^n P^2 AB$
for every $n \in \mathbb{Z}$.

Proof. i) We have $W_n = \frac{2W_{n+1}-X_n}{P}$ by (2). Substituting this value of W_n into the equation $W_n^2 - PW_nW_{n-1} - QW_{n-1}^2 = (-Q)^{n-1}AB$ given in Corollary 8, we get

$$(-Q)^{n-1}AB = \left(\frac{2W_{n+1}-X_n}{P}\right)^2 - P\left(\frac{2W_{n+1}-X_n}{P}\right)W_{n-1} - QW_{n-1}^2.$$

Then it follows that

$$\begin{aligned}(-Q)^{n-1}P^2AB &= 4W_{n+1}^2 - 4W_{n+1}X_n + X_n^2 - 2P^2W_{n-1}W_{n+1}\\ &\quad + P^2W_{n-1}X_n - P^2QW_{n-1}^2\\ &= 4W_{n+1}^2 - 4W_{n+1}X_n + X_n^2\\ &\quad - P^2W_{n-1}(2W_{n+1} - X_n + QW_{n-1})\\ &= 4W_{n+1}^2 - 4W_{n+1}X_n + X_n^2 - P^2(\frac{X_n - W_{n+1}}{Q})W_{n+1}\end{aligned}$$

using (2). It is seen that

$$\begin{aligned}-(-Q)^nP^2AB &= 4QW_{n+1}^2 - 4QW_{n+1}X_n + QX_n^2 - P^2(X_n - W_{n+1})W_{n+1}\\ &= 4QW_{n+1}^2 - 4QW_{n+1}X_n + QX_n^2 - P^2X_nW_{n+1} + P^2W_{n+1}^2\\ &= QX_n^2 - (P^2+4Q)W_{n+1}X_n + (P^2+4Q)W_{n+1}^2.\end{aligned}$$

ii) We have $W_{n+1} = \frac{X_{n+1}-2QW_n}{P}$ by (2). Substituting this value of W_{n+1} into the equation $W_{n+1}^2 - PW_{n+1}W_n - QW_n^2 = (-Q)^nAB$ given in Corollary 8, a similar argument shows that

$$(-Q)^nP^2AB = X_{n+1}^2 - (P^2+4Q)X_{n+1}W_n + Q(P^2+4Q)W_n^2.$$ □

If we take $Q = 1$ and respectively $Q = -1$ in the above theorem, we get:

Corollary 12. *Let a, b, and P be integers. Then the Diophantine equations $x^2-(P^2+4)xy+(P^2+4)y^2 = P^2(b^2-Pab-a^2)$ and $x^2-(P^2+4)xy+(P^2+4)y^2 = -P^2(b^2-Pab-a^2)$ have infinitely many integer solutions given by $(x,y) = (X_{2n+1}, W_{2n})$ or (X_{2n-1}, W_{2n}) and $(x,y) = (X_{2n}, W_{2n-1})$ or (X_{2n}, W_{2n+1}) with $n \in \mathbb{Z}$, respectively, where $W_n = W_n(a,b;P,1)$ and $X_n = X_n(a,b;P,1)$.*

Corollary 13. *Let a, b, and $P \geq 3$ be integers. Then the Diophantine equations $x^2-(P^2-4)xy-(P^2-4)y^2 = P^2(b^2-Pab+a^2)$ and $-x^2-(P^2-4)xy+(P^2-4)y^2 = -\,P^2(b^2-Pab+a^2)$ have infinitely many integer solutions given by $(x,y) = (X_{n+1}, W_n)$ and $(x,y) = (X_n, W_{n+1})$ with $n \in \mathbb{Z}$, respectively, where $W_n = W_n(a,b;P,-1)$ and $X_n = X_n(a,b;P,-1)$.*

Theorem 14. *$X_{m+n} = X_mU_{n+1} + QX_{m-1}U_n$ and $(-Q)^{n-1}X_{m-n} = X_{m-1}U_n - X_mU_{n-1}$ for every $m, n \in \mathbb{Z}$.*

Proof. If we take $X = \left[\begin{smallmatrix} P & Q \\ 1 & 0 \end{smallmatrix}\right]$, then we have $X^n = \left[\begin{smallmatrix} U_{n+1} & QU_n \\ U_n & QU_{n-1} \end{smallmatrix}\right]$ by (9). It can be seen that $EX^m = \left[\begin{smallmatrix} X_{m+1} & QX_m \\ X_m & QX_{m-1} \end{smallmatrix}\right]$ by (2) and (5), where

$$E = \begin{bmatrix} X_1 & QX_0 \\ X_0 & QX_{-1} \end{bmatrix} = \begin{bmatrix} bP + 2aQ & Q(2b - aP) \\ (2b - aP) & aP^2 + 2aQ - bP \end{bmatrix}.$$

Thus, it follows that

$$EX^{m+n} = \begin{bmatrix} X_{m+n+1} & QX_{m+n} \\ X_{m+n} & QX_{m+n-1} \end{bmatrix} \text{ and } EX^{m-n} = \begin{bmatrix} X_{m-n+1} & QX_{m-n} \\ X_{m-n} & QX_{m-n-1} \end{bmatrix}.$$

Moreover, we get

$$EX^{m+n} = (EX^m)X^n = \begin{bmatrix} X_{m+1} & QX_m \\ X_m & QX_{m-1} \end{bmatrix} \begin{bmatrix} U_{n+1} & QU_n \\ U_n & QU_{n-1} \end{bmatrix}$$

and

$$EX^{m-n} = (EX^m)X^{-n} = \begin{bmatrix} X_{m+1} & QX_m \\ X_m & QX_{m-1} \end{bmatrix} \begin{bmatrix} U_{-n+1} & QU_{-n} \\ U_{-n} & QU_{-n-1} \end{bmatrix}.$$

So, from (7), the proof follows. □

Corollary 15. *$2X_{m+n} = X_mV_n + (P^2 + 4Q)W_mU_n$ and $2W_{m+n} = W_mV_n + X_mU_n$ for every $m, n \in \mathbb{Z}$.*

Proof. Using (2), we get

$$\begin{aligned} 2X_{m+n} &= X_mU_{n+1} + QX_{m-1}U_n + X_mU_{n+1} + QX_{m-1}U_n \\ &= X_mU_{n+1} + QX_{m-1}U_n + X_m(PU_n + QU_{n-1}) + QX_{m-1}U_n \\ &= X_m(U_{n+1} + QU_{n-1}) + (PX_m + 2QX_{m-1})U_n \\ &= X_mV_n + (P^2 + 4Q)W_mU_n \end{aligned}$$

by Theorem 14. Similarly, it is seen that $2W_{m+n} = W_mV_n + X_mU_n$ by using (3) and Theorem 6. □

Corollary 16. *$X_{m+1}X_{m-1} - X_m^2 = (-Q)^{m-1}(P^2 + 4Q)AB$ for every $m \in \mathbb{Z}$.*

Proof. Since $\det(EX^m) = \det(E)(\det X)^m$, the proof follows easily. □

From the above corollary, we can give the following.

Corollary 17. *$X_m^2 - PX_mX_{m-1} - QX_{m-1}^2 = -(-Q)^{m-1}(P^2 + 4Q)AB$ for every $m \in \mathbb{Z}$.*

Corollary 18. *Let a, b, and P be integers. Then the Diophantine equations $x^2 - Pxy - y^2 = (b^2 - Pab - a^2)(P^2 + 4)$ and $x^2 - Pxy - y^2 = -(b^2 - Pab - a^2)(P^2 + 4)$ have infinitely many integer solutions given by $(x, y) = (X_{2n}, X_{2n-1})$ and $(x, y) = (X_{2n+1}, X_{2n})$ with $n \in \mathbb{Z}$, respectively, where $X_n = X_n(a, b; P, 1)$.*

Corollary 19. *Let a, b, and $P \geq 3$ be integers. Then the Diophantine equation $x^2 - Pxy + y^2 = -(b^2 - Pab + a^2)(P^2 - 4)$ has infinitely many integer solutions given by $(x, y) = (X_n, X_{n-1})$ with $n \in \mathbb{Z}$, where $X_n = X_n(a, b; P, -1)$.*

Theorem 20. *$2(-Q)^n W_{m-n} = W_m V_n - X_m U_n$ and $2(-Q)^n X_{m-n} = X_m V_n - (P^2+4Q) W_m U_n$ for every $m, n \in \mathbb{Z}$.*

Proof. Let $X = \begin{bmatrix} P/2 & (P^2+4Q)/2 \\ 1/2 & P/2 \end{bmatrix}$. Then $X^2 = PX + QI$ and therefore we have $DX^{m-n-1} = W_{m-n}X + QW_{m-n-1}I$ by Theorem 3, where $D = bX + aQI$. Hence, using (2), we obtain

$$\begin{aligned} DX^{m-n-1} &= W_{m-n}X + QW_{m-n-1}I \\ &= \begin{bmatrix} (PW_{m-n} + 2QW_{m-n-1})/2 & (P^2+4Q)W_{m-n}/2 \\ W_{m-n}/2 & (PW_{m-n} + 2QW_{m-n-1})/2 \end{bmatrix} \\ &= \begin{bmatrix} X_{m-n}/2 & (P^2+4Q)W_{m-n}/2 \\ W_{m-n}/2 & X_{m-n}/2 \end{bmatrix}. \end{aligned}$$

On the other hand, using (10), we get

$$\begin{aligned} DX^{m-n-1} &= (DX^{m-1})X^{-n} \\ &= \begin{bmatrix} \frac{X_m}{2} & \frac{(P^2+4Q)W_m}{2} \\ \frac{W_m}{2} & \frac{X_m}{2} \end{bmatrix} \begin{bmatrix} V_{-n}/2 & (P^2+4Q)U_{-n}/2 \\ U_{-n}/2 & V_{-n}/2 \end{bmatrix} \end{aligned}$$

and from (7), the proof follows. □

Corollary 21. *$X_m V_n = X_{m+n} + (-Q)^n X_{m-n}$, $W_m V_n = W_{m+n} + (-Q)^n W_{m-n}$ and $(P^2+4Q)W_m U_n = X_{m+n} - (-Q)^n X_{m-n}$ for every $m, n \in \mathbb{Z}$.*

Proof. From Theorem 20 and Corollary 15, the proof is obvious. □

Corollary 22. *$W_{2n} = X_n U_n + a(-Q)^n = W_n V_n - a(-Q)^n$ and $X_{2n} = X_n V_n + (aP - 2b)(-Q)^n$ for every $n \in \mathbb{Z}$.*

Proof. From Theorem 20 and Corollary 21, the proof follows. □

Theorem 23. *$X_m X_n - (P^2+4Q) W_m W_n = 2(-Q)^n AB V_{m-n}$ for every $m, n \in \mathbb{Z}$.*

Proof. Using Binet formulae in (1) and (6), we get

$$\begin{aligned} (LHS) &= (A\alpha^m + B\beta^m)(A\alpha^n + B\beta^n) \\ &\quad - (P^2+4Q)\frac{(A\alpha^m - B\beta^m)}{\alpha - \beta}\frac{(A\alpha^n - B\beta^n)}{\alpha - \beta} \\ &= (A\alpha^m + B\beta^m)(A\alpha^n + B\beta^n) - (A\alpha^m - B\beta^m)(A\alpha^n - B\beta^n) \\ &= 2AB(\alpha^m\beta^n + \alpha^n\beta^m) = 2AB(\alpha\beta)^n(\alpha^{m-n} + \beta^{m-n}) \\ &= 2AB(-Q)^n V_{m-n}. \end{aligned}$$

□

Taking $m = n$ in the above theorem, we get the following corollary.

Corollary 24. *$X_n^2 - (P^2+4Q)W_n^2 = 4(-Q)^n AB$ for every $n \in \mathbb{Z}$.*

Corollary 25. *Let a, b, and P be integers. Then the Pell equations $x^2 - (P^2+4)y^2 = 4(b^2 - Pab - a^2)$ and $x^2 - (P^2+4)y^2 = -4(b^2 - Pab - a^2)$ have infinitely many integer solutions given by $(x,y) = (X_{2n}, W_{2n})$ and $(x,y) = (X_{2n-1}, W_{2n-1})$ with $n \in \mathbb{Z}$, respectively, where $X_n = X_n(a,b;P,1)$, $W_n = W_n(a,b;P,1)$.*

Corollary 26. *Let a, b, and $P \geq 3$ be integers. Then the Pell equation $x^2 - (P^2-4)y^2 = 4(b^2 - Pab + a^2)$ has infinitely many integer solutions given by $(x,y) = (X_n, W_n)$ with $n \in \mathbb{Z}$, where $X_n = X_n(a,b;P,-1)$, $W_n = W_n(a,b;P,-1)$.*

The identities given in the following two theorems will be used in the next section to give some new angle addition formulas for trigonometric functions.

Theorem 27. *Let m, n, and $r \in \mathbb{Z}$ with $r \neq 0$. Then*

$$U_r W_{m+n+r} = W_{m+r} U_{n+r} - (-Q)^r W_m U_n,$$

$$U_r W_{m+n-r} = W_m U_n - (-Q)^r W_{m-r} U_{n-r},$$

and

$$U_r W_{m+n} = W_m U_{n+r} - (-Q)^r W_{m-r} U_n.$$

Proof. If we consider the matrix $X = \left[\begin{smallmatrix} x & y \\ z & P-x \end{smallmatrix}\right]$ with $\det X = -Q$ and take $x = \frac{U_{r+1}}{U_r}$, then by Corollary 2.3 in [11] and Theorem 3, we get

$$DX^n = \begin{bmatrix} b\dfrac{U_{r+1}}{U_r} + aQ & by \\ bz & bP - b\dfrac{U_{r+1}}{U_r} + aQ \end{bmatrix} \begin{bmatrix} \dfrac{U_{r+1}}{U_r} U_n + QU_{n-1} & yU_n \\ zU_n & U_{n+1} - \dfrac{U_{r+1}}{U_r} U_n \end{bmatrix},$$

where $D = bX + aQI$. Using (4), (7), (8), and Theorem 6, we see that

$$\begin{aligned} DX^n &= \begin{bmatrix} \dfrac{W_{r+1}}{U_r} & by \\ bz & \dfrac{-b(QU_{r-1}) + aQU_r}{U_r} \end{bmatrix} \begin{bmatrix} \dfrac{U_{n+r}}{U_r} & yU_n \\ zU_n & \dfrac{-(-Q)^r U_{n-r}}{U_r} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{W_{r+1}}{U_r} & by \\ bz & \dfrac{-(-Q)^r W_{1-r}}{U_r} \end{bmatrix} \begin{bmatrix} \dfrac{U_{n+r}}{U_r} & yU_n \\ zU_n & \dfrac{-(-Q)^r U_{n-r}}{U_r} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{W_{r+1}U_{n+r}}{U_r^2} - byzU_n & y\left(\dfrac{W_{r+1}U_n - b(-Q)^r U_{n-r}}{U_r}\right) \\ z\left(\dfrac{bU_{n+r} - (-Q)^r W_{1-r} U_n}{U_r}\right) & byzU_n + \dfrac{(-Q)^{2r} W_{1-r} U_{n-r}}{U_r^2} \end{bmatrix}. \end{aligned}$$

Since $\det X = -Q$ and $x = \frac{U_{r+1}}{U_r}$, it follows that

$$yz = \frac{PU_rU_{r+1} + QU_r^2 - U_{r+1}^2}{U_r^2} = \frac{U_r(PU_{r+1} + QU_r) - U_{r+1}^2}{U_r^2}$$

$$= \frac{U_r U_{r+2} - U_{r+1}^2}{U_r^2} = \frac{-(-Q)^r}{U_r^2}$$

by Corollary 7. Thus

$$DX^n = \begin{bmatrix} \frac{W_{r+1}U_{n+r}}{U_r^2} - byzU_n & y\left(\frac{W_{r+1}U_n - b(-Q)^r U_{n-r}}{U_r}\right) \\ z\left(\frac{bU_{n+r} - (-Q)^r W_{1-r}U_n}{U_r}\right) & byzU_n + \frac{(-Q)^{2r}W_{1-r}U_{n-r}}{U_r^2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{W_{n+r+1}}{U_r} & yW_{n+1} \\ zW_{n+1} & \frac{-(-Q)^r W_{n-r+1}}{U_r} \end{bmatrix}$$

by (4) and Theorem 6. If we consider the matrix multiplication $DX^{m+n-1} = \left(DX^{m-1}\right)X^n$, then we get the result. □

Corollary 28. *$W_{n+r}W_{n-r} - W_n^2 = -AB(-Q)^{n-r}U_r^2$ for all n, $r \in \mathbb{Z}$.*

Proof. By the proof of Theorem 27, we see that

$$\det DX^{n-1} = \frac{-(-Q)^r W_{n+r}W_{n-r} + (-Q)^r W_n^2}{U_r^2}$$
$$= \frac{-(-Q)^r\left(W_{n+r}W_{n-r} - W_n^2\right)}{U_r^2}.$$

On the other hand, since $\det DX^{n-1} = (\det D)\,(\det X)^{n-1}$, it follows that

$$\det DX^{n-1} = \left(\frac{-(-Q)^r W_{r+1}W_{1-r} + b^2(-Q)^r}{U_r^2}\right)(-Q)^{n-1}$$
$$= \left[\frac{-(-Q)^r W_{r+1}\left(\frac{aU_r - bU_{r-1}}{(-Q)^{r-1}}\right) + b^2(-Q)^r}{U_r^2}\right](-Q)^{n-1}$$
$$= \left[\frac{aQU_rW_{r+1} - bQ\left(W_{r+1}U_{r-1} + b(-Q)^{r-1}\right)}{U_r^2}\right](-Q)^{n-1}$$
$$= \left[\frac{aQU_rW_{r+1} - bQU_rW_r}{U_r^2}\right](-Q)^{n-1}$$
$$= \left(\frac{Q\left(aW_{r+1} - bW_r\right)}{U_r}\right)(-Q)^{n-1}$$
$$= AB(-Q)^n$$

by (4), (8), and Theorem 6. Thus, we get the equality $W_{n+r}W_{n-r} - W_n^2 = -AB(-Q)^{n-r}U_r^2$. □

Theorem 29. *Let $m, n,$ and $r \in \mathbb{Z}$. Then*

$$V_rX_{m+n+r} = X_{m+r}V_{n+r} + (-Q)^r(P^2 + 4Q)W_mU_n,$$

$$V_r X_{m+n-r} = (P^2+4Q) W_m U_n + (-Q)^r X_{m-r} V_{n-r},$$

and

$$V_r W_{m+n} = W_m V_{n+r} + (-Q)^r X_{m-r} U_n.$$

Proof. If we consider the matrix $X = \left[\begin{smallmatrix} x & y \\ z & P-x \end{smallmatrix}\right]$ with $\det X = -Q$ and take $x = \frac{V_{r+1}}{V_r}$, then by Corollary 2.3 in [11] and Theorem 3, we get

$$DX^n = \begin{bmatrix} b\dfrac{V_{r+1}}{V_r} + aQ & by \\ bz & bP - b\dfrac{V_{r+1}}{V_r} + aQ \end{bmatrix} \begin{bmatrix} \dfrac{V_{r+1}}{V_r} U_n + QU_{n-1} & yU_n \\ zU_n & U_{n+1} - \dfrac{V_{r+1}}{V_r} U_n \end{bmatrix},$$

where $D = bX + aQI$. Using (5), (7), (8), and Theorem 14, we see that

$$\begin{aligned} DX^n &= \begin{bmatrix} \dfrac{X_{r+1}}{V_r} & by \\ bz & \dfrac{-b(QV_{r-1}) + aQV_r}{V_r} \end{bmatrix} \begin{bmatrix} \dfrac{V_{n+r}}{V_r} & yU_n \\ zU_n & \dfrac{(-Q)^r V_{n-r}}{V_r} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{X_{r+1}}{V_r} & by \\ bz & \dfrac{(-Q)^r X_{1-r}}{V_r} \end{bmatrix} \begin{bmatrix} \dfrac{V_{n+r}}{V_r} & yU_n \\ zU_n & \dfrac{(-Q)^r V_{n-r}}{V_r} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{X_{r+1} V_{n+r}}{V_r^2} + byzU_n & y\left(\dfrac{X_{r+1}U_n + b(-Q)^r V_{n-r}}{V_r}\right) \\ z\left(\dfrac{bV_{n+r} + (-Q)^r X_{1-r} U_n}{V_r}\right) & byzU_n + \dfrac{(-Q)^{2r} X_{1-r} V_{n-r}}{V_r^2} \end{bmatrix}. \end{aligned}$$

Since $\det X = -Q$ and $x = \frac{V_{r+1}}{V_r}$, it follows that

$$\begin{aligned} yz &= \frac{PV_r V_{r+1} + QV_r^2 - V_{r+1}^2}{V_r^2} = \frac{V_r(PV_{r+1} + QV_r) - V_{r+1}^2}{V_r^2} \\ &= \frac{V_r V_{r+2} - V_{r+1}^2}{V_r^2} = \frac{(-Q)^r (P^2+4Q)}{V_r^2} \end{aligned}$$

by Corollary 16. Thus, a simple computation shows that

$$\begin{aligned} DX^n &= \begin{bmatrix} \dfrac{X_{r+1} V_{n+r}}{V_r^2} + byzU_n & y\left(\dfrac{X_{r+1}U_n + b(-Q)^r V_{n-r}}{V_r}\right) \\ z\left(\dfrac{bV_{n+r} + (-Q)^r X_{1-r} U_n}{V_r}\right) & byzU_n + \dfrac{(-Q)^{2r} X_{1-r} V_{n-r}}{V_r^2} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{X_{n+r+1}}{V_r} & yW_{n+1} \\ zW_{n+1} & \dfrac{(-Q)^r X_{n-r+1}}{V_r} \end{bmatrix}. \end{aligned}$$

If we consider the matrix multiplication $DX^{m+n-1} = \left(DX^{m-1}\right) X^n$, then we get the result. □

Corollary 30. $X_{n+r} X_{n-r} - (P^2+4Q) W_n^2 = AB(-Q)^{n-r} V_r^2$ *for all* n*,* $r \in \mathbb{Z}$*.*

Proof. By the proof of Theorem 29, we see that

$$\begin{aligned}\det DX^{n-1} &= \frac{(-Q)^r X_{n+r}X_{n-r} - (-Q)^r(P^2+4Q)W_n^2}{V_r^2} \\ &= \frac{(-Q)^r\left(X_{n+r}X_{n-r} - (P^2+4Q)W_n^2\right)}{V_r^2}.\end{aligned}$$

On the other hand, since $\det DX^{n-1} = (\det D)(\det X)^{n-1}$, it follows that

$$\begin{aligned}\det DX^{n-1} &= \left(\frac{(-Q)^r X_{r+1}X_{1-r} - b^2(-Q)^r(P^2+4Q)}{V_r^2}\right)(-Q)^{n-1} \\ &= \left[\frac{(-Q)^r X_{r+1}\left(\frac{bV_{r-1}-aV_r}{(-Q)^{r-1}}\right) - b^2(-Q)^r(P^2+4Q)}{V_r^2}\right](-Q)^{n-1} \\ &= \left[\frac{aQV_rX_{r+1} - bQ\left(X_{r+1}V_{r-1} - b(-Q)^{r-1}(P^2+4Q)\right)}{V_r^2}\right](-Q)^{n-1} \\ &= \left[\frac{aQV_rX_{r+1} - bQV_rX_r}{V_r^2}\right](-Q)^{n-1} \\ &= \left(\frac{Q\left(aX_{r+1} - bX_r\right)}{V_r}\right)(-Q)^{n-1} \\ &= AB(-Q)^n\end{aligned}$$

by (5), (8), and Theorem 29. Thus, we get the equality $X_{n+r}X_{n-r} - (P^2 + 4Q)W_n^2 = AB(-Q)^{n-r}V_r^2$. □

4. An application of the sequences $\{W_n\}$ and $\{X_n\}$ to trigonometric functions

We consider the following recurrence relations, known as Simpson's Formulae (see [4]) related to trigonometric functions:

$$\begin{aligned}\sin(n+2)\theta &= 2\cos\theta\sin(n+1)\theta - \sin n\theta, \\ \cos(n+2)\theta &= 2\cos\theta\cos(n+1)\theta - \cos n\theta.\end{aligned}$$

It is clear that these relations satisfy the characteristic equation $x^2-Px-Q=0$ for $P = 2\cos\theta$ and $Q = -1$. In this case, if we take $b = 2\cos\theta$, $P = 2\cos\theta$, and $Q = -1$, then we get

$$\alpha = \frac{P+\sqrt{P^2+4Q}}{2} = \frac{2\cos\theta+\sqrt{4\cos^2\theta-4}}{2} = \cos\theta + i\sin\theta$$

and

$$\beta = \frac{P-\sqrt{P^2+4Q}}{2} = \frac{2\cos\theta-\sqrt{4\cos^2\theta-4}}{2} = \cos\theta - i\sin\theta,$$

and therefore $\alpha - \beta = 2i\sin\theta$, $\alpha + \beta = 2\cos\theta$. Thus, from the Binet formula of $\{W_n\}$, we have (see also [4])

$$\begin{aligned} W_n &= W_n(a, 2\cos\theta; 2\cos\theta, -1) = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \\ &= \frac{(b - a\beta)[\cos n\theta + i\sin n\theta] - (b - a\alpha)[\cos n\theta - i\sin n\theta]}{\alpha - \beta} \\ &= a\cos n\theta + (2 - a)\sin n\theta\cot\theta \\ &= (-a\sin(n-1)\theta + 2\sin n\theta\cos\theta)/\sin\theta. \end{aligned} \tag{11}$$

Moreover, from the equality $X_n = W_{n+1} + QW_{n-1}$, we have

$$\begin{aligned} X_n &= W_{n+1} + QW_{n-1} \\ &= a\cos(n+1)\theta + (2-a)\sin(n+1)\theta\cot\theta \\ &\quad - a\cos(n-1)\theta - (2-a)\sin(n-1)\theta\cot\theta \\ &= -2a\sin n\theta\sin\theta + (2-a)\cot\theta\,(2\sin\theta\cos n\theta) \\ &= -2a\sin n\theta\sin\theta + 4\cos n\theta\cos\theta - 2a\cos n\theta\cos\theta \\ &= -2a\cos(n-1)\theta + 4\cos n\theta\cos\theta. \end{aligned} \tag{12}$$

From the above equations, it can be seen that

$$U_n = U_n(P, -1) = W_n(0, 1; 2\cos\theta, -1) = \frac{\sin n\theta}{\sin\theta} \tag{13}$$

and

$$V_n = V_n(P, -1) = X_n(0, 1; 2\cos\theta, -1) = 2\cos n\theta. \tag{14}$$

In view of the above identities, now we can give an application for each of the Corollaries 22, 28, 30, and Theorems 27, 29.

Theorem 31.

$$\sin(2n-1)\theta = 2\sin(n-1)\theta\cos n\theta + \sin\theta,$$
$$\sin(2n-1)\theta = 2\cos(n-1)\theta\sin n\theta - \sin\theta,$$

and

$$\cos(2n-1)\theta = 2\cos(n-1)\theta\cos n\theta - \cos\theta$$

for every $n \in \mathbb{Z}$.

Proof. Substituting the equations (11), (12), (13), and (14) into the equation $W_{2n} = X_nU_n + a(-Q)^n$ given in Corollary 22, we get

$$\begin{aligned} &[(-a\sin(2n-1)\theta + 2\sin 2n\theta\cos\theta)/\sin\theta] \\ &= [-2a\cos(n-1)\theta + 4\cos n\theta\cos\theta]\frac{\sin n\theta}{\sin\theta} + a \\ &= [(-2a\cos(n-1)\theta\sin n\theta + 2\sin 2n\theta\cos\theta)/\sin\theta] + a \end{aligned}$$

and so

$$\sin(2n-1)\theta = 2\cos(n-1)\theta\sin n\theta - \sin\theta.$$

Similarly, from the equations $W_{2n} = W_n V_n - a(-Q)^n$ and $X_{2n} = X_n V_n + (aP - 2b)(-Q)^n$ given in Corollary 22, it follows that

$$\sin(2n-1)\theta = 2\sin(n-1)\theta\cos n\theta + \sin\theta,$$

and

$$\cos(2n-1)\theta = 2\cos(n-1)\theta\cos n\theta - \cos\theta,$$

respectively. □

In the following theorem, we will have been get some new formulas, which are general form of angle addition formulas

$$\cos(x+y) = \cos x\cos y - \sin x\sin y,$$

and

$$\sin(x+y) = \sin x\cos y + \sin y\cos x.$$

Theorem 32.

$$\sin r\theta\sin(m+n+r)\theta = \sin(m+r)\theta\sin(n+r)\theta - \sin m\theta\sin n\theta,$$

$$\cos r\theta\cos(m+n+r)\theta = \cos(m+r)\theta\cos(n+r)\theta - \sin m\theta\sin n\theta,$$

and

$$\cos r\theta\sin(m+n)\theta = \cos(n+r)\theta\sin m\theta + \cos(m-r)\theta\sin n\theta$$

for every $m, n, r \in \mathbb{Z}$.

Proof. If $r = 0$, then the proof is obvious. Assume that $r \neq 0$. If we take $a = 0$, then we have $W_n = 2\sin n\theta\cos\theta/\sin\theta$ and $X_n = 4\cos n\theta\cos\theta$. Also we know that $U_n = \frac{\sin n\theta}{\sin\theta}$ and $V_n = 2\cos n\theta$. Substituting these values into equation

$$U_r W_{m+n+r} = W_{m+r}U_{n+r} - (-Q)^r W_m U_n$$

given in Theorem 27, one gets

$$\begin{aligned}&\frac{\sin r\theta}{\sin\theta}\left(2\sin\left(m+n+r\right)\theta\cos\theta/\sin\theta\right)\\ = \;&\left(2\sin(m+r)\theta\cos\theta/\sin\theta\right)\frac{\sin(n+r)\theta}{\sin\theta} - \left(2\sin m\theta\cos\theta/\sin\theta\right)\frac{\sin n\theta}{\sin\theta}\end{aligned}$$

and thus

$$\sin r\theta\sin(m+n+r)\theta = \sin(m+r)\theta\sin(n+r)\theta - \sin m\theta\sin n\theta.$$

Similarly, from the equations

$$V_r X_{m+n+r} = X_{m+r}V_{n+r} + (-Q)^r(P^2+4Q)W_m U_n,$$

and

$$V_r W_{m+n} = W_m V_{n+r} + (-Q)^r X_{m-r}U_n,$$

given in Theorem 29, it follows that

$$\cos r\theta\cos(m+n+r)\theta = \cos(m+r)\theta\cos(n+r)\theta - \sin m\theta\sin n\theta,$$

and

$$\cos r\theta\sin(m+n)\theta = \cos(n+r)\theta\sin m\theta + \cos(m-r)\theta\sin n\theta,$$

respectively. □

The relations in the following theorem are also given in [4].

Theorem 33.

$$\sin(n+r)\theta\sin(n-r)\theta - \sin^2 n\theta = -\sin^2 r\theta$$

and

$$\cos(n+r)\theta\cos(n-r)\theta + \sin^2 n\theta = \cos^2 r\theta$$

for every $m, n, r \in \mathbb{Z}$.

Proof. If we take $a=0$, we have $W_n = 2\sin n\theta\cos\theta/\sin\theta$ and $X_n = 4\cos n\theta\cos\theta$. Also we know that $U_n = \frac{\sin n\theta}{\sin\theta}$ and $V_n = 2\cos n\theta$. Substituting these values into equation

$$W_{n+r}W_{n-r} - W_n^2 = -(-Q)^{n-r}ABU_r^2$$

given in Corollary 28, we get

$$\left(\frac{2\sin(n+r)\theta\cos\theta}{\sin\theta}\right)\left(\frac{2\sin(n-r)\theta\cos\theta}{\sin\theta}\right) - \left(\frac{4\sin^2 n\theta\cos^2\theta}{\sin^2\theta}\right)$$

$$= -4\cos^2\theta\frac{\sin^2 r\theta}{\sin^2\theta}$$

and it follows that

$$\sin(n+r)\theta\sin(n-r)\theta - \sin^2 n\theta = -\sin^2 r\theta.$$

Similarly, from the equation

$$X_{n+r}X_{n-r} - (P^2+4Q)W_n^2 = (-Q)^{n-r}ABV_r^2,$$

given in Corollary 30, we get

$$\cos(n+r)\theta\cos(n-r)\theta + \sin^2 n\theta = \cos^2 r\theta.$$ □

References

[1] G. Cerda-Morales, *On generalized Fibonacci and Lucas numbers by matrix methods*, Hacet. J. Math. Stat. **42** (2013), no. 2, 173–179.

[2] T.-X. He and P. J.-S. Shiue, *On sequences of numbers and polynomials defined by linear recurrence relations of order 2*, Int. J. Math. Math. Sci. **2009** (2009), Art. ID 709386, 21 pp.

[3] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, Fibonacci Quart. **3** (1965), 161–176.

[4] ———, *Tschebyscheff and other functions associated with the sequence* $\{W_n(a,b;p,q)\}$, Fibonacci Quart. **7** (1969), no. 1, 14–22.

[5] Y. H. Jang and S. P. Jun, *Linearlization of generalized Fibonacci sequences*, Korean J. Math. **3** (2014), 443–454.

[6] D. Kalman and R. Mena, *The Fibonacci numbers—exposed*, Math. Mag. **76** (2003), no. 3, 167–181.

[7] R. S. Melham, *Certain classes of finite sums that involve generalized Fibonacci and Lucas numbers*, Fibonacci Quart. **42** (2004), no. 1, 47–54.

[8] R. S. Melham and A. G. Shannon, *Some congruence properties of generalized second-order integer sequences*, Fibonacci Quart. **32** (1994), no. 5, 424–428.

[9] S. Rabinowitz, *Algorithmic manipulation of second-order linear recurrences*, Fibonacci Quart. **37** (1999), no. 2, 162–177.

[10] A. G. Shannon and A. F. Horadam, *Special recurrence relations associated with the* $\{W_n(a, b; p, q)\}$, Fibonacci Quart. **17** (1979), no. 4, 294–299.

[11] Z. Şiar and R. Keskin, *Some new identities concerning generalized Fibonacci and Lucas numbers*, Hacet. J. Math. Stat. **42** (2013), no. 3, 211–222.

Refik Keskin
Sakarya University
Department of Mathematics
Sakarya, Turkey
Email address: rkeskin@sakarya.edu.tr

Zafer Şiar
Bingöl University
Department of Mathematics
Bingöl, Turkey
Email address: zsiar@bingol.edu.tr