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Some Results on Null Hypersurfaces in (LCS)-manifolds

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Abstract. We prove that a Lorentzian concircular structure (LCS)-manifold does not admit any null hypersurface which is tangential or transversal to its characteristic vector field. Due to the above, we focus on its ascreen null hypersurfaces and show that such hypersurfaces admit a symmetric Ricci tensor. Furthermore, we prove that there are no totally geodesic ascreen null hypersurfaces of a conformally flat (LCS)-manifold.

1. Introduction

Since the middle of the twentieth century, Riemannian geometry has had a substantial influence on several main areas of mathematical sciences. Primarily, semi-Riemannian (in particular, Lorentzian [3]) geometry has its roots in Riemannian geometry. On the other hand, the situation is quite different for null manifolds with a degenerate metric, since one fails to use, in the usual way, the theory of non-null geometry. Null submanifolds have numerous applications in mathematical physics and General relativity (see [16] for details). This prompted Duggal-Bejancu [6] and Duggal-Sahin [8] to introduce the geometry of null submanifolds. They introduced a non-degenerate screen distribution to construct a null transversal vector bundle which is non-intersecting to its null tangent bundle and developed local geometry of null curves, hypersurfaces, and submanifolds. Their approach is extrinsic contrary to the intrinsic approach of Kupeli [12]. Based on the above books, many authors picked interest in null geometry, for instance, see [1, 2, 5, 9, 10, 11, 13, 15] and many more references cited therein.

In [17, 18], the geometry of Lorentzian concircular structure (LCS)-manifolds is extensively studied. As these spaces are semi-Riemannian, they naturally ad-

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mit null subspaces. The aim of this paper is to introduce the geometry of null subspaces of (LCS)-manifolds. In particular, we study the geometry of null hypersurfaces of these spaces. Unlike the null hypersurfaces of Sasakian manifolds, null hypersurfaces (LCS)-manifolds are never tangent to the characteristic vector field of (LCS)-manifolds. Moreover, an ascreen null hypersurface admits a symmetric Ricci tensor. In case of a conformally flat (LCS)-manifolds, we prove that no totally geodesic ascreen null hypersurfaces can be admitted. Moreover, a screen conformal or screen totally umbilic null hypersurface is proper totally umbilic. The paper is arranged as follows; In Section 2, we quote some basic notions on (LCS)-manifolds and null hypersurfaces of semi-Riemannian manifolds. Section 3, we prove several new characterization theorems on null hypersurfaces of (LCS)-manifolds.

2. Preliminaries

An $(n+2)$-dimensional Lorentzian manifold $\overline{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $\overline{g}$, that is, $\overline{M}$ admits a smooth symmetric tensor field $\overline{g}$ of type $(0,2)$ such that for each point $p \in \overline{M}$, the tensor $\overline{g} : T_p\overline{M} \times T_p\overline{M} \longrightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,\dots,+)$, where $T_p\overline{M}$ denotes the tangent vector space of $\overline{M}$ at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_p\overline{M}$ is said to be *timelike* (resp., *non-spacelike*, *null*, *spacelike*) if it satisfies $\overline{g}_p(v,v) < 0$ (resp., $\leq 0, = 0, > 0$) [6, 8, 16]. Furthermore, the category to which a given vector falls is called its *causal character*. From now on, we denote by $\Gamma(E)$ the module of smooth sections of a vector bundle E over $\overline{M}$.

Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold. A vector field V defined by $\overline{g}(X,V) = A(X)$, for any $X \in \Gamma(T\overline{M})$, is said to be a *concircular* [17, 18] vector field if, for any $X, Y \in \Gamma(T\overline{M})$, we have $(\overline{\nabla}_X A)Y = \alpha[\overline{g}(X,Y) - \omega(X)A(Y)]$, where α is a non-vanishing smooth function and ω is a closed 1-form. Here, $\overline{\nabla}$ denotes the Levi-Civita connection of $\overline{M}$ with respect to $\overline{g}$. Suppose that $\overline{M}$ admits a unit timelike concircular vector field ζ, called the *characteristic* vector field of the manifold. Then, we have

$$\overline{g}(\zeta,\zeta) = -1. \tag{2.1}$$

Since ζ is a unit concircular vector field, it follows that there exists a non-zero 1-form θ such that for

$$\overline{g}(X,\zeta) = \theta(X), \;\; \forall\, X \in \Gamma(T\overline{M}), \tag{2.2}$$

and the following relation holds

$$(\overline{\nabla}_X \theta)Y = \alpha[\overline{g}(X,Y) + \theta(X)\theta(Y)], \tag{2.3}$$

for all $X, Y \in \Gamma(T\overline{M})$, and α is a non-vanishing smooth function satisfying

$$X\alpha = d\alpha(X) = \rho\theta(X). \tag{2.4}$$

Here, ρ is a smooth function given by $\rho = -\zeta\alpha$.

Let us put

$$\overline{\phi}X = (1/\alpha)\overline{\nabla}_X\zeta, \quad \forall\, X \in \Gamma(T\overline{M}). \tag{2.5}$$

Then, by (2.3) and (2.5), we have

$$\overline{\phi}X = X + \theta(X)\zeta, \tag{2.6}$$

which follows that $\overline{\phi}$ is a symmetric $(1,1)$ tensor field called the *structure* tensor field of the manifold. Thus, the Lorentzian manifold $\overline{M}$ together with the unit timelike concircular vector field ζ, its associated 1-form θ and a $(1,1)$ tensor field $\overline{\phi}$ is said to be a *Lorentzian concircular structure* manifold (briefly, an (LCS)-manifold) [17, 18]. In particular, if $\alpha = 1$, then we obtain the LP-Sasakian structure of Matsumoto [14]. In an (LCS)-manifold, the following relations hold for all vector fields $X, Y \in \Gamma(T\overline{M})$:

$$\overline{\phi}^2 X = X + \theta(X)\zeta, \quad \overline{\phi}\zeta = 0, \quad \theta \circ \overline{\phi} = 0, \quad \theta(\zeta) = -1, \tag{2.7}$$

$$\overline{g}(\overline{\phi}X, \overline{\phi}Y) = \overline{g}(X,Y) + \theta(X)\theta(Y), \tag{2.8}$$

$$(\overline{\nabla}_X\overline{\phi})Y = \alpha[\overline{g}(X,Y)\zeta + 2\theta(X)\theta(Y)\zeta + \theta(Y)X], \quad \overline{\nabla}_X\zeta = \alpha\overline{\phi}X. \tag{2.9}$$

Let (M, g) be a null hypersurface of $\overline{M}$. This means that, at each $x \in M$ the restriction $g = \overline{g}|_{T_xM}$ is *degenerate*. That is; there exists a non-zero $U \in T_xM$ such that $g(U, X) = 0$, for all $X \in T_xM$. Therefore, in null setting, the normal bundle $TM^{\perp}$ of the null hypersurface M is a rank 1 vector subbundle of the tangent bundle TM, contrary to the classical theory of non-degenerate hypersurfaces for which the normal bundle has trivial intersection $\{0\}$ with the tangent one and plays an important role in the introduction of the main induced geometric objects on M. The approach of [6, 8], which we adopt here, consist of fixing, on the null hypersurface, a geometric data formed by a null section and a *screen distribution*. By screen distribution on M, we mean a complimentary bundle of $TM^{\perp}$ in TM. It is then a rank n non-degenerate distribution over M. In fact, there are infinitely many possibilities of choices for such a distribution provided the hypersurface M be paracompact, but each of them is canonically isomorphic to the factor vector bundle $TM/TM^{\perp}$ [12]. We denote by $S(TM)$ the screen distribution over M. Then we have the decomposition

$$TM = S(TM) \perp TM^{\perp}, \tag{2.10}$$

where $\perp$ denotes the orthogonal direct sum. From [6] or [8], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \;\; \overline{g}(N, N) = \overline{g}(N, W) = 0, \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.11}$$

It then follows that

$$T\overline{M}|_M = S(TM) \perp \{TM^\perp \oplus tr(TM)\} = TM \oplus tr(TM), \tag{2.12}$$

where $\oplus$ denote the direct (non-orthogal) sum. We call $tr(TM)$ a (null) transversal vector bundle along M. In fact, from (2.11) and (2.12) one shows that, conversely, a choice of a transversal bundle $tr(TM)$ determines uniquely the screen distribution $S(TM)$. A vector field N as in (2.11) is called a null transversal vector field of M. It is then noteworthy that the choice of a null transversal vector field N along M determines both the null transversal vector bundle, the screen distribution $S(TM)$ and a unique radical vector field, say ξ, satisfying (2.11). The name screen distribution is justified as follows; in the case M is a null cone of a 4-dimensional Lorentzian manifold, the integral curves of vector fields in $TM^\perp$ are null (lightlike) rays and fibers of $S(TM)$ can be visualized as screen that are transversal to these rays.

Let ∇ and ∇^* denote the induced connections on M and $S(TM)$, respectively, and P be the projection of TM onto $S(TM)$, then the local Gauss-Weingarten equations of M and $S(TM)$ are the following [6]

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.13}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.14}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.15}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \quad A^*_\xi \xi = 0, \tag{2.16}$$

for all $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$. In the above setting, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively, while τ is a 1-form on TM. The above shape operators are related to their local fundamental forms by

$$g(A^*_\xi X, Y) = B(X,Y), \quad g(A_N X, PY) = C(X,PY), \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$. Moreover, $\overline{g}(A^*_\xi X, N) = 0$, and $\overline{g}(A_N X, N) = 0$, for all $X \in \Gamma(TM)$. From these relations, we notice that A^*_ξ and A_N are both screen-valued operators. Let $\vartheta = \overline{g}(N,\cdot)$ be a 1-form metrically equivalent to N defined on $\overline{M}$. Take $\eta = i^*\vartheta$ to be its restriction on M, where $i : M \to \overline{M}$ is the inclusion map. Then it is easy to show that

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.18}$$

for all $X, Y, Z \in \Gamma(TM)$. Consequently, ∇ is generally *not* a metric connection with respect to g. However, the induced connection ∇^* on $S(TM)$ is a metric connection.

Denote by $\overline{R}$ the curvature tensor of the connection $\overline{\nabla}$ on $\overline{M}$. Using the Gauss-Weingarten formulae (2.13)–(2.16), we obtain the following curvature relations (see details in [6, 8]).

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,\xi) =&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z)\\ &- \tau(Y)B(X,Z),\end{aligned} \tag{2.19}$$

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)PZ,N) =&(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \tau(X)C(Y,PZ)\\ &+ \tau(Y)C(X,PZ),\end{aligned} \tag{2.20}$$

for all $X, Y, Z \in \Gamma(TM)$, $\xi \in \Gamma(TM^{\perp})$ and $N \in \Gamma(tr(TM))$.

3. Classification Results

Let us start off with the following important observation. Let (M, g) be a null hypersurface of a (LCS)-manifold $(\overline{M}, \overline{g})$. As $\overline{M}$ is Lorenztian, any screen distribution over M is Riemannian. Next, since ζ is a global vector field of $\overline{M}$, we decompose it as follows;

$$\zeta = W + a\xi + bN, \tag{3.1}$$

where a and b are smooth functions given by

$$a = \theta(N) \quad \text{and} \quad b = \theta(\xi), \tag{3.2}$$

and W a smooth section tangent to $S(TM)$. Using (3.1) and (2.1), we have $g(W, W) + 2ab = -1$. Suppose that a or b vanishes, then it follows that $g(W, W) = -1$. Consequently, W is a unit timelike vector field of $S(TM)$. This is a contradiction as $S(TM)$ is Riemannian. Thus, we have the following.

Lemma 3.1. *There exist no null hypersurface of a (LCS)-manifold $(\overline{M}, \overline{g})$ such that ζ is tangent or transversal to M.*

In the theory of null hypersurfaces of Sasakian manifolds, it is possible to select a screen distribution $S(TM)$ containing $\overline{\phi}TM^{\perp}$ and $\overline{\phi}tr(TM)$ as subbundles. See [8, 11, 10] for details. This does not hold for null hypersurfaces of a (LCS)-manifold $(\overline{M}, \overline{g})$. More precisely, we have the following result.

Theorem 3.2. *Let (M, g) be a null hypersurface of a (LCS)-manifold $(\overline{M}, \overline{g})$. Then $\overline{\phi}TM^{\perp} \not\subset S(TM)$ and $\overline{\phi}tr(TM) \not\subset S(TM)$. Moreover, $\overline{\phi}TM^{\perp} \cap TM^{\perp} = \{0\}$ and $\overline{\phi}tr(TM) \cap tr(TM) = \{0\}$.*

Proof. First, we not that $\overline{\phi}\xi$ or $\overline{\phi}N$ are non-zero sections. In fact, suppose that $\overline{\phi}\xi = 0$ then using (2.8) and (3.2), we have $0 = \overline{g}(\overline{\phi}\xi, \overline{\phi}\xi) = b^2$. This, together with (3.1), suggests that ζ is tangent to M. This is a contraction in view of Lemma 3.1. In the same way, one can show that $\overline{\phi}N \neq 0$. Next, assume that $\overline{\phi}\xi$ belongs to $S(TM)$. Then, by (2.6) and (2.10), we have $0 = \overline{g}(\overline{\phi}\xi, \xi) = b^2$, from which ζ turns out to be tangent to M which is a contradiction by Lemma 3.1. In the same way, if $\overline{\phi}N$ belongs to $S(TM)$ we see from (2.11) and (2.6) that $0 = \overline{g}(\overline{\phi}N, N) = a^2$. In this case (3.1) gives $\zeta = W + bN$. As ζ is timelike, we obtain $g(W, W) = -1$.

This is a contradiction as $S(TM)$ is Riemannian. Therefore, $\overline{\phi}TM^{\perp} \not\subset S(TM)$ and $\overline{\phi}tr(TM) \not\subset S(TM)$. Next, suppose that $\overline{\phi}TM^{\perp} \cap TM^{\perp} \neq \{0\}$. Then there exist a non-vanishing smooth function β such that $\overline{\phi}\xi = \beta\xi$. Applying $\overline{\phi}$ to this equation and using (2.7), we get $(\beta^2-1)\overline{\phi}\xi = b\zeta$. Applying $\overline{\phi}$ to this last relation and considering (2.7), one gets $\beta(\beta^2-1)\xi = 0$, which implies that $\beta = \pm 1$ since $\beta \neq 0$. Now let $\beta = -1$ such that $\overline{\phi}\xi + \xi = 0$. Taking the inner product of this relation with ξ and using (2.6), we get $b^2 = 0$. This is a contradiction since by Lemma 3.1, ζ can not be tangent to M. On the other hand, if $\beta = 1$ we get $\overline{\phi}\xi = \xi$. But (2.6) implies $\overline{\phi}\xi = \xi + b\zeta$. From these two relations we deduce that $b\zeta = 0$. Since ζ is a unit timelike vector field, the last relation gives $b = 0$, which is a contradiction as before. Thus, $\overline{\phi}TM^{\perp} \cap TM^{\perp} = \{0\}$. In the same way, one can show that $\overline{\phi}tr(TM) \cap tr(TM) = \{0\}$, which completes the proof. □

Using the language of [11], we will say that a null hypersurface (M, g) of a (LCS)-manifold $(\overline{M}, \overline{g})$ is *ascreen* if the characteristic vector field ζ belongs to $S(TM)^{\perp}[= TM^{\perp} \oplus tr(TM)]$. Equivalently, M is ascreen if $W = 0$. On an ascreen null hypersurface, the following holds.

Theorem 3.3. *Let (M, g) be a null hypersurface of a (LCS)-manifold $\overline{M}$. Then M is an ascreen null hypersurface of $\overline{M}$ if and only if $\overline{\phi}TM^{\perp} = \overline{\phi}tr(TM)$.*

Proof. Following the method of [11], suppose that M is ascreen null hypersurface, then (2.1) reduces to $\zeta = a\xi + bN$, where $a = \theta(N)$ and $b = \theta(\xi)$ are non-vanishing smooth functions. Applying $\overline{\phi}$ to this relation and using the fact that $\overline{\phi}\zeta = 0$, we get $\overline{\phi}\xi + b\overline{\phi}N = 0$. Thus, one gets $\overline{\phi}\xi = \omega\overline{\phi}N$, where $\omega = -\frac{b}{a} \neq 0$, a non vanishing smooth function. This implies that $\overline{\phi}TM^{\perp} \cap \overline{\phi}tr(TM) \neq \{0\}$. Since rank $\overline{\phi}TM^{\perp} =$ rank $\overline{\phi}tr(TM) = 1$, it follows that $\overline{\phi}TM^{\perp} = \overline{\phi}tr(TM)$. Conversely, suppose that $\overline{\phi}TM^{\perp} = \overline{\phi}tr(TM)$. Then, there exists a non-vanishing smooth function ω such that $\overline{\phi}\xi = \omega\overline{\phi}N$. Taking the inner product of this relation with respect to $\overline{\phi}\xi$ and $\overline{\phi}N$ in turn, we get $b^2 = \omega(ab+1)$ and $\omega a^2 = ab + 1$, respectively. Since $\omega \neq 0$, we have $a \neq 0$, $b \neq 0$ and $b^2 = (\omega a)^2$. The latter gives $b = \pm\omega a$. The case $b = \omega a$ implies that $ab = \omega a^2 = ab + 1$, which is a contradiction. Thus $b = -\omega a$, from which $2ab = -1$. Since $\omega = -\frac{b}{a}$, $a \neq 0$ and $\overline{\phi}\xi = \omega\overline{\phi}N$, it is easy to see that $a\overline{\phi}\xi + b\overline{\phi}N = 0$. Applying $\overline{\phi}$ to this equation, and using $b^2 = \omega(ab+1)$ together with $2ab = -1$, we get $\zeta = a\xi + bN$. Therefore, M is ascreen null hypersurface of $\overline{M}$, which completes the proof. □

Now, we will construct an example of this class of hypersurface in (LCS)-manifold.

Example 3.4. Consider a 3-dimensional manifold $\overline{M} = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on $\overline{M}$ given by $E_1 = e^{-z}(\partial_x + y\partial_y)$, $E_2 = e^{-z}\partial_y$ and $E_3 = e^{-2z}\partial_z$. Let $\overline{g}$ be the Lorentzian metric defined by $\overline{g}(E_1, E_2) = \overline{g}(E_1, E_3) = \overline{g}(E_2, E_3) = 0$, $\overline{g}(E_1, E_1) = \overline{g}(E_2, E_2) = 1$ and $\overline{g}(E_3, E_3) = -1$. Let θ be the 1-form defined by $\theta(X) = \overline{g}(X, E_3)$, for any $X \in \Gamma(T\overline{M})$. Let $\overline{\phi}$ be a (1,1) tensor field defined by

$\overline{\phi}E_1 = E_1$, $\overline{\phi}E_2 = E_2$ and $\overline{\phi}E_3 = E_3$. Then using the linearity of $\overline{\phi}$ and $\overline{g}$ we have $\theta(E_3) = -1$, $\overline{\phi}^2 X = X + \theta(X)E_3$ and $\overline{g}(\overline{\phi}X, \overline{\phi}Y) = \overline{g}(X, Y) + \theta(X)\theta(Y)$, for any $X, Y \in \Gamma(T\overline{M})$. Thus, for $\zeta = E_3$, $(\overline{\phi}, \zeta, \theta, \overline{g})$ defines a Lorentzian paracontact structure on $\overline{M}$, which is also a (LCS)-structure. Consequently, $(\overline{M}, \overline{\phi}, \zeta, \theta, \overline{g})$ is a (LCS)-manifold with $\alpha = e^{-2z}$ such that $X\alpha = \rho\theta(X)$, where $\rho = 2e^{-4z}$ (see [18]). Now, we define a null hypersurface M of $(\overline{M}, \overline{\phi}, \zeta, \theta, \overline{g})$ as $M = \{(x, y, z) \in \mathbb{R}^3 : y = z\}$. Thus, TM is spanned by $U_1 = E_1$ and $U_2 = \partial_y + \partial_z$, in which $TM^\perp$ is spanned by $\xi = U_2$. The transversal bundle $tr(TM)$ is spanned by N, where $N = (\alpha/2)(\partial_y - \partial_z)$. By using the definition of $\overline{\phi}$, we have $\overline{\phi}N = (\alpha/2)\overline{\phi}\xi$. Hence, by Theorem 3.3 we have $\overline{\phi}TM^\perp = \overline{\phi}tr(TM)$. Next, applying $\overline{\phi}$ to $\zeta = a\xi + bN$ and considering the previous relations, we get $2a + \alpha b = 0$. Since $2ab + 1 = 0$, we obtain $\zeta = (\sqrt{\alpha}/2)\xi - (1/\sqrt{\alpha})N$. Therefore, M is an ascreen null hypersurface of $\overline{M}$.

A null hypersurface (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called *totally umbilic* [6, p. 106] if there exist a smooth function μ on a coordinate neighborhood $\mathcal{U} \subset M$ such that $A^*_E X = \mu PX$, or equivalently,

$$B(X, PY) = \mu g(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.3}$$

In case $\mu = 0$ on $\mathcal{U}$, we say that M is *totally geodesic* otherwise it is *proper* totally umbilic in $\overline{M}$. Furthermore, M is *screen totally umbilic* [6, p. 109] if there exist a smooth function ϱ on a coordinate neighborhood $\mathcal{U} \subset M$ such that $A_N X = \varrho PX$, or equivalently,

$$C(X, PY) = \varrho g(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.4}$$

In case $\varrho = 0$ on $\mathcal{U}$, we say that M is *screen totally geodesic* otherwise it is *proper* screen totally umbilic in $\overline{M}$.

Let (M, g) be a null hypersurface. We say that M is *screen conformal* [8, p. 60] if there exist a non-vanishing smooth function ψ on a coordinate neighborhood $\mathcal{U}$ such that $A_N X = \psi A^*_E X$, or equivalently,

$$C(X, PY) = \psi B(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.5}$$

In case ψ is a constant function, we say that M is *screen homothetic*.

Suppose that M is an ascreen null hypersurface of $\overline{M}$. In view of (3.1), we have $\zeta = a\xi + bN$, where a and b are given by (3.2). Differentiating this relation and using (2.14) and (2.16), we get

$$\alpha\overline{\phi}X = -aA^*_\xi X - bA_N X + [Xa - a\tau(X)]\xi + [Xb + b\tau(X)]N, \tag{3.6}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (3.6) with N and ξ, in turn, we get

$$Xa - a\tau(X) = \alpha\eta(X) + \alpha a\theta(X), \quad Xb + b\tau(X) = \alpha b\theta(X), \tag{3.7}$$

in which we have used the fact $X = PX + \eta(X)\xi$, for any $X \in \Gamma(TM)$. Replacing X with ξ in (3.7), gives

$$\xi a - a\tau(\xi) = (1/2)\alpha \quad \text{and} \quad \xi b + b\tau(\xi) = \alpha b^2. \tag{3.8}$$

On the other hand, taking the inner product of (3.6) with PY, where $Y \in \Gamma(TM)$, we have

$$aB(X, PY) + bC(X, PY) = -\alpha g(PX, PY). \tag{3.9}$$

Notice from (3.9) that C is symmetric on $S(TM)$. Setting $X = \xi$ in (3.9) and using (2.16) together with the fact $b \neq 0$, we get

$$C(\xi, PY) = 0, \quad \forall Y \in \Gamma(TM). \tag{3.10}$$

In view of (3.3), (3.4), (3.5), (3.7), (3.9) and (3.10), we have the following result.

Theorem 3.5. *Let (M, g) be an ascreen null hypersurface of a (LCS)-manifold $\overline{M}$. Then the following holds;*

(1) *zero is an eigenvalue of A_N with respect to ξ,*

(2) *M is a screen integrable null hypersurface and locally isometric to $\mathcal{C}_\xi \times M'$, where $\mathcal{C}_\xi$ is a null curve tangent to $TM^\perp$ and M' is a leaf of $S(TM)$,*

(3) *a screen conformal or screen totally umbilic M is proper totally umbilic,*

(4) *a screen totally geodesic M is never totally geodesic,*

(5) *there exist no M with a or b constant and $\tau = 0$.*

Proof. Parts (1), (3), (4) and (5) are obvious. Part (2) follows from (3.9) and similar arguments as in [7], and the proof is completed. □

Remark 3.6. In [11], it was proved (see Theorem 3.4 therein) that a Sasakian manifold does not admit any ascreen null hypersurfaces which are screen totally umbilic or screen conformal. We remark, based on Theorem 3.5, that this is not the case with ascreen null hypersurfaces of (LCS)-manifolds. Futhermore, it was shown, by the same author in Theorem 3.5, that $S(TM)$ is never parallel, and if $\dim \overline{M} > 3$ then $S(TM)$ is not integrable. We note once again that this is not the same with ascreen null hypersurfaces of (LCS)-manifolds.

It is well-known that the Ricci tensor of null hypersurface (and generally of null submanifold) is not symmetric. This is because the induced connection is not a metric connection (see relation (2.18)). In line with the above, the authors in [6, p. 99] (also see [8]) proves the following result.

Theorem 3.7.([6]) *Let (M, g) be a null hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the Ricci tensor of the induced connection ∇ is symmetric, if and only if, each 1-form τ induced by $S(TM)$ is closed, i.e., $d\tau = 0$, on $\mathcal{U} \subset M$.*

With reference to Theorem 3.7, we have the following result.

Theorem 3.8. *The Ricci tensor of an ascreen null hypersurface (M, g) of a (LCS)-manifold $(\overline{M}, \overline{g})$ is symmetric.*

Proof. By the second relation in (3.7) and the fact that $b \neq 0$, we have

$$\tau(X) = \alpha\theta(X) - \frac{Xb}{b}, \quad \forall\, X \in \Gamma(TM). \tag{3.11}$$

Differentiating (3.11) and using (2.9) and (2.13), we get

$$\begin{aligned} Y\tau(X) &= (Y\alpha)\theta(X) + \alpha\theta(\nabla_Y X) + a\alpha B(X, Y) \\ &\quad + \alpha^2\overline{g}(X, \overline{\phi}Y) - \frac{Y(Xb)}{b} + \frac{(Xb)(Yb)}{b^2}, \end{aligned} \tag{3.12}$$

for any $X, Y \in \Gamma(TM)$. Interchanging X and Y in (3.12) and subtracting (3.12) from the new relation, we get

$$\begin{aligned} X\tau(Y) - Y\tau(X) &= (X\alpha)\theta(Y) - (Y\alpha)\theta(X) + \alpha\theta([X, Y]) \\ &\quad + \frac{Y(Xb)}{b} - \frac{X(Yb)}{b}, \end{aligned} \tag{3.13}$$

in which we have used the symmetries of B and $\overline{\phi}$. On the other hand, (3.11) gives

$$\tau([X, Y]) = \alpha\theta([X, Y]) - \frac{[X, Y]b}{b}. \tag{3.14}$$

Then, by (3.13), (3.14) and the definition of $d\tau$, we get

$$\begin{aligned} 2d\tau(X, Y) &= [X\tau(Y) - Y\tau(X) - \tau([X, Y])] \\ &= (X\alpha)\theta(Y) - (Y\alpha)\theta(X), \end{aligned} \tag{3.15}$$

in which we have considered the fact $Y(Xb) - X(Yb) + [X, Y]b = 0$. In view of (2.4), we have $X\alpha = \rho\theta(X)$, and hence (3.15) reduces to

$$2d\tau(X, Y) = \rho[\theta(X)\theta(Y) - \theta(Y)\theta(X)] = 0. \tag{3.16}$$

Finally, considering (3.16) and Theorem 3.7, we see that the Ricci tensor of M is symmetric. This completes the proof. □

Remark 3.9. A similar conclusion, as in Theorem 3.8, can be arrived at if the first relation of (3.7), i.e., $\tau(X) = [Xa - \alpha\eta(X) - a\alpha\theta(X)]/a$ is used. In fact, differentiating this relation with respect to $Y \in \Gamma(TM)$ and considering (2.9), (2.13), (2.14) and (2.17), we derive

$$\begin{aligned} Y\tau(X) &= \frac{Y(Xa)}{a} - \frac{(Xa)(Ya)}{a^2} - \eta(X)Y\left(\frac{\alpha}{a}\right) - \frac{\alpha}{a}\eta(\nabla_Y X) \\ &\quad - \frac{\alpha}{a}\tau(Y)\eta(X) + \frac{\alpha}{a}C(Y, PX) - a\alpha B(X, Y) - \alpha\theta(\nabla_Y X) \\ &\quad - \alpha^2\overline{g}(X, \overline{\phi}Y) - (Y\alpha)\theta(X). \end{aligned} \tag{3.17}$$

Interchanging X and Y in (3.17) and subtracting, while considering the facts B, $\overline{\phi}$ are symmetric and $X\alpha = \rho\theta(X)$ (see (2.4)), we get

$$\begin{aligned} X\tau(Y) - Y\tau(X) &= \frac{X(Ya)}{a} - \frac{Y(Xa)}{a} + Y\left(\frac{\alpha}{a}\right)\eta(X) - X\left(\frac{\alpha}{a}\right)\eta(Y) \\ &\quad - \frac{\alpha}{a}\eta([X,Y]) + \frac{\alpha}{a}\tau(X)\eta(Y) - \frac{\alpha}{a}\tau(Y)\eta(X) \\ &\quad + \frac{\alpha}{a}C(X,PY) - \frac{\alpha}{a}C(Y,PX) - \alpha\theta([X,Y]). \end{aligned} \tag{3.18}$$

On the other hand, using the expression for Xa in (3.7), we derive

$$\begin{aligned} Y\left(\frac{\alpha}{a}\right)\eta(X) - X\left(\frac{\alpha}{a}\right)\eta(Y) &= \left[\frac{Y\alpha}{a} - \frac{\alpha^2}{a}\theta(Y)\right]\eta(X) \\ &\quad - \left[\frac{X\alpha}{a} - \frac{\alpha^2}{a}\theta(X)\right]\eta(Y) + \frac{\alpha}{a}\tau(X)\eta(Y) - \frac{\alpha}{a}\tau(Y)\eta(X). \end{aligned} \tag{3.19}$$

Also, by (3.9), we have

$$\tau([X,Y]) = \frac{[X,Y]a}{a} - \frac{\alpha}{a}\eta([X,Y]) - \alpha\theta([X,Y]) \tag{3.20}$$

Thus, putting (3.18), (3.19), (3.20) and (2.4) together, we derive

$$\begin{aligned} 2d\tau(X,Y) &= \left[\frac{Y\alpha}{a} - \frac{\alpha^2}{a}\theta(Y)\right]\eta(X) - \left[\frac{X\alpha}{a} - \frac{\alpha^2}{a}\theta(X)\right]\eta(Y) \\ &+ \frac{\alpha}{a}[C(X,PY) - C(Y,PX)]. \\ &= \frac{\alpha^2 - \rho}{a}[\theta(X)\eta(Y) - \theta(Y)\eta(X)] + \frac{\alpha}{a}[C(X,PY) - C(Y,PX)]. \end{aligned} \tag{3.21}$$

Observe that the right hand side of (3.21) vanishes on M since $S(TM)$ is integrable (see Theorem 3.5) and that $C(\xi, PX) = 0$, for all $X \in \Gamma(TM)$, by (3.10). That is $d\tau = 0$ and thus, by Theorem 3.8, M admits a symmetric Ricci tensor.

Let $\tilde{\xi} = \lambda\xi$, then it follows that $\tilde{N} = (1/\lambda)N$. Moreover, $\tilde{B} = B$ and

$$\tau(X) = \tilde{\tau}(X) + X\ln(\lambda), \quad \forall\, X \in \Gamma(TM_{|\mathcal{U}}). \tag{3.22}$$

This shows that B and τ depend on the section ξ on $\mathcal{U}$. By Theorem 3.8 and Poincaré's lemma we obtain $\tau(X) = Xf$, where f is a smooth function on $\mathcal{U}$. Let $\lambda = \exp(f)$ in (3.22), then, $\tilde{\tau} = 0$ on $\mathcal{U}$. Thus, we have

Corollary 3.10. *Let (M, g) be an ascreen null hypersurface of a (LCS)-manifold $(\overline{M}, \overline{g})$. There exist a pair $\{\xi, N\}$ on $\mathcal{U}$ such that the corresponding 1-form τ from the Weingarten equation vanishes.*

A semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant sectional curvature c is called a semi-Riemannian space form (see [16, p. 80]) and denoted by $\overline{M}(c)$. The curvature tensor field $\overline{R}$ of $\overline{M}(c)$ is given by

$$\overline{R}(X,Y)Z = c\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\}, \ \ \forall\, X,Y,Z \in \Gamma(T\overline{M}). \tag{3.23}$$

Let $\overline{M}(c)$ be a (LCS)-manifold of constant curvature c. Then by Proposition 3.1 of [18, p. 451], c satisfies $c = (\alpha^2 - \rho)$. Using this fact together with (3.9) and (3.23), we get the following result.

Theorem 3.11. *Let (M, g) be a screen totally geodesic ascreen null hypersurface of a (LCS)-space form $\overline{M}(\alpha^2 - \rho)$ then $\alpha^2 = \rho$. Moreover, M is a proper totally umbilic null hypersurface whose immersion into $\overline{M}$ is affinely equivalent to the graph immersion of a certain function $F : M \longrightarrow \mathbb{R}$.*

Proof. A proof follows easily as in [6]. □

The notion of *quasi-constant curvature* was introduced by Chen and Yano [4]. Moreover, it was shown in [18] that a *conformally flat* (LCS)-manifold $(\overline{M}, \overline{g})$ is of quasi-constant curvature and its curvature tensor $\overline{R}$ satisfies

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= c_1[\overline{g}(Y,Z)\overline{g}(X,W) - \overline{g}(X,Z)\overline{g}(Y,W)]\\ &+c_2[\overline{g}(X,W)\theta(Y)\theta(Z) - \overline{g}(Y,W)\theta(X)\theta(Z) + \overline{g}(Y,Z)\theta(X)\theta(W)\\ &- \overline{g}(X,Z)\theta(Y)\theta(W)], \quad \forall\, X,Y,Z,W \in \Gamma(T\overline{M}),\end{aligned} \tag{3.24}$$

where c_1 and $c_2(\neq 0)$ are smooth functions. We denote such a manifold by $\overline{M}(c_1, c_2)$. The following result holds.

Theorem 3.12. *Let (M, g) be an ascreen null hypersurface of a conformally flat (LCS)-manifold $\overline{M}(c_1, c_2)$ of dimension > 3. Then, M is totally umbilic if and only if the umbiliciy factor μ in (3.3) is a solution of the following differential equations*

$$\xi\mu + \mu\tau(\xi) - \mu^2 - c_2 b^2 = 0 \ \ \textit{and} \ \ PX\mu + \mu\tau(PX) = 0, \ \ \forall\, X \in \Gamma(TM).$$

Moreover, if M is totally umbilic then it is a screen totally umbilic null hypersurface with screen umbilicity factor $\varrho := -(\alpha + a\mu)/b$ in (3.4) satisfying the following pair of differential equations

$$\xi\varrho - \varrho\tau(\xi) - \mu\varrho - c_1 + \frac{c_2}{2} = 0 \ \ \textit{and} \ \ PX\varrho - \varrho\tau(PX) = 0, \ \ \forall\, X \in \Gamma(TM).$$

Proof. Replacing W, Z by ξ, PZ, respectively, in (3.24) and using the fact $b = \theta(\xi)$, we get

$$\overline{R}(X,Y,PZ,\xi) = c_2 b\theta(X)g(Y,PZ) - c_2 b\theta(Y)g(X,PZ), \tag{3.25}$$

for all $X, Y, Z \in \Gamma(TM)$. On the other hand, using (2.19), (2.15), (2.16), (2.18) and (3.3), we get

$$\overline{R}(X,Y,PZ,\xi) = [X\mu + \mu\tau(X) - \mu^2\eta(X)]g(Y,PZ) - [Y\mu + \mu\tau(Y) - \mu^2\eta(Y)]g(X,PZ). \tag{3.26}$$

In view of (3.25) and (3.26), we have

$$[X\mu + \mu\tau(X) - \mu^2\eta(X) - c_2 b\theta(X)]g(Y,PZ) = [Y\mu + \mu\tau(Y) - \mu^2\eta(Y) - c_2 b\theta(Y)]g(X,PZ), \tag{3.27}$$

for any $X, Y, Z \in \Gamma(TM)$. Replacing X (or Y) by ξ, we get $\xi\mu+\mu\tau(\xi)-\mu^2-c_2b^2 = 0$. Again replacing X, Y by PX, PY, respectively, in (3.27) and using the fact that $S(TM)$ is non-degenerate, we have

$$[PX\mu + \mu\tau(PX)]PY = [PY\mu + \mu\tau(PY)]PX. \tag{3.28}$$

Next, suppose that there exist $X' \in \Gamma(TM_{|\mathcal{U}})$ such that $PX'\mu + \mu\tau(PX') \neq 0$ at $x \in M$. It then follows from (3.28) that all vectors from the fiber $S(TM)_x$ are collinear with $(PX')_x$. This is a contradiction as $\dim(S(TM))_x > 1$, which proves the first part of the theorem.

Observe, from (3.9), that M is also screen totally umbilic with $\varrho := -(\alpha+a\mu)/b$. Following the same steps as above, this time with (2.20), (2.15), (2.18), (3.4), (3.10) and (3.24), we get

$$[X\varrho - \varrho\tau(X) - \varrho\mu\eta(X) - c_1\eta(X) - c_2 a\theta(X)]g(Y,PZ) = [Y\varrho - \varrho\tau(Y) - \varrho\mu\eta(Y) - c_1\eta(Y) - c_2 a\theta(Y)]g(X,PZ), \tag{3.29}$$

for all $X, Y, Z \in \Gamma(TM)$. Replacing X (or Y) by ξ in (3.29) and using the fact $b = \theta(\xi)$, we get $\xi\varrho - \varrho\tau(\xi) - \varrho\mu - c_1 - c_2ab = 0$. As M is ascreen, we have $2ab+1 = 0$. Considering these two relations, we have $\xi\varrho - \varrho\tau(\xi) - \mu\varrho - c_1 + \frac{c_2}{2} = 0$. The last differential equation follows as in (3.28) and the following arguments. This completes the proof. □

It then follows from Theorem 3.12 that the following result holds.

Corollary 3.13.

(1) *There exist no totally geodesic ascreen null hypersurfaces of a conformally flat (LCS)-manifold $\overline{M}(c_1, c_2)$.*

(2) *There exist no screen totally geodesic ascreen null hypersurfaces of a conformally flat (LCS)-manifold $\overline{M}(c_1, c_2)$ with $c_1 = 0$.*

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