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Global Nonexistence of Solutions for a Quasilinear Wave Equation with Time Delay and Acoustic Boundary Conditions

Yong Han Kang*
Francisco College, Daegu Catholic University, Gyeongsan-si 712-702, Republic of Korea
e-mail : yonghann@cu.ac.kr

Jong-Yeoul Park
Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea
e-mail : jyepark@pusan.ac.kr

Abstract. In this paper, we prove the global nonexistence of solutions for a quasilinear wave equation with time delay and acoustic boundary conditions. Further, we establish the blow up result under suitable conditions.

1. Introduction

In this paper, we consider the following quasilinear wave equation with time delay and acoustic boundary conditions:

$$
\begin{aligned}
&(|u_t(x,t)|^{l-2}u_t(x,t))_t - \triangle u_t(x,t) - div(a(x)|\nabla u(x,t)|^{\alpha-2}\nabla u(x,t)) \\
&\quad -div(|\nabla u_t(x,t)|^{\beta-2}\nabla u_t(x,t)) + Q(x,t,u_t) + \mu_1 u_t(x,t) \\
&\quad +\mu_2 u_t(x,t-\tau) = f(x,u(x,t)) \text{ in } \Omega\times[0,T), \qquad (1.1)
\end{aligned}
$$

$$
u = 0 \text{ on } \Gamma_0\times[0,T), \qquad (1.2)
$$

$$
\begin{aligned}
&\frac{\partial u_t(x,t)}{\partial\nu} + a(x)|\nabla u(x,t)|^{\alpha-2}\frac{\partial u(x,t)}{\partial\nu} \\
&\quad +|\nabla u_t(x,t)|^{\beta-2}\frac{\partial u_t(x,t)}{\partial\nu} = h(x)y_t(x,t) \text{ on } \Gamma_1\times[0,T), \qquad (1.3)
\end{aligned}
$$

$$
u_t(x,t) + k(x)y_t(x,t) + q(x)y(x,t) = 0 \text{ on } \Gamma_1\times[0,T), \qquad (1.4)
$$

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$$(1.5)\qquad u(x,0)=u_0(x),\;\; u_t(x,0)=u_1(x) \text{ in } \Omega,$$
$$(1.6)\qquad u_t(x,t-\tau)=f_0(x,t-\tau) \text{ in } \Omega\times(0,\tau),$$
$$(1.7)\qquad y(x,0)=y_0(x) \text{ on } \Gamma_1.$$

Here, $J=[0,T)$, $0<T\leq\infty$, $a:\Omega\longrightarrow R^+$ is a positive function, $l,\alpha,\beta\geq 2$, $\mu_1>0,\mu_2$ is a real number, and $\tau>0$ represents the time delay. Further, Ω is a regular and bounded domain of $R^n(n\geq 1)$ and $\partial\Omega(:=\Gamma)=\Gamma_0\cup\Gamma_1$, where Γ_0 and Γ_1 are closed and disjoint and $\frac{\partial}{\partial\nu}$ denotes the outer normal derivative. The functions $k,q,h:\Gamma_1\longrightarrow R^+(:=[0,\infty])$ are essentially bounded and $0<q_0\leq q(x)$ on Γ_1.

The acoustic boundary conditions were introduced by Morse and Ingard [16] and developed by Beale and Rosencrans in [1], where the authors proved the global existence and regularity of the linear problem. Other authors have studied the existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions (see [3, 4, 6, 7, 12, 13, 15, 19, 20, 23] and the references therein).

The time delay arises in many physical, chemical, biological and economical phenomena because these phenomena depend not only on the present state but also on the past history of the system in a more complicated way. In particular, the effects of time delay strikes on our system have a significant effect on the range of existence and the stability of the system. The differential equations with time delay effects have become an active area of research, see for example [9, 11, 17, 18]. In [14], without the delay term and the acoustic boundary condition, Liu and Wang considered the global nonexistence of solutions with the positive initial energy for a class of wave equations:

$$\begin{aligned}
&(|u_t(x,t)|^{l-2}u_t(x,t))_t-\triangle u_t(x,t)-div(a(x)|\nabla u(x,t)|^{\alpha-2}\nabla u(x,t))\\
&\qquad -div(|\nabla u_t(x,t)|^{\beta-2}\nabla u_t(x,t))+Q(x,t,u_t)\\
&\qquad = f(x,u(x,t)) \text{ in } J\times\Omega,\\
&u(x,t)=0 \text{ on } J\times\partial\Omega,\\
&u(x,0)=u_0(x),\;\; u_t(x,0)=u_1(x) \text{ in } \Omega,
\end{aligned}$$

where $J=[0,T)$, $0<T\leq\infty$, Ω is a bounded regular open subset of $R^n(n\geq 1)$, $l,\alpha,\beta\geq 2$ and a,Q,f satisfy some conditions. Recently, for $l=2, a(x)=1, Q(u_t)=a|u_t|^{m-2}u_t, \mu_1=\mu_2=0$, $f(u)=b|u|^{p-2}u$, and without the time delay term in our system, Jeong at al [8] investigated the global nonexistence of solutions for a quasilinear wave equation with acoustic boundary conditions

$$\begin{aligned}
&u_{tt}-\triangle u_t-div(|\nabla u|^{\alpha-2}\nabla u)-div(|\nabla u_t|^{\beta-2}\nabla u_t)\\
&\qquad +a|u_t|^{m-2}u_t=b|u|^{p-2}u \text{ in } \Omega\times(0,\infty),\\
&u=0 \text{ on } \Gamma_0\times(0,\infty),\\
&\frac{\partial u_t}{\partial\nu}+|\nabla u|^{\alpha-2}\frac{\partial u}{\partial\nu}\\
&\qquad +|\nabla u_t|^{\beta-2}\frac{\partial u_t}{\partial\nu}=h(x)y_t \text{ on } \Gamma_1\times(0,\infty),
\end{aligned}$$

$$u_t + f(x)y_t + q(x)y = 0 \text{ on } \Gamma_1 \times (0,\infty),$$
$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \text{ in } \Omega,$$
$$y(x,0) = y_0(x) \text{ on } \Gamma_1,$$

where $a, b > 0$, $\alpha, \beta, m, p > 2$ are constants and Ω is a regular and bounded domain of $R^n (n \geq 1)$ and $\partial\Omega(=\Gamma) = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint. The functions $h, f, q : \Gamma_1 \to R^+$ are essentially bounded. Moreover, for $a(x) = 1, l = 2, div(|\nabla u_t|^{\beta-2}\nabla u_t) = 0, Q = 0$, and without boundary conditions, Kafini and Messaoudi [10] studied the following nonlinear damped wave equation

$$\begin{aligned}
&u_{tt}(x,t) - div(|\nabla u(x,t)|^{m-2}\nabla u(x,t)) \\
&\qquad + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b|u(x,t)|^{p-2}u(x,t) \text{ in } \Omega \times (0,\infty), \\
&u_t(x,t-\tau) = f_0(x,t-\tau) \text{ on } (0,\tau), \\
&u(x,t) = 0 \text{ on } \partial\Omega \times (0,\infty), \\
&u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \text{ in } \Omega,
\end{aligned}$$

where $p > m \geq 2$, b, μ_1 are positive constants, μ_2 is a real number, and $\tau > 0$ represents the time delay. They proved the blow-up result in a nonlinear wave equation with time delay and without acoustic boundary conditions.

Motivated by the previous works, we consider an equation in a broader and more generalized form than the system discussed above. So we study the global nonexistence of solutions for a quasilinear wave equation with the time delay and acoustic boundary conditions. To the best of our knowledge. there are no results of a quasilinear wave equations with the time delay and acoustic boundary conditions. Thus the result in this work is very meaningful. The main result will be proved in Section 3.

2. Preliminaries

In this section, we shall give some notations, assumptions and a theorem which will be used throughout this paper. We denote by m' the Hölder conjugate of m, $||u||_p = ||u||_{L^p(\Omega)}$, $||u||_{p,\Gamma} = ||u||_{L^p(\Gamma)}$, $||u||_{1,s} = ||u||_{W^{1,s}(\Omega)}$, where $L^p(\Omega)$ and $W^{1,s}(\Omega)$ stand for the Lebesgue spaces and the classical Sobolev spaces, respectively. Specially we introduce the set

$$W^{1,s}_{\Gamma_0}(\Omega) = \{u \in W^{1,s} \mid u = 0 \text{ on } \Gamma_0\}, \ \ W^{1,s}_0(\Omega) = \{u \in W^{1,s} \mid u = 0 \text{ on } \Gamma\}.$$

We make the following same assumptions on a, Q, f as section 4.2 of [22].

(H1) $a(x) \in L^\infty(\Omega)$ such that $a(x) \geq a_0$ a.e. in Ω for some $a_0 > 0$.

(H2) $f(x,u) \in C(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ and $f(x,u) = \nabla_u \Phi(x,u)$, with normalizing condition $\Phi(x,0) = 0$.

There are constants $d_1 > 0, p > \alpha$ and $\mu < \mu_0 a_0$ such that

$$|f(x,u)| \leq \mu|u|^{\alpha-1} + d_1|u|^{p-1} \tag{2.1}$$

for all $x \in \Omega$ and $u \in \mathbb{R}^n$. Moreover, there is $\epsilon_1 > 0$ such that for all $\epsilon \in (0, \epsilon_1]$ there exists $d_2 = d_2(\epsilon) > (p - \alpha)d_1/p$ such that

$$f(x,u)u - (p-\epsilon)\Phi(x,u) \geq d_2|u|^p \tag{2.2}$$

for all $x \in \Omega$.

(H3) There are $m > 1$ and a measurable function $d = d(x,t)$ defined on $\Omega \times J$ such that $d(\cdot, t) \in L^{p/(p-m)}(\Omega)$ for a.e. $t \in J$ and

$$Q(x,t,v)v \geq 0 \tag{2.3}$$

$$|Q(x,t,v)| \leq [d(x,t)]^{1/m}[Q(x,t,v)v]^{1/m'} \tag{2.4}$$

for all values of the arguments x, t, v, where

$$d(x,t) \geq 0, \ \ ||d(\cdot,t)||_{p/(p-m)} \in L^\infty_{loc}(J). \tag{2.5}$$

Remark 2.1. We note that when $Q(x,t,u_t) = b(1+t)^\rho |u_t|^{m-2}u_t$, $-\infty < \rho \leq m-1$, condition (**H3**) holds.

Now, we transform the equation (1.1)–(1.7) to the system, using the idea of [21] and introduce the associated energy. So, we introduce the new variable:

$$z(x,\rho,t) = u_t(x, t - \tau\rho), \quad x \in \Omega, \ \rho \in (0,1), \ t > 0.$$

Thus, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x \in \Omega, \ \rho \in (0,1), \ t > 0.$$

Then problem (1.1)–(1.7) takes the following form:

$$\begin{aligned} &(|u_t(x,t)|^{l-2}u_t(x,t))_t - \triangle u_t(x,t) - div(a(x)|\nabla u(x,t)|^{\alpha-2}\nabla u(x,t)) \\ &\quad -div(|\nabla u_t(x,t)|^{\beta-2}\nabla u_t(x,t)) + Q(x,t,u_t) \\ &+\mu_1 u_t(x,t) + \mu_2 z(x,1,t) = f(x,u(x,t)) \text{ in } \Omega \times J, \end{aligned} \tag{2.6}$$

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in} \ \ \Omega \times (0,1) \times J, \tag{2.7}$$

$$u = 0 \text{ on } \Gamma_0 \times J, \tag{2.8}$$

$$\begin{aligned} &\frac{\partial u_t(x,t)}{\partial \nu} + a(x)|\nabla u(x,t)|^{\alpha-2}\frac{\partial u(x,t)}{\partial \nu} \\ &\quad +|\nabla u_t(x,t)|^{\beta-2}\frac{\partial u_t(x,t)}{\partial \nu} = h(x)y_t(x,t) \text{ on } \Gamma_1 \times J, \end{aligned} \tag{2.9}$$

$$u_t(x,t) + k(x)y_t(x,t) + q(x)y(x,t) = 0 \text{ on } \Gamma_1 \times J, \tag{2.10}$$

$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \text{ in } \Omega, \tag{2.11}$$

$$z(x,\rho,0) = f_0(x, -\rho\tau) \text{ in } \Omega \times (0,1), \tag{2.12}$$

$$y(x,0) = y_0(x) \text{ on } \Gamma_1. \tag{2.13}$$

We introduce the following space

$$Z = L^{\infty}([0,T]; W^{1,\alpha}_{\Gamma_0}(\Omega)) \cap W^{1,\infty}([0,T]; L^2(\Omega))$$
$$\cap W^{1,\beta}([0,T]; W^{1,\beta}_{\Gamma_0}(\Omega)) \cap W^{1,m}([0,T]; L^m(\Omega)), \tag{2.14}$$

for some $T > 0$.

We state, without a proof, a local existence which can be established by combining arguments of [2, 5, 24].

Theorem 2.1. *Let $u_0 \in W^{1,\alpha}_{\Gamma_0}(\Omega), u_1 \in L^2(\Omega), f_0 \in L^2(\Omega \times (0,1))$ and $y_0 \in L^2(\Gamma_1)$ be given. Suppose that $l, \alpha, \beta, m, p > 2$, $\max\{l, \beta, m\} < \alpha < p < n\alpha/(n-\alpha)$, $\mu_1 > |\mu_2|$ and **(H1)–(H3)** hold. Then problem(2.6)–(2.13) has a unique local solution $(u, z, y) \in Z \times L^2([0,T]; L^2(\Omega \times (0,1))) \times L^2([0,T]; L^2(\Gamma_1))$ for some $T > 0$.*

In order to state and prove our result, we introduce the energy functional

$$E(t) = \frac{l-1}{l}\int_{\Omega} |u_t(x,t)|^l dx + \frac{1}{\alpha}\int_{\Omega} a(x)|\nabla u(x,t)|^{\alpha} dx - \int_{\Omega} \Phi(x, u(x,t)) dx$$
$$+\frac{\xi}{2}\int_{\Omega}\int_0^1 z^2(x,\rho,t) d\rho dx + \frac{1}{2}\int_{\Gamma_1} h(x)q(x)y^2(x,t) d\Gamma, \tag{2.15}$$

where

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \quad \mu_1 > |\mu_2|. \tag{2.16}$$

We set

$$\lambda_1 = (A_0 - \tfrac{\mu}{\mu_0})^{1/(p-\alpha)}(d_1 B_1^p)^{-1/(p-\alpha)}, \tag{2.17}$$

$$E_1 = (\tfrac{1}{\alpha} - \tfrac{1}{p})(a_0 - \tfrac{\mu}{\mu_0})^{p/(p-\alpha)}(d_1 B_1^p)^{-\alpha/(p-\alpha)}, \tag{2.18}$$

where B_1 is the best constant of the Sobolev embedding $W^{1,\alpha}_0(\Omega) \hookrightarrow L^p(\Omega)$ given by

$$B_1^{-1} = \inf\{||\nabla u||_{\alpha} : u \in W^{1,\alpha}_0(\Omega), ||u||_p = 1\}.$$

We also set

$$\Sigma = \{(\lambda, E) \in \mathbb{R}^2 | \lambda > \lambda_1, \quad E < E_1\}.$$

3. Proof of Main Result

In this section, we state and prove our main result. Our main result as follows.

Theorem 3.1. *Let $u_0 \in W^{1,\alpha}_{\Gamma_0}(\Omega), u_1 \in L^2(\Omega), f_0 \in L^2(\Omega \times (0,1))$ and $y_0 \in L^2(\Gamma_1)$ be given. Suppose that $l, \alpha, \beta, m, p > 2$, $\max\{l, \beta, m\} < \alpha < p < n\alpha/(n-\alpha)$, $\mu_1 > |\mu_2|$ and **(H1)–(H3)** hold. Assume further that*

$$(||\nabla u||_{\alpha}, E(0)) \in \Sigma.$$

Then the solution $(u, z, y) \in Z \times L^2(R_+); L^2(\Omega \times (0,1))) \times L^2(R_+); L^2(\Gamma_1))$ *of problem (2.6)–(2.13) can not exist for all time.*

In this section, we shall prove Theorem 3.1. We start with a series of lemmas. We denote

$$\lambda_0 = ||\nabla u_0||_\alpha, \quad E_0 = E(0). \tag{3.1}$$

Theorem 3.1 will be proved by contradiction, so we shall suppose that the solution of (2.6)–(2.13) exists on the whole interval $[0, \infty)$, i.e. $T = \infty$.

Proof of Theorem 3.1. We use the idea of Vitillaro [22].

Lemma 3.1. *Let* (u, z, y) *be the solution of (2.6)–(2.13). Then the energy functional defined by (2.15) satisfies, for some constant* $c_0 > 0$,

$$\begin{aligned} \tfrac{d}{dt}E(t) \quad &\le (l-1)\int_\Omega |u_t(t)|^{l-2}u_t(t)u_{tt}(t)dx + \int_\Omega a(x)|\nabla u(t)|^{\alpha-2}\nabla u(t)\nabla u_t(t)dx \\ &\quad - \int_\Omega Q(x,t,u_t(t))u_t(t)dx - c_0\int_\Omega (u_t^2(t) + z(x,1,t))dx \\ &\quad - \int_{\Gamma_1} h(x)k(x)y_t^2(t)d\Gamma \le 0. \end{aligned} \tag{3.2}$$

Proof. Multiplying the equation (2.6) by $u_t(t)$, integrating over Ω, using Green's formula and exploiting the equation (2.9), we obtain

$$\begin{aligned} &\frac{d}{dt}\{\frac{l-1}{l}\int_\Omega |u_t(t)|^l dx + \frac{1}{\alpha}\int_\Omega a(x)|\nabla u(t)|^\alpha dx \\ &\qquad - \int_\Omega \Phi(x,u(t))dx\} - \int_{\Gamma_1} h(x)y_t(t)u_t(t)d\Gamma \\ &\quad = (l-1)\int_\Omega |u_t(t)|^{l-2}u_t(t)u_{tt}(t)dx + \int_\Omega a(x)|\nabla u(t)|^{\alpha-2}\nabla u(t)\nabla u_t(t)dx \\ &\qquad - \int_\Omega |\nabla u_t(t)|dx - \int_\Omega \nabla_u\Phi(x,u)u_t(t)dx \\ &\qquad -\mu_1\int_\Omega u_t^2(t)dx - \mu_2\int_\Omega z(x,1,t)u_t(t)dx. \end{aligned} \tag{3.3}$$

On the other hand, we have from the equation in (2.10) that

$$-\int_{\Gamma_1} h(x)y_t(t)u_t(t)d\Gamma = \int_{\Gamma_1} h(x)k(x)y_t^2(t)d\Gamma + \int_{\Gamma_1} h(x)q(x)y(t)y_t(t)d\Gamma. \tag{3.4}$$

Also, multiplying the equation (2.7) by $\frac{\xi}{2}z(x,\rho,t)$ and integrating over $\Omega \times (0,1)$, we deduce

$$
\begin{aligned}
\tfrac{d}{dt}\{\tfrac{\xi}{2}\textstyle\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\} &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx \\
&= -\frac{\xi}{2\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho dx \\
&= \frac{\xi}{2\tau}\int_\Omega [z^2(x,0,t)-z^2(x,1,t)]dx \\
&= \frac{\xi}{2\tau}[\int_\Omega u_t^2(t)dx - \int_\Omega z^2(x,1,t)dx],
\end{aligned} \tag{3.5}
$$

and

$$
-\mu_2\textstyle\int_\Omega z(x,1,t)u_t(t)dx \le \frac{|\mu_2|}{2}[\int_\Omega u_t^2(t)dx + \int_\Omega z^2(x,1,t)dx]. \tag{3.6}
$$

Hence, from (2.15) and (3.3)–(3.6), we arrive at

$$
\begin{aligned}
\tfrac{d}{dt}E(t) \le & -\int_\Omega |\nabla u_t(t)|^2 dx - \int_\Omega |\nabla u_t(t)|^\beta dx \\
& -\int_\Omega Q(x,t,u_t(t))u_t(t)dx - (\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2})\int_\Omega u_t^2(t)dx \\
& -(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2})\int_\Omega z^2(x,1,t)dx - \int_{\Gamma_1} h(x)k(x)y_t^2(t)d\Gamma.
\end{aligned}
$$

By using (2.16), we get, for some $c_0 > 0$,

$$
\begin{aligned}
\tfrac{d}{dt}E(t) \le & -\int_\Omega |\nabla u_t(t)|^2 dx - \int_\Omega |\nabla u_t(t)|^\beta dx \\
& -\int_\Omega Q(x,t,u_t(t))u_t(t)dx - c_0\int_\Omega [u_t^2(t) + z^2(x,1,t)]dx \\
& -\int_{\Gamma_1} h(x)k(x)y_t^2(t)d\Gamma \le 0.
\end{aligned} \tag{3.7}
$$

Hence we get $E(t) \le E(0)$ for all $t \in J$. □

Lemma 3.2. *If* $(\lambda_0, E(0)) \in \Sigma$*, then we have*

(3.8) (i) $||\nabla u(t)||_\alpha \ge \lambda_2$ *for all* $t \in J$*, for some* $\lambda_2 > \lambda_1$*,*

(3.9) (ii) $||u(t)||_p \ge B_1\lambda_2$ *for all* $t \in J$*, for the some* λ_2 *in* (i).

Proof. First, we will prove the (i). From (2.15), we see that

$$
E(t) \ge \frac{1}{\alpha}\int_\Omega a(x)|\nabla u(t)|^\alpha dx - \int_\Omega \Phi(x,u(t))dx. \tag{3.10}
$$

Using (2.1), since $f(x, u(t)) = \nabla_u \Phi(x, u(t))$, it follows that

$$\Phi(x, u(t)) = \int_0^1 f(x, \tau u(t))u(t)d\tau \leq \frac{\mu}{\alpha}|u(t)|^\alpha + \frac{d_1}{p}|u(t)|^p,$$

and then

$$\int_\Omega \Phi(x, u(t))dx = \int_0^1 f(x, \tau u(t))u(t)d\tau \leq \frac{\mu}{\alpha}||u(t)||_\alpha^\alpha + \frac{d_1}{p}||u(t)||_p^p. \tag{3.11}$$

Therefore

$$\begin{aligned} E(t) &\geq \frac{a_0}{\alpha}||\nabla u(t)||_\alpha^\alpha - \frac{\mu}{\alpha}||u(t)||_\alpha^\alpha - \frac{d_1}{p}||u(t)||_p^p \\ &\geq (a_0 - \frac{\mu}{\mu_0})\frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha - \frac{d_1}{p}||u(t)||_p^p \\ &\geq (a_0 - \frac{\mu}{\mu_0})\frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha - d_1 B_1^p \frac{1}{p}||u(t)||_\alpha^p \\ &= (a_0 - \frac{\mu}{\mu_0})\frac{1}{\alpha}\lambda^\alpha - d_1 B_1^p \frac{1}{p}\lambda^p := \varphi(\lambda), \end{aligned} \tag{3.12}$$

where $\lambda = ||\nabla u(t)||_\alpha$. It is easy to verify that φ is increasing for $0 < \lambda < \lambda_1$, decreasing for $\lambda > \lambda_1$, $\varphi(\lambda) \to -\infty$ as $\lambda \to +\infty$ and $\varphi(\lambda) = E_1$, where λ_1 is given in (2.18). Therefore, since $E_0 < E_1$, there exists $\lambda_2 > \lambda_1$ such that $\varphi(\lambda_2) = E(0)$. From (3.12) we have $\varphi(\lambda_0) \leq E(0) = \varphi(\lambda_2)$, which implies that $\lambda_0 \geq \lambda_2$ since $\lambda_0 > \lambda_1$. To proof the result, we suppose by contradiction that $||\nabla u_0||_\alpha < \lambda_2$, for some $t_0 > 0$ and by the continuity of $||\nabla u(t)||_\alpha$ we can choose such that $||\nabla u(t_0)||_\alpha > \lambda_1$. Again the use of (3.12) leads to

$$E(t_0) \geq \varphi(||\nabla u(t_0)||_\alpha) > \varphi(\lambda_2) = E(0).$$

This is impossible since $E(t) \leq E(0)$, for all $t \geq 0$. Thus (i) is established.
Next, we will prove the (ii). From (3.12), we get

$$\begin{aligned} \tfrac{d_1}{p}||u(t)||_p^p &\geq (a_0 - \frac{\mu}{\mu_0})\frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha - E(t) \\ &\geq (a_0 - \frac{\mu}{\mu_0})\frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha - E_0 \\ &\geq (a_0 - \frac{\mu}{\mu_0})\frac{1}{\alpha}\lambda_2^\alpha - \varphi(\lambda_2) = d_1 B_1^p \frac{1}{p}\lambda_2^p. \end{aligned}$$

Thus, the proof is complete. □

In the remainder of this section, we consider initial values $(\lambda_0, E_0) \in \Sigma$. We set

$$H(t) = E_1 - E(t), \quad t \geq 0. \tag{3.13}$$

Then we have the following Lemma.

Lemma 3.3. *For all $t \in J$, we have*

$$0 < H(0) \le H(t) \le \frac{d_1}{p}||u(t)||_p^p. \tag{3.14}$$

Proof. From Lemma 3.1, we see that $H'(t) \ge 0$. Thus, we deduce

$$H(t) \ge H(0) = E_1 - E(0) > 0, \quad \forall t \ge 0. \tag{3.15}$$

From (3.12), we obtain

$$\begin{aligned} H(t) &= E_1 - E(t) \\ &\ge \varphi(\lambda_1) - (a_0 - \frac{\mu}{\mu_0})\frac{1}{\alpha}||\nabla u(t)||_\alpha^\alpha + \frac{d_1}{p}||u(t)||_p^p \\ &= (a_0 - \frac{\mu}{\mu_0})\frac{1}{\alpha}(\lambda_1^\alpha - ||\nabla u(t)||_\alpha^\alpha) - d_1 B_1^p \frac{1}{p}\lambda_1^p + \frac{d_1}{p}||u(t)||_p^p. \end{aligned}$$

From (3.8), $||\nabla u(t)||_\alpha > \lambda_1$, we get

$$H(t) \le \frac{d_1}{p}||u(t)||_p^p. \tag{3.16}$$

Thus, combing (3.15) and (3.16) we obtain (3.14). □

Now, we define

$$\begin{aligned} L(t) = {} & H^{1-\sigma}(t) + \varepsilon \int_\Omega u(t)|u_t(t)|^{l-2}u_t(t)dx \\ & + \frac{\mu_1\varepsilon}{2}\int_\Omega u_t^2(t)dx - \frac{\varepsilon}{2}\int_{\Gamma_1} h(x)k(x)y^2(t)d\Gamma \\ & - \varepsilon\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma, \end{aligned} \tag{3.17}$$

for ε small to be chosen later and

$$0 < \sigma \le \min\{\frac{\alpha-2}{p}, \frac{\alpha-\beta}{p(\beta-1)}, \frac{\alpha-m}{p(m-1)}, \frac{\alpha-l}{\alpha l}, \frac{k}{\varepsilon\alpha} - 1\}. \tag{3.18}$$

By taking a derivative of (3.17) we have

$$\begin{aligned} L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon||u_t(t)||_l^l + \varepsilon\int_\Omega u(t)(|u_t(t)|^{l-2}u_t(t))_t dx \\ & + \mu_1\varepsilon\int_\Omega u(t)u_t(t)dx - \varepsilon\int_{\Gamma_1} h(x)k(x)y(t)y_t(t)d\Gamma \\ & - \varepsilon\int_{\Gamma_1} h(x)u_t(t)y(t)d\Gamma - \varepsilon\int_{\Gamma_1} h(x)u(t)y_t(t)d\Gamma. \end{aligned} \tag{3.19}$$

By using (2.6)–(2.10) and estimate (3.19), we find

$$
\begin{aligned}
L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon||u_t(t)||_l^l \\
&+\varepsilon\int_\Omega \Big(\triangle u_t(t) + div(a(x)|\nabla u(t)|^{\alpha-2}\nabla u(t)) \\
&\quad +div(|\nabla u_t(t)|^{\beta-2}\nabla u_t(t)) - Q(x,t,u_t) \\
&\quad -\mu_1 u_t(t) - \mu_2 z(x,1,t) + f(x,u(t))\Big)u(t)dx \\
&+\mu_1\varepsilon\int_\Omega u(t)u_t(t)dx - \varepsilon\int_{\Gamma_1} h(x)k(x)y(t)y_t(t)d\Gamma \\
&-\varepsilon\int_{\Gamma_1} h(x)u_t(t)y(t)d\Gamma - \varepsilon\int_{\Gamma_1} h(x)u(t)y_t(t)d\Gamma \\
&= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon||u_t(t)||_l^l \\
&-\varepsilon\int_\Omega \nabla u_t(t)\nabla u(t)dx - \varepsilon\int_\Omega a(x)|\nabla u(x,t)|^\alpha dx \\
&-\varepsilon\int_\Omega |\nabla u_t(t)|^{\beta-2}\nabla u_t(t)\nabla u(t)dx \\
&+\varepsilon\int_{\Gamma_1}\Big(\frac{\partial u_t(t)}{\partial\nu} + |\nabla u(t)|^{\alpha-2}\frac{\partial u(t)}{\partial\nu} + |\nabla u_t(x,t)|^{\beta-2}\frac{\partial u_t(x,t)}{\partial\nu}\Big)u(t)d\Gamma \\
&-\varepsilon\int_\Omega Q(x,t,u_t)u(t)dx + \varepsilon\int_\Omega f(x,u(x,t))u(t)dx \\
&-\mu_1\varepsilon\int_\Omega u(t)u_t(t)dx - \mu_2\varepsilon\int_\Omega z(x,1,t)u(t)dx \\
&+\mu_1\varepsilon\int_\Omega u(t)u_t(t)dx - \varepsilon\int_{\Gamma_1} h(x)k(x)y(t)y_t(t)d\Gamma \\
&-\varepsilon\int_{\Gamma_1} h(x)u_t(t)y(t)d\Gamma - \varepsilon\int_{\Gamma_1} h(x)u(t)y_t(t)d\Gamma \\
&= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon||u_t(t)||_l^l \\
&-\varepsilon\int_\Omega \nabla u_t(t)\nabla u(t)dx - \varepsilon\int_\Omega a(x)|\nabla u(x,t)|^\alpha dx \\
&-\varepsilon\int_\Omega |\nabla u_t(t)|^{\beta-2}\nabla u_t(t)\nabla u(t)dx \\
&-\varepsilon\int_\Omega Q(x,t,u_t)u(t)dx + \varepsilon\int_\Omega f(x,u(x,t))u(t)dx \\
&-\mu_2\varepsilon\int_\Omega z(x,1,t)u(t)dx + \varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma \qquad (3.20)
\end{aligned}
$$

Exploiting Hölder's and Young's inequality and **(H3)**, for any $\delta, \mu, \eta, \rho > 0$, we deduce

$$
\begin{aligned}
\int_\Omega Q(x,t,u)u(t)dx &\le \int_\Omega |u(t)|[d(x,t)]^{1/m}[Q(x,t,u_t(t))u_t(t)]^{1/m'}dx \\
&\le \frac{\delta^m}{m}\int_\Omega |u(t)|^m d(x,t)dx + \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\int_\Omega Q(x,t,u_t(t))u_t(t)dx \\
&\le \frac{\delta^m}{m}||u(t)||_p^m ||d(t)||_{p/(p-m)} + \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\int_\Omega Q(x,t,u_t(t))u_t(t)dx \\
&\le \frac{\delta^m C}{m}||u(t)||_p^m + \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\int_\Omega Q(x,t,u_t(t))u_t(t)dx. \qquad (3.21)
\end{aligned}
$$

By Young's inequality, we get

$$
\int_\Omega \nabla u_t(t)\nabla u(t)dx \le \frac{1}{4\mu}\int_\Omega |\nabla u(t)|^2dx + \mu\int_\Omega |\nabla u_t(t)|^2dx, \qquad (3.22)
$$

$$
\begin{aligned}
\int_\Omega |\nabla u_t(t)|^{\beta-2}\nabla u_t(t)\nabla u(t)dx \le \frac{\eta^\beta}{\beta}\int_\Omega |\nabla u(t)|^\beta dx \\
+ \frac{\beta-1}{\beta}\eta^{-\frac{\beta}{\beta-1}}\int_\Omega |\nabla u_t(t)|^\beta dx, \qquad (3.23)
\end{aligned}
$$

$$
\mu_2\int_\Omega u(t)z(x,1,t)dx \le \frac{|\mu_2|}{4\rho}\int_\Omega u^2(t)dx + |\mu_2|\rho\int_\Omega z^2(x,1,t)dx. \qquad (3.24)
$$

A substitution of (3.21) – (3.24) into (3.20) yields

$$
\begin{aligned}
L'(t) \ge\ & (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon||u_t(t)||_l^l \\
& -\frac{\varepsilon}{4\mu}\int_\Omega |\nabla u(t)|^2dx - \varepsilon\mu\int_\Omega |\nabla u_t(t)|^2dx \\
& -\varepsilon\int_\Omega a(x)|\nabla u(t)|^\alpha dx \\
& -\varepsilon\frac{\eta^\beta}{\beta}\int_\Omega |\nabla u(t)|^\beta dx - \frac{\varepsilon(\beta-1)}{\beta}\eta^{-\frac{\beta}{\beta-1}}\int_\Omega |\nabla u_t(t)|^\beta dx \\
& -\frac{\varepsilon\delta^m C}{m}||u(t)||_p^m - \frac{\varepsilon(m-1)}{m}\delta^{-\frac{m}{m-1}}\int_\Omega Q(x,t,u_t(t))u_t(t)dx \\
& -\frac{\varepsilon|\mu_2|}{4\rho}\int_\Omega u^2(t)dx - \varepsilon|\mu_2|\rho\int_\Omega z^2(x,1,t)dx \\
& +\varepsilon\int_\Omega f(x,u(x,t))u(t)dx + \varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma \qquad (3.25)
\end{aligned}
$$

Therefore, we choose δ, μ, η, and ρ so that

$$
\begin{aligned}
&\delta^{-\frac{m}{m-1}} = M_1H^{-\sigma}(t), \quad \mu = M_2H^{-\sigma}(t) \\
&\eta^{-\frac{\beta}{\beta-1}} = M_3H^{-\sigma}(t), \quad \rho = M_4H^{-\sigma}(t), \qquad (3.26)
\end{aligned}
$$

for M_1, M_2, M_3, M_4 to be specified later. Using (2.10), (3.25) and (3.26), we arrive at

$$
\begin{aligned}
L'(t) \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon||u_t(t)||_l^l - \frac{\varepsilon}{4M_2}H^{\sigma}(t)\int_\Omega |\nabla u(t)|^2 dx \\
& -\varepsilon\int_\Omega a(x)|\nabla u(t)|^\alpha dx - \varepsilon\frac{M_3^{-(\beta-1)}}{\beta}H^{\sigma(\beta-1)}(t)\int_\Omega |\nabla u(t)|^\beta dx \\
& -\frac{\varepsilon M_1^{(m-1)}C}{m}H^{\sigma/(m-1)}(t)||u(t)||_p^m - \frac{\varepsilon|\mu_2|}{4M_4}H^{\sigma}(t)\int_\Omega u^2(t)dx \\
& -\varepsilon\Big[M_2\int_\Omega |\nabla u_t(t)|^2 dx + \frac{(\beta-1)}{\beta}M_3\int_\Omega |\nabla u_t(t)|^\beta dx \\
& \quad + \frac{(m-1)}{m}M_1\int_\Omega Q(x,t,u_t(t))u_t(t)dx + |\mu_2|M_4\int_\Omega z^2(x,1,t)dx\Big]H^{-\sigma}(t) \\
& +\varepsilon\int_\Omega f(x,u(x,t))u(t)dx + \varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma. \qquad (3.27)
\end{aligned}
$$

If $M = M_2 + \frac{(\beta-1)M_3}{\beta} + \frac{(m-1)M_1}{m} + |\mu_2|M_4$, then (3.27) takes the form

$$
\begin{aligned}
L'(t) \geq\ & (1-\sigma-\varepsilon M)H^{-\sigma}(t)H'(t) + \varepsilon||u_t(t)||_l^l - \frac{\varepsilon}{4M_2}H^{\sigma}(t)\int_\Omega |\nabla u(t)|^2 dx \\
& -\varepsilon\int_\Omega a(x)|\nabla u(t)|^\alpha dx - \varepsilon\frac{M_3^{-(\beta-1)}}{\beta}H^{\sigma(\beta-1)}(t)\int_\Omega |\nabla u(t)|^\beta dx \\
& -\frac{\varepsilon C}{m}M_1^{(m-1)}H^{\sigma/(m-1)}(t)||u(t)||_p^m - \frac{\varepsilon|\mu_2|}{4M_4}H^{\sigma}(t)\int_\Omega u^2(t)dx \qquad (3.28) \\
& +\varepsilon M H^{-\sigma}(t)\int_{\Gamma_1} h(x)k(x)y_t^2(t)d\Gamma \\
& +\varepsilon\int_\Omega f(x,u(t))u(t)dx + \varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma.
\end{aligned}
$$

From(3.14),(3.18), the embedding $W^{1,\alpha}(\Omega) \hookrightarrow L^p(\Omega)$ and

$$
z^\delta \leq (1+1/a)(z+a), \forall z > 0, \quad 0 < \delta \leq 1, \quad a > 0,
$$

we have (see[15])

$$
\begin{aligned}
H^{\sigma}(t)\int_\Omega |\nabla u(t)|^2 dx &\leq c(\Omega)\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma}\Big(\int_\Omega |\nabla u(t)|^\alpha dx\Big)^{(p\sigma+2)/\alpha} \\
&\leq d\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma}\Big(\int_\Omega |\nabla u(t)|^\alpha dx + H(t)\Big), \qquad (3.29)
\end{aligned}
$$

$$H^{\sigma(\beta-1)}(t)\int_\Omega|\nabla u(t)|^\beta dx \le c(\Omega)\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma(\beta-1)}\Big(\int_\Omega|\nabla u(t)|^\alpha dx\Big)^{(p\sigma(\beta-1)+\beta)/\alpha}$$
$$\le d\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma(\beta-1)}\Big(\int_\Omega|\nabla u(t)|^\alpha dx + H(t)\Big), \tag{3.30}$$

$$H^{\sigma(m-1)}||u(t)||_p^m \le c(\Omega)\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma(m-1)}B_1^m\Big(\int_\Omega|\nabla u(t)|^\alpha dx\Big)^{(\sigma p(m-1)+m)/\alpha}$$
$$\le d\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma(m-1)}B_1^m\Big(\int_\Omega|\nabla u(t)|^\alpha dx + H(t)\Big), \tag{3.31}$$

and

$$H^{\sigma}(t)\int_\Omega|\nabla u(t)|^2 dx \le c(\Omega)\Big(\tfrac{B_1^p d_1}{p}\Big)^{\sigma}B_1^2\Big(\textstyle\int_\Omega|\nabla u(t)|^\alpha dx\Big)^{(\sigma p)/\alpha}$$
$$\le d\Big(\tfrac{B_1^p d_1}{p}\Big)^{\sigma}B_1^2\Big(\textstyle\int_\Omega|\nabla u(t)|^\alpha dx + H(t)\Big), \tag{3.32}$$

for all $t \ge 0$, where $d = c(\Omega)[1 + 1/H(0)]$. Inserting estimates (3.29)-(3.32) into (3.28), we obtain

$$\begin{aligned}
L'(t) \ \ge\ & \Big(1-\sigma) - \varepsilon M\Big)H^{-\sigma}(t)H'(t) + kH(t) + (\varepsilon + \frac{k(l-1)}{l})||u_t(t)||_l^l \\
& -\frac{\varepsilon c_2}{M_2}\Big(\int_\Omega|\nabla u(t)|^\alpha dx + H(t)\Big) - \varepsilon\int_\Omega a(x)|\nabla u(t)|^\alpha dx \\
& -\frac{\varepsilon c_3}{M_3^{\beta-1}}\Big(\int_\Omega|\nabla u(t)|^\alpha dx + H(t)\Big) + \frac{k}{\alpha}\int_\Omega a(x)|\nabla u(t)|^\alpha dx \\
& -\frac{\varepsilon c_1}{M_1^{m-1}}\Big(\int_\Omega|\nabla u(t)|^\alpha dx + H(t)\Big) \\
& -\frac{\varepsilon c_4}{M_4}\Big(\int_\Omega|\nabla u(t)|^\alpha dx + H(t)\Big) \\
& +\varepsilon\int_\Omega f(x,u(t))u(t)dx + \varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma \\
& -k\int_\Omega \Phi(x,u(t))dx + k\frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \\
& +\frac{k}{2}\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma - kE_1 + \varepsilon M H^{-\sigma}(t)\int_{\Gamma_1} h(x)k(x)y_t^2(t)d\Gamma,
\end{aligned}$$

for some constant k and

$$c_1 = \frac{cd}{m}\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma/(m-1)}B_1^m, \quad c_2 = \frac{d}{4}\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma},$$
$$c_3 = \frac{d}{\beta}\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma(\beta-1)}, \quad c_4 = d\Big(\frac{B_1^p d_1}{p}\Big)^{\sigma}B_1^2.$$

From (2.17),(2.18) and Lemma 3.2, we have

$$-kE_1 \geq -kE_1 B_1^{-p}\lambda_1^{-p}||u(t)||_p^p = -kd_1(\frac{1}{\alpha}-\frac{1}{p})||u(t)||_p^p.$$

From (2.2), we can choose k satisfying

$$\alpha\varepsilon \leq k < p\varepsilon \min\Big\{\frac{\alpha d_2}{(p-\alpha)d_1}, 1\Big\}$$

and

$$\varepsilon\int_\Omega f(x,u(t))u(t)dx - k\int_\Omega \Phi(x,u(t))dx - kE_1 \\ \geq \varepsilon d_2||u(t)||_p^p - kd_1(\frac{1}{\alpha}-\frac{1}{p})||u(t)||_p^p \geq 0.$$

Thus, it follows that

$$\begin{aligned} L'(t) \quad &\geq \Big(1-\sigma)-\varepsilon M\Big)H^{-\sigma}(t)H'(t) + (\varepsilon + \frac{k(l-1)}{l})||u_t(t)||_l^l \\ &\varepsilon\Big(\frac{k}{\varepsilon} - \frac{c_2}{M_2} - \frac{c_3}{M_3^{\beta-1}} - \frac{c_1}{M_1^{m-1}} - \frac{c_4}{M_4}\Big)H(t) \\ &+\varepsilon\Big((\frac{k}{\varepsilon\alpha}-1)a_0 - \frac{c_2}{M_2} - \frac{c_3}{M_3^{\beta-1}} - \frac{c_1}{M_1^{m-1}} - \frac{c_4}{M_4}\Big)\int_\Omega |\nabla u(t)|^\alpha dx \\ &+\varepsilon\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma + \frac{k\xi}{2}\int_\Omega\int_0^1 z(x,\rho,t)d\rho dx \\ &+\frac{k}{2}\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma + \varepsilon M H^{-\sigma}(t)\int_{\Gamma_1} h(x)k(x)y_t^2(t)d\Gamma. \end{aligned}$$

At this point, choosing M_1, M_2, M_3, M_4 large enough and ε sufficiently small and using

$$\varepsilon M H^{-\sigma}(t)\int_{\Gamma_1} h(x)k(x)y_t^2(t)d\Gamma \geq 0,$$

we deduce

$$\begin{aligned} L'(t) \quad &\geq \Big(1-\sigma)-\varepsilon M\Big)H^{-\sigma}(t)H'(t) + \gamma\varepsilon\Big(H(t) + ||u_t(t)||_l^l \\ &+\int_\Omega |\nabla u(t)|^\alpha dx + \int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma + \int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\Big), \end{aligned} \tag{3.33}$$

where γ is a positive constant (it is possible since $k > \varepsilon\alpha$). We choose ε sufficiently small and $0 < \varepsilon < (1-\sigma)/M$ so that

$$L(0) = H^{1-\sigma}(0) + \varepsilon\int_\Omega u_0|u_1|^{l-2}u_1 dx + \frac{\mu_1\varepsilon}{2}\int_\Omega u_0^2 dx \\ -\frac{\varepsilon}{2}\int_{\Gamma_1} h(x)k(x)y_0^2 d\Gamma - \varepsilon\int_{\Gamma_1} h(x)u_0 y_0 d\Gamma > 0.$$

Then from(3.33) we get

$$L(t) \geq L(0) \geq 0, \quad \forall t \geq 0,$$

and

$$\begin{aligned} L'(t) \quad &\geq \gamma\varepsilon\Big(H(t) + ||u_t(t)||_l^l \\ &+ \int_\Omega |\nabla u(t)|^\alpha dx + \int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma + \int_\Omega \int_0^1 z^2(x,\rho,t)d\rho dx\Big). \end{aligned} \tag{3.34}$$

On the other hand, from(3.17) and $h(x), q(x) > 0$, we have

$$L(t) \leq H^{1-\sigma}(t) + \varepsilon \int_\Omega u(t)|u_t(t)|^{l-2}u_t(t)dx + \frac{\mu_1\varepsilon}{2}\int_\Omega u^2(t)dx - \varepsilon\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma.$$

Then the above inequality leads to

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &\leq \Big[H^{1-\sigma}(t) + \varepsilon\int_\Omega u(t)|u_t(t)|^{l-2}u_t(t)dx \\ &\quad + \frac{\mu_1\varepsilon}{2}\int_\Omega u^2(t)dx - \varepsilon\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma\Big]^{1/(1-\sigma)} \\ &\leq C(\varepsilon,\mu_1,\sigma)\Big[H(t) + |\int_\Omega u(t)|u_t(t)|^{l-2}u_t(t)dx|^{\frac{1}{1-\sigma}} \\ &\quad + (\int_\Omega u^2(t)dx)^{\frac{1}{1-\sigma}} + |\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma|^{\frac{1}{1-\sigma}}\Big]. \end{aligned} \tag{3.35}$$

Next, using Hölder's inequality, the embedding $W^{1,\alpha}(\Omega) \hookrightarrow L^l(\Omega), \alpha > l$ and Young's inequality, we derive

$$\begin{aligned} &\Big|\int_\Omega u(t)|u_t(t)\Big|^{l-2}u_t(t)dx| \leq (\int_\Omega |u(t)|^l dx)^{1/l}(\int_\Omega |u_t(t)|^l dx)^{(l-1)/l} \\ &\quad \leq (\int_\Omega |\nabla u(t)|^\alpha dx)^{1/\alpha}(\int_\Omega |u_t(t)|^l dx)^{(l-1)/l} \\ &\quad \leq c\Big[(\int_\Omega |\nabla u(t)|^\alpha dx)^{l(1-\sigma)/[l(1-\sigma)-(l-1)]\alpha} + (\int_\Omega |u_t(t)|^l dx)^{(1-\sigma)}\Big]. \end{aligned}$$

From (3.18) and (3.29), we obtain

$$\begin{aligned} &\Big|\int_\Omega u(t)|u_t(t)|^{l-2}u_t(t)dx\Big|^{1/(1-\sigma)} \\ &\quad \leq c\Big[(\int_\Omega |\nabla u(t)|^\alpha dx)^{l/[l(1-\sigma)-(l-1)]\alpha} + \int_\Omega |u_t(t)|^l dx\Big] \\ &\quad \leq c\Big[(1+\frac{1}{H(0)})(\int_\Omega |\nabla u(t)|^\alpha dx + H(t)) + \int_\Omega |u_t(t)|^l dx\Big]. \end{aligned}$$

Therefore, there exists a positive constant C' such that for all $t \geq 0$,

$$\Big| \int_\Omega u(t)|u_t(t)|^{l-2}u_t(t)dx\Big|^{1/(1-\sigma)} \leq C'\Big[H(t)) + ||\nabla u(t)||_\alpha^\alpha + ||u_t(t)||_l^l\Big]. \tag{3.36}$$

Furthermore, by the same method, we deduce

$$\begin{aligned} \textstyle\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma &= \Big| \int_{\Gamma_1} \frac{h(x)q(x)}{q(x)} u(t)y(t)d\Gamma\Big| \\ &\leq \frac{||h||_\infty^{\frac{1}{2}}||q||_\infty^{\frac{1}{2}}}{q_0}\Big(\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma\Big)^{\frac{1}{2}}\Big(\int_{\Gamma_1} u^2(t)d\Gamma\Big)^{\frac{1}{2}}. \end{aligned}$$

Similarly, we find

$$\begin{aligned} \textstyle\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma &= \Big| \int_{\Gamma_1} \frac{h(x)q(x)}{q(x)} u(t)y(t)d\Gamma\Big| \\ &\leq \frac{||h||_\infty^{\frac{1}{2}}||q||_\infty^{\frac{1}{2}}}{q_0}\Big(\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma\Big)^{\frac{1}{2}}\Big(\int_{\Gamma_1} u^2(t)d\Gamma\Big)^{\frac{1}{2}}. \end{aligned}$$

Using the embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Gamma_1)$ and Hölder's inequality, we get

$$\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma \leq c_5 \frac{||h||_\infty^{\frac{1}{2}}||q||_\infty^{\frac{1}{2}}}{q_0}\Big(\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla u(t)|^\alpha dx\Big)^{\frac{1}{\alpha}}.$$

where c_5 is a embedding constant. Consequently, there exists a positive constant $c_6 = c(||h||_\infty, ||q||_\infty, q_0, \sigma, \alpha)$ such that

$$\Big(\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma\Big)^{\frac{1}{1-\sigma}} \leq c_6\Big(\int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma\Big)^{\frac{1}{2(1-\sigma)}}\Big(\int_\Omega |\nabla u(t)|^\alpha dx\Big)^{\frac{1}{\alpha(1-\sigma)}}.$$

Using Young's inequality, we write

$$\Big(\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma\Big)^{\frac{1}{1-\sigma}} \leq c_7\Big[\Big(\int_\Omega |\nabla u(t)|^\alpha dx\Big)^{\frac{2}{\alpha(1-2\sigma)}} + \int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma\Big],$$

where c_7 is a positive constant depending on c_6 and α. Applying once again the algebraic inequality (3.29) with $z = ||\nabla u(t)||_\alpha^\alpha, \nu = 2/[\alpha(1-2\sigma)]$ and making use of (3.18), we see that by the same method as above

$$\Big(\int_{\Gamma_1} h(x)u(t)y(t)d\Gamma\Big)^{\frac{1}{1-\sigma}} \leq c_8\Big[H(t) + ||\nabla u(t)||_\alpha^\alpha + \int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma\Big], \tag{3.37}$$

where c_8 is a positive constant. Hence combining (3.35) − (3.37) and using $\alpha > 2$, we arrive at

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &\leq C_*\Big[H(t) + ||u_t(t)||_l^l + ||\nabla u(t)||_\alpha^\alpha \\ &\quad + \int_{\Gamma_1} h(x)q(x)y^2(t)d\Gamma + + \int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\Big], \forall\ t \geq 0, \end{aligned} \tag{3.38}$$

where C_* is a positive constant. Consequently a combining of (3.34) and (3.38), for some $\xi > 0$, we obtain

$$L'(t) \geq \xi L^{\frac{1}{1-\sigma}}(t), \quad \forall\ t \geq 0. \tag{3.39}$$

Integration of (3.9) over $(0, t)$ yield

$$L^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{L^{\frac{-\sigma}{1-\sigma}}(0) - \frac{\xi\sigma}{1-\sigma}t}, \forall\ t \geq 0.$$

Therefore $L(t)$ blow up in finite time

$$T \leq T^* = \frac{1-\sigma}{\xi\sigma L^{\frac{\sigma}{1-\sigma}}(0)}.$$

Thus the proof of Theorem 2.1 is complete. □

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