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Ore Extension Rings with Constant Products of Elements

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Abstract. Let R be an associative unital ring with an endomorphism α and α-derivation δ. The constant products of elements in Ore extension rings, when the coefficient ring is reversible, is investigated. We show that if $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ be non-zero elements in Ore extension ring $R[x;\alpha,\delta]$ such that $g(x)f(x) = c \in R$, then there exist non-zero elements $r, a \in R$ such that $rf(x) = ac$, when R is an (α,δ)-compatible ring which is reversible. Among applications, we give an exact characterization of the unit elements in $R[x;\alpha,\delta]$, when the coeficient ring R is (α,δ)-compatible. Furthermore, it is shown that if R is a weakly 2-primal ring which is (α,δ)-compatible, then $J(R[x;\alpha,\delta]) = Ni\ell(R)[x;\alpha,\delta]$. Some other applications and examples of rings with this property are given, with an emphasis on certain classes of NI rings. As a consequence we obtain generalizations of the many results in the literature. As the final part of the paper we construct examples of rings that explain the limitations of the results obtained and support our main results.

1. Introduction and Preliminary Definitions

Throughout, unless mentioned otherwise, R denotes an associative ring with unity. Letting R be a ring, α be an endomorphism of R and δ be an α-derivation of R (so δ is an additive map satisfying $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$), the general (left) Ore extension $R[x;\alpha,\delta]$ is the ring of polynomials over R in the variable x, with termwise addition and with coefficients written on the left of x, subject to the skew-multiplication rule $xr = \alpha(r)x + \delta(r)$ for $r \in R$. If α is an identity map on R or $\delta = 0$, then we denote $R[x;\alpha,\delta]$ by $R[x;\delta]$ and $R[x;\alpha]$, respectively. We use $Ni\ell_*(R)$, $Ni\ell^*(R)$, $L\text{-}rad(R)$, $Ni\ell(R)$ and $J(R)$ to denote the prime radical, upper nil radical, Levitzki radical, the set of all nilpotent elements of R and the Jacobson

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radical of R, respectively. Given a polynomial $f(x)$ over a ring R, we denote by $\deg(f(x))$ the degree of $f(x)$.

Recall that a ring R is *reduced* if it has no non-zero nilpotent element. According to Krempa [14], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. R is called an α-rigid ring [11] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced by Hong et al. [11]. Properties of α-rigid rings have been studied in Krempa [14], Hirano [10] and Hong et al. [11, 12].

According to Hong et al. [12], for an endomorphism α of a ring R, an α-ideal I is called to be an *α-rigid ideal* if $a\alpha(a) \in I$ implies $a \in I$ for $a \in R$. Hong et al. [12] studied connections between the α-rigid ideals of R and the related ideals of some ring extensions.

In [9], the authors defined α-compatible rings, which are a generalization of α-rigid rings. A ring R is called *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is said to be *δ-compatible* if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both α-compatible and δ-compatible, we say that R is *(α, δ)-compatible.* In [9, Lemma 2.2], the authors showed that R is α-rigid if and only if R is α-compatible and reduced. Thus the α-compatible ring is a generalization of an α-rigid ring to the more general case where R is not assumed to be reduced. In [7], the author defined α-compatible ideals, which are a generalization of α-rigid ideals. An ideal I is called an *α-compatible ideal* if for each $a, b \in R$, $ab \in I \Leftrightarrow a\alpha(b) \in I$. Moreover, I is said to be a *δ-compatible ideal* if for each $a, b \in R$, $ab \in I \Rightarrow a\delta(b) \in I$. If I is both α-compatible and δ-compatible, we say that I is an *(α, δ)-compatible ideal.* In [7, Proposition 2.4], the author showed that an ideal I is α-rigid if and only if I is α-compatible and *completely semiprime*(i.e., if $a^2 \in I$, then $a \in I$).

In [16], Nasr-Isfahani studied Ore extensions of 2-primal rings. He showed that if R is an (α, δ)-compatible ring, then R is 2-primal if and only if $R[x; \alpha, \delta]$ is 2-primal if and only if $Ni\ell(R)[x; \alpha, \delta] = Ni\ell_*(R[x; \alpha, \delta])$ if and only if every minimal (α, δ)-prime ideal of R is completely prime.

Following [1], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Also, following [2], a ring R is called *2-primal* if $Ni\ell_*(R) = Ni\ell(R)$. Shin in [18, Proposition 1.11] showed that a ring R is 2-primal if and only if every minimal prime ideal P of R is *completely prime* (i.e. R/P is a domain). Moreover, a ring R is called *weakly 2-primal* if $Ni\ell(R) =$L$-rad(R)$. If $Ni\ell(R) = Ni\ell^*(R)$, then R is called *NI.* It is known that the following implications holds between the mentioned classes of rings:

reduced $\Rightarrow$ reversible $\Rightarrow$ *2-primal* $\Rightarrow$ weakly 2-primal $\Rightarrow$ NI.

But the converses does not hold (see [4, 13]). Moreover, a ring is *right* (resp., *left*) *duo* if every right (resp., left) ideal is an ideal. The importance of the study of these classes of rings in noncommutative ring theory is because of the famous Köthe's problem which ask whether every one-sided nil ideal of any associative ring is contained in a two-sided nil ideal of the ring. As observed by Bell [1], these rings fulfill the requirements of the Köthe's Conjecture.

In [3] Chen proved that if R is an α-compatible ring and $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ are non-zero polynomials in $R[x;\alpha]$ such that $g(x)f(x) = c \in R$, then $b_0 a_0 = c$ and there exist non-zero elements $r, a \in R$ such that $rf(x) = ac$. As an application of the above result, he showed that if R is an α-compatible and weakly 2-primal ring, then a polynomial $f(x) \in R[x;\alpha]$ is unit if and only if its constant term is unit and other coefficients are all nilpotent. In [4] Chen and Cui proved that if R is an α-compatible and weakly 2-primal ring, then $R[x;\alpha]$ is weakly 2-primal.

In this paper we prove some results which concern the constant products of elements in Ore extension rings over reversible ring. Roughly speaking, our main theorems give a characterization of the unit elements in Ore extension ring. We will also pay a particular attention to stable range one property of Ore extension rings. As the final part of the paper we construct examples of rings that support our main results and explain the limitations of the results obtained in former sections.

2. Ore Extension of Reversible Rings with Constant Products of Elements

In this section we prove some results which concern the constant products of elements in Ore extension rings over reversible ring. Note that the methods used for the "unmixed" Ore extensions do not apply to the general case. We also note that in the investigation of Ore extension rings $R[x;\alpha,\delta]$, our results based on the twist property in multiplication of polynomials. We start this section by the following lemma, which will be useful in the sequel.

Lemma 2.1. *Let R be an (α,δ)-compatible ring and $a, b \in R$. Then we have the following:*

(1) *If $ab = 0$, then $a\alpha^n(b) = 0 = \alpha^n(a)b$ for any non-negative integer n.*

(2) *If $\alpha^k(a)b = 0$ for some non-negative integer k, then $ab = 0$.*

(3) *If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)$ for any non-negative integers m, n.*

(4) *If $ab = 0$, then $\alpha(a)\alpha(b) = 0 = \delta(a)\delta(b)$.*

(5) *If $a^n = 0$, then $(\alpha(a))^n = 0 = (\delta(a))^n$ for any positive integer n.*

(6) *If $ab = 0$ then $ax^m b = 0$ for each $m \geq 0$.*

(7) *If $ax^m b = 0$ in $R[x;\alpha,\delta]$, for some $m \geq 0$, then $ab = 0$.*

Proof. (1), (2) and (3) are proved in [9, Lemma 2.1].

(4) Since $ab = 0$, then by (1) and (2) we have $\alpha(a)b = 0 = \delta(a)b$. Hence $\alpha(a)\alpha(b) = 0 = \delta(a)\delta(b)$, since R is (α,δ)-compatible.

(5), (6) and (7) follow from (4), (3) and (1), respectively. □

Lemma 2.2.([8, Lemma 2.2]) *Let R be an (α,δ)-compatible ring, $f(x)=a_0+a_1x+\cdots+a_nx^n\in R[x;\alpha,\delta]$ and $c,r\in R$. Then $f(x)r=c$ if and only if $a_0r=c$ and $a_ir=0$ for each $1\le i\le n$.*

Lemma 2.3.([8, Lemma 2.3]) *Let R be an α-rigid ring and also $f(x)=a_0+a_1x+\cdots+a_nx^n$ and $g(x)=b_0+b_1x+\cdots+b_mx^m$ be non-zero elements of $R[x;\alpha,\delta]$ such that $f(x)g(x)=c\in R$. Then $a_0b_0=c$ and $a_ib_j=0$, for each i,j with $i+j\ge 1$*

Recall that an ideal I of R is called an *α-ideal* if $\alpha(I)\subseteq I$; I is called an *α-invariant* if $\alpha^{-1}(I)=I$; I is called a *δ-ideal* if $\delta(I)\subseteq I$; I is called an *(α,δ)-ideal* if it is both an α- and a δ-ideal. Clearly, each α-compatible ideal is an α-invariant ideal, and each δ-compatible ideal is δ-ideal.

If I is an (α,δ)-ideal, then $\overline{\alpha}:R/I\to R/I$ defined by $\overline{\alpha}(a)=\alpha(a)+I$ is an endomorphism and $\overline{\delta}:R/I\to R/I$ defined by $\overline{\delta}(a)=\delta(a)+I$ is an $\overline{\alpha}$-derivation.

Recall also that, an ideal $\mathcal{P}$ of R is *completely prime* if $ab\in\mathcal{P}$ implies $a\in\mathcal{P}$ or $b\in\mathcal{P}$ for $a,b\in R$.

Lemma 2.4. *Let R be a 2-primal ring and $\mathcal{P}$ a minimal prime ideal of R. If R is an (α,δ)-compatible ring, then $\mathcal{P}$ is an α-invariant and a δ-ideal of R. Moreover, $\mathcal{P}$ is an (α,δ)-compatible ideal of R.*

Proof. Since R is 2-primal, $Ni\ell(R)=Ni\ell_*(R)$. Then $\overline{R}=R/Ni\ell(R)$ is a reduced ring. By Lemma 2.1, $Ni\ell(R)$ is an (α,δ)-compatible ideal of R, and so $\overline{R}$ is $(\overline{\alpha},\overline{\delta})$-compatible, by [7, Proposition 2.1]. Clearly $\mathcal{P}/Ni\ell(R)$ is a minimal prime ideal of $\overline{R}$. Then, by [14, Lemma 1.5], $\mathcal{P}/Ni\ell(R)$ is a collection of some right annihilators of subsets of $\overline{R}$. Thus $\mathcal{P}/Ni\ell(R)$ is an $\overline{\alpha}$-invariant and a $\overline{\delta}$-ideal of $\overline{R}$, since $\overline{R}$ is $(\overline{\alpha},\overline{\delta})$-compatible. Therefore $\mathcal{P}$ is an α-invariant and a δ-ideal of R.

Now, since $\mathcal{P}$ is completely prime, α-invariant and a δ-ideal of R, one can easily show that $\mathcal{P}$ is an (α,δ)-compatible ideal of R. □

Now we are in a position to give one of our main theorems in this paper. Recall that following [17], an associative ring R with unity is called *left McCoy* when the equation $g(x)f(x)=0$ implies $rf(x)=0$ for some non-zero element $r\in R$, where $f(x)$ and $g(x)$ are non-zero polynomials in $R[x]$. *Right McCoy rings* are defined dually and they satisfy dual properties. A ring R is called *McCoy* if it is both left and right McCoy. This name for them was chosen by Nielsen in [17] in recognition of McCoy's proof in [15, Theorem 2] that commutative rings satisfy this condition. McCoy rings are unified generalization of a reversible and right duo rings. These rings, though may look a bit specific, were studied by many authors and are related to important ring theory problems. Systematic studies of these rings were started in [17] and next continued in a number of papers, generalizing the McCoy condition in many different ways. The following two theorems are generalizations of [3, Theorems 2.4 and 2.5]. When $Ni\ell(R)$ forms an ideal of a ring R, we say an element $r\in R$ is unit modulo $Ni\ell(R)$ if $r+Ni\ell(R)$ is unit in $R/Ni\ell(R)$.

Theorem 2.5. *Let R be a reversible (α,δ)-compatible ring and $f(x)=a_0+a_1x+\cdots+a_nx^n$ be a non-zero element of $R[x;\alpha,\delta]$. If there is a non-zero element $g(x)=$*

$b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha,\delta]$ *such that* $g(x)f(x) = c$ *is a constant, then there exist non-zero elements* $r, a \in R$ *such that* $rf(x) = ac$. *In particular* $r = ab_p$ *for some* p, $0 \leq p \leq m$, *and* a *is either one or a product of at most* m *coefficients from* $f(x)$. *Furthermore, if* b_0 *is a unit element modulo* $Ni\ell(R)$, *then* $a_1, a_2, \ldots, a_n$ *are all nilpotent.*

Proof. Let $\deg(f(x)) = 0$. Then $f(x) = a_0$. Hence $b_ja_0 = 0$ for each $1 \leq j \leq m$ and $b_0a_0 = c$, by Lemma 2.2. If $c \neq 0$, then $b_0 \neq 0$ and so $r = b_0$ and $a = 1$ are desired non-zero elements. If $c = 0$, then $b_ja_0 = 0$ for each $0 \leq j \leq m$. Since $g(x) \neq 0$, there exists $0 \leq s \leq m$ such that $b_s \neq 0$. Thus $r = b_s$ and $a = 1$ are desired non-zero elements.

Now let $\deg(f(x)) = n \geq 1$. We proceed by induction on degree of $g(x)$. If $\deg(g(x)) = m = 0$, then $g(x) = b_0 \neq 0$. Since $g(x)f(x) = c$, we have $b_0a_0 = c$ and $b_0a_i = 0$ for each $1 \leq i \leq n$. Thus $r = b_0$ and $a = 1$ are desired elements. Assume that the conclusion is true for all polynomials of degree less that m. Let $g(x)f(x) = c$ and $\deg(g(x)) = m$. We proceed by dividing the proof into two cases:

Case 1: Let $b_0 = b_1 = \cdots = b_{m-1} = 0$. Since $c = g(x)f(x) = b_mx^mf(x)$ we have $b_ma_i = 0$ for each $0 \leq i \leq n$, by Lemma 2.1, and so $c = 0$. Thus $r = b_m$ and $a = 1$ are desired elements.

Case 2: Assume that there exits $0 \leq j \leq m-1$ such that $b_j \neq 0$. If $b_mf(x) = 0$, then $b_mx^mf(x) = 0$, by Lemma 2.1, and so $c = g(x)f(x) = (b_0 + b_1x + \cdots + b_{m-1}x^{m-1})f(x)$. Since $g_1(x) = b_0 + b_1x + \cdots + b_{m-1}x^{m-1} \neq 0$, hence $\deg(g_1(x)) \leq m-1$ and $g_1(x)f(x) = c$, so the result follows from induction hypothesis. Now assume that $b_mf(x) \neq 0$. Then $b_mx^mf(x) \neq 0$, by Lemma 2.1. Hence $c = g(x)f(x) = g(x)(a_{i_1}x^{i_1} + \cdots + a_{i_t}x^{i_t})$ such that $g(x)a_{i_k} \neq 0$ for each $1 \leq k \leq t$, where $0 \leq i_1$, $i_t \leq n$. Thus $b_m\alpha^m(a_{i_t}) = 0$ and so $b_ma_{i_t} = 0$, by Lemma 2.1. Hence $a_{i_t}b_m = 0$, since R is reversible. Thus $a_{i_t}c = (a_{i_t}g(x))f(x)$. Since $a_{i_t}g(x) \neq 0$, $\deg(a_{i_t}g(x)) \leq m-1$ and $(a_{i_t}g(x))f(x) = a_{i_t}c$, hence by induction hypothesis there exist elements $0 \neq r_1, a_1 \in R$ such that $r_1f(x) = a_1a_{i_t}c$. Thus $r = r_1$ and $a = a_1a_{i_t}$ are desired elements.

Now we will show that the elements $a_1, a_2, \ldots, a_n$ are all nilpotent, when b_0 is unit modulo $Ni\ell(R)$. Since R is reversible and hence $Ni\ell(R)$ is an ideal of R, then $\overline{R} = R/Ni\ell(R)$ is reduced. On the other hand, since $Ni\ell(R)$ is an (α,δ)-compatible ideal of R, hence by [7, Proposition 2.1], $\overline{R} = R/Ni\ell(R)$ is $(\bar{\alpha},\bar{\delta})$-compatible, and therefore $\overline{R}$ is $\bar{\alpha}$-rigid, by [9, Lemma 2.2]. Now from $g(x)f(x) = c$, we get $\bar{g}(x)\bar{f}(x) = \bar{c} \in \bar{R}$, and hence by Lemma 2.3 we have $\bar{b}_0\bar{a}_0 = \bar{c}$ and $\bar{b}_j\bar{a}_i = \bar{0}$ for each i, j with $i + j \geq 1$. Now since b_0 is a unit element of R, then $d_0b_0 = 1$ for some $d_0 \in R$. Multiplying $\bar{b}_0\bar{a}_i = \bar{0}$ from left by $\bar{d}_0$ we get $\bar{a}_i = \bar{0}$ for each $i \geq 1$. This yields that $a_i \in Ni\ell(R)$ for each $i \geq 1$, completing the proof. □

The same idea as that in the proof of Theorem 2.5, can be used to prove the following theorem.

Theorem 2.6. *Let* R *be a reversible* (α,δ)*-compatible ring and* $f(x) = a_0 + a_1x + \cdots + a_nx^n$ *be a non-zero element of* $R[x;\alpha,\delta]$. *If there is a non-zero element* $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha,\delta]$ *such that* $f(x)g(x) = c$ *is a constant, then there* *exist non-zero elements $r, a \in R$ such that $f(x)r = ca$. In particular $r = b_p a$ for some p, $0 \leq p \leq m$, and a is either one or a product of at most m coefficients from $f(x)$. Furthermore, if b_0 is a unit element modulo $Ni\ell(R)$, then $a_1, \ldots, a_n$ are nilpotent.*

Theorems 2.5 and 2.6, have the following immediate corollary.

Corollary 2.7. *Let R be a reversible (α, δ)-compatible ring. Then $f(x) \in R[x; \alpha, \delta]$ is a right or left zero-divisor, if and only if there exists a non-zero constant $r \in R$ such that $rf(x) = 0 = f(x)r$.*

Proof. It follows from Theorems 2.5, 2.6 and Lemma 2.2. □

In particular, taking $c = 0$, α to be identity and δ equal to zero, we obtain the following main result of [17], as another corollary of Theorem 2.5.

Corollary 2.8.([17, Theorem 2]) *Let R be a reversible ring. Then R is right and left McCoy ring.*

Let δ be an α-derivation of R. For integers $0 \leq i \leq j$, let us write f_i^j for the set of all "words" in α and δ in which there are i factors of α and $j - i$ factors of δ. For instance, $f_j^j = \{\alpha^j\}$, $f_0^j = \{\delta^j\}$ and $f_{j-1}^j = \{\alpha^{j-1}\delta, \alpha^{j-2}\delta\alpha, \ldots, \delta\alpha^{j-1}\}$. Clearly each element of f_i^j is an additive map on R. Also for each $a \in R$ and each integers $0 \leq i \leq j$, we write $f_i^j(a) = \{\beta(a) \mid \beta \in f_i^j\}$.

We say also that a set $S \subseteq R$ is *locally nilpotent* if for any subset $\{s_1, \ldots, s_n\} \subseteq S$, there exists an integer t, such that any product of t elements from $\{s_1, \ldots, s_n\}$ is zero.

Lemma 2.9 *Let R be an (α, δ)-compatible ring and $Ni\ell(R)$ a locally nilpotent ideal of R. Then $Ni\ell(R)[x; \alpha, \delta]$ is a nil ideal of R.*

Proof. Since R is (α, δ)-compatible ideal of R, $Ni\ell(R)$ is an (α, δ)-compatible ideal of R, by Lemma 2.1. Hence $Ni\ell(R)[x; \alpha, \delta]$ is an ideal of $R[x; \alpha, \delta]$. Let $f(x) = a_0 + a_1x + \cdots + a_mx^m \in Ni\ell(R)[x; \alpha, \delta]$. Assume that $M = \{a_0, a_1, \ldots, a_m\} \subseteq Ni\ell(R)$. Since $Ni\ell(R)$ is locally nilpotent, there exists a positive integer t such that any product of t elements from M is zero. Let $N = \bigcup f_i^j(a_r)$, where $0 \leq r \leq m$ and $0 \leq i \leq j$ are integers. Then any product of t elements of N is zero, by Lemma 2.1. Thus $(f(x))^t = 0$, since each coefficient of $(f(x))^t$ is a finite sum of the product of t elements from N. Therefore $Ni\ell(R)[x; \alpha, \delta]$ is a nil ideal of $R[x; \alpha, \delta]$. □

We continue by proving the second main result of this paper, which investigates the equivalences of the weakly 2-primal property of Ore extension ring with the coefficent ring R, when R is an (α, δ)-compatible ring.

Theorem 2.10. *Let R be an (α, δ)-compatible ring. Then R is a weakly 2-primal ring if and only if $R[x; \alpha, \delta]$ is a weakly 2-primal ring. In this case $R[x; \alpha, \delta]$ is a weakly semicommutative ring.*

Proof. For the forward direction, since R is weakly 2-primal, we have $L - rad(R) =$

$Ni\ell(R)$. We will show that $L-rad(R[x;\alpha,\delta]) = Ni\ell(R[x;\alpha,\delta])$. It is enough to show that $Ni\ell(R[x;\alpha,\delta]) \subseteq L-rad(R[x;\alpha,\delta])$, since the reverse inclusion is obvious. First we show that $Ni\ell(R[x;\alpha,\delta]) = Ni\ell(R)[x;\alpha,\delta] = L-rad(R)[x;\alpha,\delta]$. By Lemma 2.9, we have $Ni\ell(R)[x;\alpha,\delta]$ is a nil ideal of $R[x;\alpha,\delta]$, and hence $Ni\ell(R)[x;\alpha,\delta] \subseteq Ni\ell(R[x;\alpha,\delta])$. For the reverse inclusion, assume that $f(x) = \sum_{i=0}^{m} a_i x^i$ is a nilpotent element with the nilpotency index t. Therefore we get $a_m\alpha^m(a_m)\alpha^{2m}\ldots\alpha^{(t-1)m}(a_m) = 0$, since it is the leading coefficient of $(f(x))^t = 0$. Then $a_m^t = 0$, by Lemma 2.1, and so $a_m \in Ni\ell(R)$. Since $Ni\ell(R)$ is an (α,δ)-compatible ideal, $f_i^j(a_m) \subseteq Ni\ell(R)$ for each integers $0 \le i \le j$. Thus $0 = (f(x))^t = (a_0 + a_1x + \cdots + a_{m-1}x^{m-1})^t + h(x)$, where $h(x) \in Ni\ell(R)[x;\alpha,\delta]$. Then $(a_0 + a_1x + \cdots + a_{m-1}x^{m-1})^t \in Ni\ell(R)[x;\alpha,\delta]$. Then $f(x) \in Ni\ell(R[x;\alpha,\delta])$, since $Ni\ell(R)[x;\alpha,\delta]) \subseteq Ni\ell(R[x;\alpha,\delta])$. Now by induction hypothesis on degree of $f(x)$ we conclude that $a_0, a_1, \ldots, a_{m-1} \in Ni\ell(R)$, which implies that $f(x) \in Ni\ell(R)[x;\alpha,\delta]$. Therefore $Ni\ell(R[x;\alpha,\delta]) = Ni\ell(R)[x;\alpha,\delta] = L-rad(R)[x;\alpha,\delta]$.

Next we show that $L-rad(R)[x;\alpha,\delta]$ is a locally nilpotent ideal of $R[x;\alpha,\delta]$. Assume that $M = \{f_1(x), \ldots, f_k(x)\}$ be a subset of $L-rad(R)[x;\alpha,\delta]$. Write $f_i(x) = a_{i0} + a_{i1}x + \cdots + a_{im}x^m$, where a_{ij} is in $L-rad(R)$ for all $i = 1, 2, \cdots, k$ and $j = 0, 1, \ldots, n$. Let $N = \{a_{i0}, a_{i1}, \ldots, a_{im} \,|\, i = 1, 2, \ldots, k\}$. Then N is a finite subset of $L-rad(R)$ and since $L-rad(R)$ is locally nilpotent, there exists a positive integer t such that any product of t elements from N is zero. Assume that $W = \bigcup f_i^j(a_{rs})$, where $1 \le r \le k$, $0 \le s \le m$ and $0 \le i \le j$ are non-negative integers. Then by Lemma 2.1, any product of t elements from W is zero, which implies that any product of t elements from M is zero. Therefore $L-rad(R)[x;\alpha,\delta]$ is a locally nilpotent ideal of $R[x;\alpha,\delta]$. The backward direction is clear, since each subring of a weakly 2-primal ring is weakly 2-primal.

Moreover it is clear that the ring $R[x;\alpha,\delta]$ is weakly semicommutative, since $R[x;\alpha,\delta]/Ni\ell(R[x;\alpha,\delta])$ is a reduced ring, and the proof is complete. □

Corollary 2.11. *Let R be a reversible ring which is (α,δ)-compatible. Then a polynomial $f(x) \in R[x;\alpha,\delta]$ is unit if and only if its constant term is a unit and the other coefficionts are nilpotent.*

Proof. Let $\bar{R} = R/Ni\ell(R)$. Then $\bar{R}$ is $\bar{\alpha}$-rigid, by [9, Lemma 2.2]. For the forward direction, let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta]$ is a unit. Then there exists $g(x) \in R[x;\alpha,\delta]$ such that $f(x)g(x) = 1 = g(x)f(x)$. Hence $\bar{f}(x)\bar{g}(x) = 1$. Then $\bar{a}_0$ is a unit element of $\bar{R}$, by Lemma 2.3, and so $a_1, \ldots, a_n$ are nilpotent, by Theorem 2.5.

For the reverse implication, let a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent. Then $a_1x + \cdots + a_nx^n \in ni\ell(R)[x;\alpha,\delta] = L-rad(R[x;\alpha,\delta]) \subseteq J(R[x;\alpha,\delta])$, by Theorem 2.10. Thus $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is a unit of $R[x;\alpha,\delta]$. □

We denote the unit group of R by $U(R)$.

Corollary 2.12. *Let R be an α-rigid ring. Then $U(R[x;\alpha,\delta]) = U(R)$.*

Corollary 2.13. *Let R be a NI ring. If $Ni\ell(R)$ is an (α,δ)-compatible ideal of R, then $U(R[x;\alpha,\delta]) \subseteq U(R) + Ni\ell(R)[x;\alpha,\delta]$.*

Proof. Since R is NI, hence $Ni\ell(R)$ is an ideal of R and then $\overline{R} = R/Ni\ell(R)$ is reduced. Since $Ni\ell(R)$ is an (α, δ)-compatible ideal of R, then the ring $\overline{R}$ is $(\overline{\alpha}, \overline{\delta})$-compatible, by [7, Proposition 2.1]. Hence $\overline{R}$ is $\overline{\alpha}$-rigid, by [9, Lemma 2.2]. Now, the result follows from Corollary 2.12. □

Corollary 2.14. *Let R be a weakly 2-primal ring which is (α, δ)-compatible. Then $U(R[x; \alpha, \delta]) = U(R) + Ni\ell(R)[x; \alpha, \delta] = U(R) + Ni\ell(R[x; \alpha, \delta])$.*

Proof. Since R is (α, δ)-compatible, $Ni\ell(R)$ is (α, δ)-compatible, by Lemma 2.1. Since a weakly 2-primal ring is NI, we have $U(R[x; \alpha, \delta]) \subseteq U(R) + Ni\ell(R)[x; \alpha, \delta]$, by Corollary 2.13. Since R is weakly 2-primal, $Ni\ell(R)[x; \alpha, \delta] = Ni\ell(R[x; \alpha, \delta])$. Thus $U(R) + Ni\ell(R)[x; \alpha, \delta] \subseteq U(R[x; \alpha, \delta])$. Therefore $U(R[x; \alpha, \delta]) = U(R) + Ni\ell(R)[x; \alpha, \delta]$. □

Corollary 2.15. *Let R be a weakly 2-primal ring which is (α, δ)-compatible. Then $J(R[x; \alpha, \delta]) = Ni\ell(R)[x; \alpha, \delta]$.*

Proof. Following Theorem 2.10, we get

$$Ni\ell(R)[x; \alpha, \delta] = Ni\ell(R[x; \alpha, \delta]) = L - rad(R)[x; \alpha, \delta] = L - rad(R[x; \alpha, \delta]).$$

Thus the result follows from Corollary 2.14. □

Our next result concerns with stable range 1 property of Ore extension rings. Recall that an element a in any ring R is said to have (*right*) *stable range* 1 (written $S_r(a) = 1$) if $aR + bR = R$ (for any $b \in R$) implies that $a + br \in U(R)$ for some $r \in R$. If $S_r(a) = 1$ for all $a \in R$, then a ring R is said to have stable range 1, written $S_r(R) = 1$. It is well known that this property is left-right symmetric.

Proposition 2.16. *If $Ni\ell(R)$ is an (α, δ)-compatible ideal of R, then we have $S_r(R[x; \alpha, \delta]) > 1$.*

Proof. We will begin by assuming, for a contradiction, that $S_r(R[x; \alpha, \delta]) = 1$. Now, since $x(-x)+1+x^2 = 1$, then there exists $f(x) \in R[x; \alpha, \delta]$ such that $x+(1+x^2)f(x)$ is a unit in $R[x; \alpha, \delta]$. For each $a \in R$ we have $x^2 a = \delta^2(a) + [\alpha\delta(a) + \delta\alpha(a)]x + \alpha^2(a)x^2$. Write $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. Then

$$\begin{aligned} x + (1 + x^2)f(x) = {} & [\delta^2(a_0) + a_0] \\ & + [\delta^2(a_1) + \alpha\delta(a_0) + \delta\alpha(a_0) + a_1 + 1]x \\ & + [\delta^2(a_2) + \alpha\delta(a_1) + \delta\alpha(a_1) + \alpha^2(a_0) + a_2]x^2 \\ & + [\delta^2(a_3) + \alpha\delta(a_2) + \delta\alpha(a_2) + \alpha^2(a_1) + a_3]x^3 \\ & \vdots \\ & + [\delta^2(a_{n-1}) + \alpha\delta(a_{n-2}) + \delta\alpha(a_{n-2}) + \alpha^2(a_{n-3}) + a_{n-1}]x^{n-1} \\ & + [\delta^2(a_n) + \alpha\delta(a_{n-1}) + \delta\alpha(a_{n-1}) + \alpha^2(a_{n-2}) + a_n]x^n \\ & + [\alpha\delta(a_n) + \delta\alpha(a_n) + \alpha^2(a_{n-1})]x^{n+1} \\ & + [\alpha^2(a_n)]x^{n+2}. \end{aligned}$$

Since $Ni\ell(R)$ is an ideal of R, hence R is weakly 2-primal and so by Corollary 2.14, the constant term of $x+(1+x^2)f(x)$ is unit and other coefficients are all nilpotent. Then nilpotency of $\alpha^2(a_n)$ and Lemma 2.1 imply that a_n is nilpotent. Since $Ni\ell(R)$ is an α-invariant and a δ-ideal, we have $\alpha\delta(a_n)+\delta\alpha(a_n) \in Ni\ell(R)$. Now nilpotency of $[\alpha\delta(a_n)+\delta\alpha(a_n)+\alpha^2(a_{n-1})]$ and Lemma 2.1 implies that $a_{n-1} \in Ni\ell(R)$. By a similar argument, one can show that $a_0, a_1, \ldots, a_n \in Ni\ell(R)$. Then $[\delta^2(a_0)+a_0] \in Ni\ell(R)$, which is a contradiction, since it is the constant term of $x+(1+x^2)f(x)$. Therefore $S_r(R[x;\alpha,\delta]) > 1$, as desired. □

3. The Examples

Below we construct examples of rings which help us to support our main results and also explain the limitations of the obtained results. The class of NI rings which are (α,δ)-compatible are quite large and important. For some other classes of examples, we direct the reader to see [6]. As a first example, we present some classes of rings with this property.

Example 3.1. Let S be any reduced ring and consider the reversible and hence NI ring $R = \{(a,b) \mid a,b \in S\}$ with addition pointwise and multiplication given by $(a,b)(c,d) = (ac, ad+bc)$. Let $\alpha : R \to R$ be an automorphism defined by $\alpha((a,b)) = (a,-b)$ and $\delta : R \to R$ be an α-derivation defined by $\delta((a,b)) = (a,b) - \alpha((a,b)) = (0,-b)$. We will show that R is (α,δ)-compatible. To see this, let $(a,b)(c,d) = 0$. Thus $ac = ad+bc = 0$. Therefore $ca = 0$, since S is reduced. Now, multiplying $ad+bc = 0$ from left by c, we get $cad+cbc = 0$ and then $cbc = 0$. Hence $(bc)^2 = bcbc = 0$ and so $bc = ad = 0$, since S is reduced. Then $(a,b)\alpha((c,d)) = (ac, bc-ad) = (0,0)$. Similarly, one can see that $(a,b)\alpha((c,d)) = (0,0)$ implies that $(a,b)(c,d) = (0,0)$. Thus R is α-compatible. On the other hand, let $(a,b)(c,d) = 0$. Then similar computations as above show that $(a,b)\delta((c,d)) = 0$. Therefore R is (α,δ)-compatible, as desired.

Let R be a ring and σ denotes an endomorphism of R with $\sigma(1) = 1$. In [5], the authors introduced *skew triangular matrix* ring as a set of all triangular matrices with addition point-wise and a new multiplication subject to the condition $E_{ij}r = \sigma^{j-i}(r)E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + \cdots + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it by $T_n(R,\sigma)$.

The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R,n,\sigma)$; and the subring of the skew triangular matrices with constant diagonals is denoted by $T(R,n,\sigma)$. We can denote $A = (a_{ij}) \in T(R,n,\sigma)$ by $(a_{11},\ldots,a_{1n})$. Then $T(R,n,\sigma)$ is a ring with addition point-wise and multiplication given by:

$$(a_0,\ldots,a_{n-1})(b_0,\ldots,b_{n-1}) = (a_0b_0, a_0*b_1+a_1*b_0, \ldots, a_0*b_{n-1}+\cdots+a_{n-1}*b_0),$$

with $a_i * b_j = a_i\sigma^i(b_j)$, for each i and j. Therefore, clearly one can see that $T(R,n,\sigma) \cong R[x;\sigma]/(x^n)$, where (x^n) is the ideal generated by x^n in $R[x;\sigma]$.

We consider the following two subrings of $S(R,n,\sigma)$, as follows:

$$A(R,n,\sigma)=\left\{\sum_{j=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{i=1}^{n-j+1}a_jE_{i,i+j-1}+\sum_{j=\lfloor\frac{n}{2}\rfloor+1}^{n}\sum_{i=1}^{n-j+1}a_{i,i+j-1}E_{i,i+j-1}\right\};$$

$$B(R,n,\sigma)=\{A+rE_{1k}\mid A\in A(R,n,\sigma)\text{ and }r\in R\}\qquad n=2k\geq 4.$$

Let α and σ be endomorphisms of a ring R and δ is an α-derivation, with $\alpha\sigma=\sigma\alpha$ and $\delta\sigma=\sigma\delta$. The endomorphism α of R is extended to the endomorphism $\bar{\alpha}:T_n(R,\sigma)\to T_n(R,\sigma)$ defined by $\bar{\alpha}((a_{ij}))=(\alpha(a_{ij}))$ and the α-derivation δ of R is also extended to $\bar{\delta}:T_n(R,\sigma)\to T_n(R,\sigma)$ defined by $\bar{\delta}((a_{ij}))=(\delta(a_{ij}))$.

Example 3.2. α be a rigid endomorphism of a ring R, δ and α-derivation and σ an endomorphism of R such that $\alpha\sigma=\sigma\alpha$ and $\sigma\delta=\delta\sigma$. If R is α-rigid, then the ring $T(R,n,\sigma)[x;\bar{\alpha},\bar{\delta}]$ is a reversible ring.

Proof. We break the proof in to some parts.

Claim 1: If R is a (α,δ) compatible and (α,δ)-skew Armendariz, then R is a reversible ring if and only if $R[x;\alpha,\delta]$ is reversible.
Proof. We only need to prove the necessity. For this, let $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{n}b_jx^j\in R[x;\alpha,\delta]$ such that $f(x)g(x)=0$. Then $a_ix^ib_jx^j=0$, for each i and j, by (α,δ)-skew Armendariz property of R. Then by Lemma 2.1, we have $a_ib_j=0$, for each i and j. Since R is reversible $b_ja_i=0$ and hence $b_j\alpha^k(a_i)=0$, for each $k\geq 0$ and $b_j\delta^l(a_i)=0$, for each $l\geq 0$, since R is (α,δ) compatible. Then $g(x)f(x)=0$ and therefore $R[x;\alpha,\delta]$ is reversible.

Claim 2: Let R be a σ-rigid ring and $A=(a_{ij})$, $B=(b_{ij})\in A(R,n,\sigma)$ such that $AB=0$, then $a_{ik}b_{kj}=0$, for each $1\leq i,j,k\leq n$.
Proof. We proceed by induction on n. The case $n=1$ is clear and hence the base case of our induction is established. So, we may assume $n>1$ and that the claim is true for all smaller values by inductive assumption. Let $A=(a_{ij})$ and $B=(b_{ij})$ in $A(R,n,\sigma)$ such that $AB=0$. Consider the following elements in $A(R,n,\sigma)$: $A'=(a'_{ij}),A''=(a''_{ij}),B'=(b'_{ij}),B''=(b''_{ij})$, where $a'_{ij}=a_{ij},a''_{ij}=a_{i+1,j+1},b'_{ij}=b_{ij},b''_{ij}=b_{i+1,j+1}$, for all $1\leq i,j\leq n-1$. Now from $AB=0$, we get $A'B'=0=A''B''$. Thus by induction hypothesis, we have

$$a_{ik}b_{kj}=0,\qquad a_{ln}b_{nn}=0,\qquad a_{lk}b_{kn}=0,$$

for all $1\leq i,j,k\leq n-1$ and $2\leq l\leq n$. Now it is sufficient to show that $a_{1k}b_{kn}=0$, for each k. From the entry $1n$ in $AB=0$, we have that

$$a_{11}b_{1n}+a_{12}\sigma(b_{2n})+\cdots+a_{1n}\sigma^{n-1}(b_{nn})=0. \tag{3.1}$$

Since $A,B\in A(R,n,\sigma)$, we have that $a_{ij}=a_{i+1,j+1}$, for $1\leq i\leq j\leq\lfloor\frac{n}{2}\rfloor$ and $b_{kn}=b_{k-1,n-1}$ for $\lceil\frac{n}{2}\rceil\leq k\leq n$, where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x and also $\lceil x\rceil$ denotes the smallest integer greater than or equal to x.

Since R is σ-rigid, we have $\sigma^{j-1}(b_{jn})a_{11} = \sigma^{j-1}(b_{jn})a_{jj} = 0$, for all $1 < j \leq n$. Thus multiplying (3.1) by a_{11} on the right, we obtain $a_{11}b_{1n}a_{11} = 0$. Since R is reduced, we get $a_{11}b_{1n} = 0$. Therefore

$$a_{12}\sigma(b_{2n}) + a_{13}\sigma^2(b_{3n}) + \cdots + a_{1n}\sigma^{n-1}(b_{nn}) = 0. \tag{3.2}$$

Since R is σ-rigid, $\sigma^{j-1}(b_{jn})a_{12} = \sigma^{j-1}(b_{jn})a_{j-1,j} = 0$, for all $2 < j \leq n$ if $2 \leq \lfloor \frac{n}{2} \rfloor$. Thus, multiplying (3.2) by a_{12} on the right, we obtain $a_{12}\sigma(b_{2n})a_{12} = 0$. Since R is reduced, we get $a_{12}\sigma(b_{2n}) = 0$. Therefore

$$a_{13}\sigma^2(b_{3n}) + \cdots + a_{1n}\sigma^{n-1}(b_{nn}) = 0, \tag{3.3}$$

and also, by σ-compatibility of R, we get $a_{12}b_{2n} = 0$. Repeating in this way, we get $a_{1k}b_{kn} = 0$, for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and also

$$a_{1,\lfloor \frac{n}{2} \rfloor+1}\sigma^{\lfloor \frac{n}{2} \rfloor}(b_{\lfloor \frac{n}{2} \rfloor+1,n}) + \cdots + a_{1n}\sigma^{n-1}(b_{nn}) = 0. \tag{3.4}$$

Since R is σ-rigid, we get $\sigma^{n-1}(b_{nn})a_{1k} = \sigma^{n-1}(b_{kk})a_{1k} = 0$, for all $1 \leq k < n$. Thus, multiplying (3.4) by $\sigma^{n-1}(b_{nn})$ on the left, we obtain $\sigma^{n-1}(b_{nn})a_{1n}\sigma^{n-1}(b_{nn}) = 0$. Using the reduced property of a ring R, we get $a_{1n}\sigma^{n-1}(b_{nn}) = 0$. Therefore

$$a_{1,\lfloor \frac{n}{2} \rfloor+1}\sigma^{\lfloor \frac{n}{2} \rfloor}(b_{\lfloor \frac{n}{2} \rfloor+1,n}) + \cdots + a_{1,n-1}\sigma^{n-2}(b_{n-1,n}) = 0 \tag{3.5}$$

and also using the σ-compatibility of R, we get $a_{1n}b_{nn} = 0$. By continuing in this way, we can show that $a_{1k}b_{kn} = 0$, for all $\lceil \frac{n}{2} \rceil \leq k \leq n$. Thus if n is even, we are done. Also, if n is odd, we obtain that

$$a_{1,\lfloor \frac{n}{2} \rfloor+1}\sigma^{\lfloor \frac{n}{2} \rfloor}(b_{\lfloor \frac{n}{2} \rfloor+1,n}) = 0,$$

and by σ-compatibility of R, we get

$$a_{1,\lfloor \frac{n}{2} \rfloor+1}b_{\lfloor \frac{n}{2} \rfloor+1,n} = 0.$$

Therefore $a_{ik}b_{kj} = 0$, for all $1 \leq i, j, k \leq n$, as desired.

Now, by Claim 2, one can see easily that $T(R, n, \sigma)$ is reversible and $(\bar{\alpha}, \bar{\delta})$-compatible. Therefore by Claim 1, it is sufficient to prove that $T(R, n, \sigma)$ is $(\bar{\alpha}, \bar{\delta})$-skew Armendariz ring. For, consider the map $\varphi : T(R, n, \sigma)[x; \bar{\alpha}, \bar{\delta}] \to T(R[x; \alpha, \delta], n, \bar{\sigma})$ defined by $\varphi(\sum_{k=0}^{m} A_k x^k) = (f_{ij})$, where $A_k = (a_{ij}^{(k)})$ and $f_{ij} = \sum_{k=0}^{m} a_{ij}^{(k)} x^k$, for each m and $1 \leq i, j \leq n$. It is easy to show that φ is an isomorphism. Let $f = \sum_{k=0}^{r} A_k x^k$ and $g = \sum_{l=0}^{s} B_l x^l \in T(R, n, \sigma)[x; \bar{\alpha}, \bar{\delta}]$ and $fg = 0$, where $A_k = (a_{ij}^{(k)})$ and $B_l = (b_{ij}^{(l)})$. So we have $(f_{ij})(g_{ij}) = 0$, where $f_{ij} = \sum_{k=0}^{r} a_{ij}^{(k)} x^k$ and $g_{ij} = \sum_{l=0}^{s} b_{ij}^{(l)} x^l$. Since R is σ-rigid, $R[x; \alpha, \delta]$ is $\bar{\sigma}$-rigid. This is because, if $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x; \alpha, \delta]$ and $f(x)\bar{\sigma}(f(x)) = 0$, then we have $a_m\alpha^m(\sigma(a_m)) = 0$ and consequently $a_m\sigma(\alpha^m(a_m)) = 0$, since $\alpha\sigma = \sigma\alpha$. Thus $a_m\alpha^m(a_m) = 0$, since R is σ-rigid and so $a_m = 0$, since R is α-rigid. Hence $f(x) = 0$

and hence $R[x;\alpha,\delta]$ is $\bar{\sigma}$-rigid. Thus $f_{it}g_{tj}=0$, for each $1\le i,j,t\le n$ and easy calculations show that $A_kx^kB_lx^l=0$, for each k and l, showing that $T(R,n,\sigma)$ is $(\bar{\alpha},\bar{\delta})$-skew Armendariz ring, and the result follows. □

The following example shows that in Theorems 2.5 and 2.6, the hypothesis that "R is (α,δ)-compatible" can not be dropped.

Example 3.3.

(1) Let R_0 be any reduced ring. Then for a reversible ring $R=R_0[x]$, consider the endomorphism $\alpha: R\to R$ given by $\alpha(f(x))=f(0)$ and α-derivation $\delta: R\to R$ given by $\delta(f(x))=xf(x)-\alpha(f(x))x$. Then one can see easily that the ring R is δ-compatible but R is not α-compatible. Now considering the elements $F(y)=a_0+a_1y$ and $G(y)=b_0+b_1y$ in $R[y;\alpha,\delta]$, where $a_0=b_0=x, a_1=0$ and $b_1=-x$, we get $G(y)F(y)=x^2-x^3=c\in R$ but $b_0a_0=x^2\neq c$.

(2) Let S and T be any reduced rings. Suppose $R=S\oplus T$ with the usual addition and multiplication. Then for reversible ring R, let $\alpha: R\to R$ be an endomorphism defined by $\alpha((a,b))=(b,a)$ and $\delta: R\to R$ be an α-derivation defined by $\delta((a,b))=(a-b,0)$. Then it can be easily seen that the ring R is neither α-compatible nor δ-compatible. Let $f(x)=(-1,0)+(1,0)x$ and $g(x)=(1,0)+(-1,0)x$ be non-zero elements in $R[x;\alpha,\delta]$. Then $f(x)g(x)=(0,0)$, but $(-1,0)(1,0)=(-1,0)\neq(0,0)$.

It is well known that when R is an NI ring and (α,δ)-compatible, then $Ni\ell(R)$ is an (α,δ)-compatible ideal of R. But, the following example shows that the converse is not true in general.

Example 3.4. Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Consider the reversible and hence NI ring $R=\{(a,b)\,|\,a,b\in\mathbb{Z}_4\}$ with addition pointwise and multiplication given by $(a,b)(c,d)=(ac,ad+bc)$. Let $\alpha: R\to R$ be an endomorphism defined by $\alpha((a,b))=(a,2b)$. We will show that R is not α-compatible ring. To see this, let $r=(2,0)$ and $s=(0,1)$ in R, then $rs=(0,2)\neq(0,0)$ whereas $r\alpha(s)=(2,0)(0,2)=(0,0)$. Thus R is not α-compatible. On the other hand, since $Ni\ell(R)=\{(0,b),(2,b)|\ b\in\mathbb{Z}_4\}$, easy calculations show that $Ni\ell(R)$ is an α-compatible ideal of R.

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