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UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING THE SHIFTS AND DERIVATIVES†

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Abstract. This paper is devoted to studying the sharing value problem for the derivative of a meromorphic function with its shift and q-difference. The results in the paper improve and generalize the recent result due to Qi, Li and Yang [28].

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1. Introduction and main results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. Let k be a positive integer or infinity and $a \in C \cup \{\infty\}$. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity k is counted k times in the set. If these zeros points are only counted once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM; if $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM. We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of distinct a-points of f with multiplicities not greater than k. We denote by $N_{k)}(r, 1/(f-a))$ the counting function for zeros of $f-a$ with multiplicity less than or equal to k, and by $\overline{N}_{k)}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. It is assumed that the reader is familiar with the notations of Nevanlinna theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $S(r,f)$ and so on, that can be found, for instance, in

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[14][36].

Around 2001, I Lahiri introduced the notion of weighted sharing, which measures how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

Definition 1.1. [16] For a complex number $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with mutiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. For a complex number $a \in C \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Mermorphic functions sharing values with their derivatives has become a subject of great interest in uniqueness theory. The paper by Rubel and Yang is the starting point of this topic, along with the following.

Theorem 1.2. [30] *Let f be a nonconstant entire function. If f and f' share two distinct finite values CM, then $f = f'$.*

Now one may ask the following question: Can we change the number 2 of shared values to 1 in the Theorem 1.1 ? The following counterexample from shows the answer is negative. Let $f = e^{e^z} \int_0^z e^{-e^t}(1 - e^t)dt$. Clearly, f and f' share 1 CM but $f \neq f'$. In a special case, we recall a well-known conjecture by Brück [4]: Let f be a nonconstant entire function such that hype order $\sigma_2(f) < \infty$ and $\sigma_2(f)$ isn't positive integer. If f and f' share the finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is nonzero constant. The conjecture has been verified in the special cases when $a = 0$ [4], or when f is of finite order [12], or when $\sigma_2(f) < \frac{1}{2}$ [7]. Many results have been obtained for this and related topics(See [1],[5],[11],[17],[18],[23]-[27],[31],[32],[34],[35],[37],[39],[41]-[44] and the references therein).

Heittokangas et al. considered analogues of Brück's conjecture for meromorphic functions concerning their shifts, and proved the following theorem.

Theorem 1.3. [15] *Let f be a meromorphic function of order $\sigma(f) < 2$ and let $c \in C$. If $f(z)$ and $f(z + c)$ share the values $a \in C$ and ∞ CM, then*

$$\frac{f(z+c) - a}{f(z) - a} = \tau\,,$$

for some constant τ.

Since then, many mathematicians considered this topic (See [6],[8],[10],[19]-[22],[29],[38] and the references therein). In 2018, Qi, Li and Yang considered the value sharing problem related to $f'(z)$ and $f(z+c)$, where c is a complex number. They obtained the following result.

Theorem 1.4. [28] *Let* $f(z)$ *be a non-constant meromorphic function of finite order,* $n \geq 9$ *be an integer. If* $[f'(z)]^n$ *and* $f^n(z+c)$ *share* $a(\neq 0)$ *and* ∞ *CM, then* $f'(z) = tf(z+c)$*, for a constant* t *that satisfies* $t^n = 1$.

It is natural to ask whether the nature of sharing values can be reduced in Theorem 1.4. Considering this question, we prove the following results.

Theorem 1.5. *Let* $f(z)$ *be a non-constant meromorphic function of finite order,* $n \geq 10$ *be an integer. If* $[f'(z)]^n$ *and* $f^n(z+c)$ *share* $(1,2)$ *and* $(\infty,0)$*, then* $f'(z) = tf(z+c)$*, for a constant* t *that satisfies* $t^n = 1$.

Theorem 1.6. *Let* $f(z)$ *be a non-constant meromorphic function of finite order,* $n \geq 9$ *be an integer. If* $[f'(z)]^n$ *and* $f^n(z+c)$ *share* $(1,2)$ *and* (∞,∞)*, then* $f'(z) = tf(z+c)$*, for a constant* t *that satisfies* $t^n = 1$.

Theorem 1.7. *Let* $f(z)$ *be a non-constant meromorphic function of finite order,* $n \geq 17$ *be an integer. If* $[f'(z)]^n$ *and* $f^n(z+c)$ *share* $(1,0)$ *and* $(\infty,0)$*, then* $f'(z) = tf(z+c)$*, for a constant* t *that satisfies* $t^n = 1$.

Corollary 1.8. *Let* $f(z)$ *be a non-constant entire function of finite order,* $n \geq 5$ *be an integer. If* $[f'(z)]^n$ *and* $f^n(z+c)$ *share* $(1,2)$*, then* $f'(z) = tf(z+c)$*, for a constant* t *that satisfies* $t^n = 1$.

Remark 1.1. It's obvious that the condition that $[f'(z)]^n$ and $f^n(z+c)$ share $(1,2)$ and (∞,∞) in Theorem 1.6 is weaker than the condition $[f'(z)]^n$ and $f^n(z+c)$ share $a(\neq 0)$ and ∞ CM in Theorem 1.4.

If the shifts $f(z+c)$ in Theorem 1.5 and 1.6 are replaced by q-difference $f(qz)$, we obtain

Theorem 1.9. *Let* $f(z)$ *be a non-constant meromorphic function of zero order,* $n \geq 10$ *be an integer. If* $[f'(z)]^n$ *and* $f^n(qz)$ *share* $(1,2)$ *and* $(\infty,0)$*, then* $f'(z) = tf(qz)$*, for a constant* t *that satisfies* $t^n = 1$.

Theorem 1.10. *Let* $f(z)$ *be a non-constant meromorphic function of zero order,* $n \geq 9$ *be an integer. If* $[f'(z)]^n$ *and* $f^n(qz)$ *share* $(1,2)$ *and* (∞,∞)*, then* $f'(z) = tf(qz)$*, for a constant* t *that satisfies* $t^n = 1$.

Theorem 1.11. *Let* $f(z)$ *be a non-constant meromorphic function of zero order,* $n \geq 17$ *be an integer. If* $[f'(z)]^n$ *and* $f^n(qz)$ *share* $(1,0)$ *and* $(\infty,0)$*, then* $f'(z) = tf(qz)$*, for a constant* t *that satisfies* $t^n = 1$.

Corollary 1.12. *Let* $f(z)$ *be a non-constant entire function of zero order,* $n \geq 5$ *be an integer. If* $[f'(z)]^n$ *and* $f^n(qz)$ *share* $(1,2)$*, then* $f'(z) = tf(qz)$*, for a constant* t *that satisfies* $t^n = 1$.

2. Some Lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

Lemma 2.1. [2] *Let F, G be two non-constant meromorphic functions. If F, G share $(1,2)$ and (∞,k), where $0\leq k\leq\infty$, and $H\not\equiv 0$, then*

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)$$
$$+\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G)\,,$$

where $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.

Lemma 2.2. [33] *Let f be a non-constant meromorphic function, and let $a_1,a_2,...,a_n$ be finite complex numbers, $a_n\neq 0$. Then*

$$T(r,a_nf^n+\cdots+a_2f^2+a_1f+a_0)=nT(r,f)+S(r,f)\,.$$

Lemma 2.3. [9] *Let $f(z)$ be a finite order meromorphic function, and let c be a nonzero constant. Then*

$$T(r,f(z+c))=T(r,f(z))+O(r^{\sigma-1+\epsilon})+O(logr)\,.$$

Lemma 2.4. [44] *Let f be a nonconstant meromorphic function, k be a positive integer, then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)\,,$$

where $N_p\left(r,\frac{1}{f^{(k)}}\right)$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity m is counted m times if $m\leq p$ and p times if $m>p$. Clearly $\overline{N}\left(r,\frac{1}{f^{(k)}}\right)=N_1\left(r,\frac{1}{f^{(k)}}\right)$.

Lemma 2.5. [13] *Let $f(z)$ be a meromorphic function of finite order, and let $c\in C$ and $\delta\in(0,1)$. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=o\left(\frac{T(r,f)}{r^{\delta}}\right)=S(r,f)\,.$$

Lemma 2.6. [39] *Suppose that two nonconstant meromorphic functions F and G share 1 and ∞ IM. Let H be given as above. If $H\not\equiv 0$, then*

$$T(r,F)+T(r,G)\leq 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)$$
$$+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G)\,.$$

Lemma 2.7. [40] *Let $f(z)$ be a zero-order meromorphic function, and $q \in C\backslash\{0\}$. Then*

$$T(r, f(qz)) = (1 + o(1))T(r, f(z))$$

and

$$N(r, f(qz)) = (1 + o(1))N(r, f(z))$$

on a set of lower logarithmic density 1.

Lemma 2.8. [3] *Let f be a zero-order meromorphic function, and $q \in C\backslash\{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S(r, f)$$

on a set of logarithmic density 1.

3. Proof of Theorem 1.5

Let

$$F = f^n(z + c), \quad G = [f'(z)]^n . \tag{1}$$

Then it is easy to verify F and G share $(1, 2)$ and $(\infty, 0)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.1 that

$$T(r, F) \le N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G) . \tag{2}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$T(r, F) = nT(r, f(z + c)) + S(r, f) = nT(r, f) + S(r, f) . \tag{3}$$

It's obvious that

$$N_2\left(r, \frac{1}{F}\right) = 2\overline{N}\left(r, \frac{1}{f(z + c)}\right) \le 2T(r, f(z + c)) = 2T(r, f) + S(r, f) , \tag{4}$$

$$\overline{N}(r, F) = \overline{N}(r, f(z + c)) \le T(r, f(z + c)) = T(r, f) + S(r, f) , \tag{5}$$

$$\overline{N}(r, G) = \overline{N}(r, f) \le T(r, f) . \tag{6}$$

$$\overline{N}_*(r, \infty; F, G) \le \overline{N}(r, F) \le T(r, f(z + c)) = T(r, f) + S(r, f) . \tag{7}$$

Lemma 2.4 gives

$$N_2\left(r, \frac{1}{G}\right) = 2\overline{N}\left(r, \frac{1}{f'}\right) \le 2N_2\left(r, \frac{1}{f}\right) + 2\overline{N}(r, f) + S(r, f) \le 4T(r, f) + S(r, f) . \tag{8}$$

Combining (2), (3), (4), (5), (6), (7) and (8), we deduce

$$(n - 9)T(r, f) \le S(r, f) , \tag{9}$$

which contradicts with $n \geq 10$. Therefore $H \equiv 0$. By integration, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B\,, \tag{10}$$

where $A \neq 0$ and B are constants. From (10) we have

$$G = \frac{(B-A)F + (A-B-1)}{BF-(B+1)}\,. \tag{11}$$

We discuss the following three cases.

Case I. Suppose that $B \neq 0, -1$. From (11), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) = \overline{N}(r, G)\,. \tag{12}$$

From the second fundamental theorem and Lemma 2.3, we have

$$\begin{aligned} nT(r,f) = T(r,F) + S(r,f) &\leq \overline{N}(r,F) + \overline{N}\left(r. \frac{1}{F}\right) \\ &+ \overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) + S(r,f) \\ \leq \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) &+ \overline{N}(r,f) + S(r,f)\,, \end{aligned} \tag{13}$$

which contradicts with $n \geq 10$.

Case II. Suppose that $B = 0$. From (11), we have

$$G = AF - (A-1)\,. \tag{14}$$

If $A \neq 1$, from (14) we obtain

$$\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right)\,. \tag{15}$$

From the second fundamental theorem and Lemma 2.4, we have

$$\begin{aligned} nT(r,f) = T(r,F) + S(r,f) &\leq \overline{N}(r,F) + \overline{N}\left(r. \frac{1}{F}\right) \\ &+ \overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) + S(r,f) \\ &\leq \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}\left(r, \frac{1}{f'}\right) \\ &\leq \overline{N}(r, f(z+c)) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) \\ &+ N_2\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + S(r,f)\,, \end{aligned} \tag{16}$$

which contradicts with $n \geq 10$. Thus $A = 1$. From (14) we have $F = G$, that is $f^n(z+c) = [f'(z)]^n$. Hence $f'(z) = tf(z+c)$, for a constant t with $t^n = 1$.

Case III. Suppose that $B = -1$. From (11) we have

$$G = \frac{(A+1)F - A}{F}\,. \tag{17}$$

If $A \neq -1$, we obtain from (17) that

$$\overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) = \overline{N}\left(r, \frac{1}{G}\right)\,. \tag{18}$$

By the same reasoning discussed in Case II, we obtain a contradiction. Hence $A = -1$. From (17), we get $FG = 1$, that is

$$f^n(z+c)[f'(z)]^n = 1\,. \tag{19}$$

Since $[f'(z)]^n$ and $f^n(z+c)$ share $(\infty, 0)$, from (19) we get

$$N(r, f') = 0, \quad T(r, f') = T(r, f(z+c)) + S(r, f)\,, \tag{20}$$

and

$$[f'(z)]^{2n} = \frac{[f'(z)]^n}{f^n(z+c)} = \frac{\frac{[f'(z)]^n}{f^n(z)}}{\frac{f^n(z+c)}{f^n(z)}}\,. \tag{21}$$

From Lemma 2.5 and the logarithmic derivative lemma, we get

$$m(r, f') = S(r, f)\,. \tag{22}$$

By (20) and (22), we know that

$$T(r, f(z+c)) = T(r, f') = S(r, f)\,, \tag{23}$$

which is a contradiction with Lemma 2.3. The proof of Theorem 1.5 is completed.

4. Proof of Theorem 1.6

Let

$$F = f^n(z+c), \quad G = [f'(z)]^n\,. \tag{24}$$

Then it is easy to verify F and G share $(1, 2)$ and (∞, ∞). Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.1 that

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G)\,. \tag{25}$$

According to Lemma 2.2 and Lemma 2.3, we have

$$T(r, F) = nT(r, f(z+c)) + S(r, f) = nT(r, f) + S(r, f)\,. \tag{26}$$

It's obvious that

$$N_2\left(r,\frac{1}{F}\right)=2\overline{N}\left(r,\frac{1}{f(z+c)}\right)\le 2T(r,f(z+c))=2T(r,f)+S(r,f)\,, \quad (27)$$

$$\overline{N}(r,F)=\overline{N}(r,f(z+c))\le T(r,f(z+c))=T(r,f)+S(r,f)\,, \quad (28)$$

$$\overline{N}(r,G)=\overline{N}(r,f)\le T(r,f)\,. \quad (29)$$

$$\overline{N}_*(r,\infty;F,G)=0\,. \quad (30)$$

Lemma 2.4 gives

$$N_2\left(r,\frac{1}{G}\right)=2\overline{N}\left(r,\frac{1}{f'}\right)\le 2N_2\left(r,\frac{1}{f}\right)+2\overline{N}(r,f)+S(r,f)$$
$$\le 4T(r,f)+S(r,f)\,. \quad (31)$$

Combining (25), (26), (27), (28), (29), (30) and (31), we deduce

$$(n-8)T(r,f)\le S(r,f)\,, \quad (32)$$

which contradicts with $n\ge 9$. Therefore $H\equiv 0$. Similar to the proof of Theorem 1.5, we can get the conclusion of Theorem 1.6.

5. Proof of Theorem 1.7

Let

$$F=f^n(z+c),\quad G=[f'(z)]^n\,. \quad (33)$$

Then it is easy to verify F and G share $(1,0)$ and $(\infty,0)$. Let H be defined as above. Suppose that $H\not\equiv 0$. It follows from Lemma 2.6 that

$$T(r,F)+T(r,G)\le 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_E^{1)}\left(r,\frac{1}{F-1}\right)$$
$$+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{F-1}\right)+3N_L\left(r,\frac{1}{G-1}\right)$$
$$+S(r,F)+S(r,G)\,. \quad (34)$$

Since

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)+2N_E^{(2}\left(r,\frac{1}{F-1}\right)+N_L\left(r,\frac{1}{F-1}\right)+2N_L\left(r,\frac{1}{G-1}\right)$$
$$\le N\left(r,\frac{1}{G-1}\right)\le T(r,G)+O(1)\,, \quad (35)$$

we get from (34) and (35) that

$$T(r,F)\le 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2N_L\left(r,\frac{1}{F-1}\right)$$
$$+N_L\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G)\,. \quad (36)$$

According to Lemma 2.2 and Lemma 2.3, we have

$$T(r,F) = nT(r,f(z+c)) + S(r,f) = nT(r,f) + S(r,f)\,. \tag{37}$$

It's obvious that

$$\overline{N}(r,F) = \overline{N}(r,f(z+c)) \le T(r,f(z+c)) = T(r,f) + S(r,f)\,, \tag{38}$$

$$N_2\left(r,\frac{1}{F}\right) = 2\overline{N}\left(r,\frac{1}{f(z+c)}\right) \le 2T(r,f(z+c)) = 2T(r,f) + S(r,f)\,, \tag{39}$$

$$\begin{aligned} N_L\left(r,\frac{1}{F-1}\right) &\le N\left(r,\frac{F}{F'}\right) \le N\left(r,\frac{F'}{F}\right) + S(r,f) \\ &\le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \\ &\le \overline{N}(r,f(z+c)) + \overline{N}\left(r,\frac{1}{f(z+c)}\right) + S(r,f) \\ &\le 2T(r,f) + S(r,f)\,. \end{aligned} \tag{40}$$

Lemma 2.4 gives

$$\begin{aligned} N_2\left(r,\frac{1}{G}\right) = 2\overline{N}\left(r,\frac{1}{f'}\right) &\le 2N_2\left(r,\frac{1}{f}\right) + 2\overline{N}(r,f) + S(r,f) \\ &\le 4T(r,f) + S(r,f)\,, \end{aligned} \tag{41}$$

$$\begin{aligned} N_L\left(r,\frac{1}{G-1}\right) &\le N\left(r,\frac{G}{G'}\right) \le N\left(r,\frac{G'}{G}\right) + S(r,f) \\ &\le \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + S(r,f) \\ &\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f'}\right) + S(r,f) \\ &\le \overline{N}(r,f) + N_2\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + S(r,f) \\ &\le 3T(r,f) + S(r,f)\,. \end{aligned} \tag{42}$$

Combining (36), (37), (38), (39), (40), (41) and (42), we deduce

$$(n-16)T(r,f) \le S(r,f)\,, \tag{43}$$

which contradicts with $n \ge 17$. Therefore $H \equiv 0$. Similar to the proof of Theorem 1.5, we can get the conclusion of Theorem 1.7.

6. Proof of Theorem 1.9

Let

$$F = f^n(qz), \quad G = [f'(z)]^n . \tag{44}$$

Then it is easy to verify F and G share $(1,2)$ and $(\infty,0)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.1 that

$$\begin{aligned} T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) \\ + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G) . \end{aligned} \tag{45}$$

According to Lemma 2.2 and Lemma 2.7, we have

$$T(r,F) = nT(r,f(qz)) + S(r,f) = nT(r,f) + S(r,f) , \tag{46}$$

$$\overline{N}(r,F) = \overline{N}(r,f(qz)) = \overline{N}(r,f(z)) + S(r,f) \le T(r,f) + S(r,f) , \tag{47}$$

$$N_2\left(r,\frac{1}{F}\right) = 2\overline{N}\left(r,\frac{1}{f(qz)}\right) \le 2T(r,f(qz)) = 2T(r,f) + S(r,f) . \tag{48}$$

It's obvious that

$$\overline{N}(r,G) = \overline{N}(r,f) \le T(r,f) . \tag{49}$$

$$\overline{N}_*(r,\infty;F,G) \le \overline{N}(r,G) = \overline{N}(r,f) \le T(r,f) . \tag{50}$$

Lemma 2.4 gives

$$\begin{aligned} N_2\left(r,\frac{1}{G}\right) = 2\overline{N}\left(r,\frac{1}{f'}\right) \le 2N_2\left(r,\frac{1}{f}\right) + 2\overline{N}(r,f) + S(r,f) \\ \le 4T(r,f) + S(r,f) . \end{aligned} \tag{51}$$

Combining (45), (46), (47), (48), (49), (50) and (51), we deduce

$$(n-9)T(r,f) \le S(r,f) , \tag{52}$$

which contradicts with $n \ge 10$. Therefore $H \equiv 0$. By integration, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B , \tag{53}$$

where $A \ne 0$ and B are constants. From (53) we have

$$G = \frac{(B-A)F + (A-B-1)}{BF-(B+1)} . \tag{54}$$

We discuss the following three cases.

Case I. Suppose that $B \ne 0, -1$. From (54), we have

$$\overline{N}\left(r,\frac{1}{F-\frac{B+1}{B}}\right) = \overline{N}(r,G) . \tag{55}$$

From the second fundamental theorem and Lemma 2.7, we have

$$\begin{aligned} nT(r,f) = T(r,F) + S(r,f) &\le \overline{N}(r,F) + \overline{N}\left(r.\frac{1}{F}\right) \\ &+\overline{N}\left(r,\frac{1}{F-\frac{B+1}{B}}\right) + S(r,f) \\ &\le \overline{N}(r,f(qz)) + \overline{N}\left(r,\frac{1}{f(qz)}\right) + \overline{N}(r,f) + S(r,f)\,, \end{aligned} \tag{56}$$

which contradicts with $n \ge 10$.

Case II. Suppose that $B = 0$. From (54), we have

$$G = AF - (A-1)\,. \tag{57}$$

If $A \ne 1$, from (57) we obtain

$$\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right) = \overline{N}\left(r,\frac{1}{G}\right)\,. \tag{58}$$

From the second fundamental theorem and Lemma 2.4, we have

$$\begin{aligned} nT(r,f) = T(r,F) + S(r,f) &\le \overline{N}(r,F) + \overline{N}\left(r.\frac{1}{F}\right) \\ &+\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right) + S(r,f) \\ &\le \overline{N}(r,f(qz)) + \overline{N}\left(r,\frac{1}{f(qz)}\right) + \overline{N}\left(r,\frac{1}{f'}\right) \\ &\le \overline{N}(r,f(qz)) + \overline{N}\left(r,\frac{1}{f(qz)}\right) + N_2\left(r,\frac{1}{f}\right) \\ &+\overline{N}(r,f) + S(r,f)\,, \end{aligned} \tag{59}$$

which contradicts with $n \ge 10$. Thus $A = 1$. From (57) we have $F = G$, that is $f^n(qz) = [f'(z)]^n$. Hence $f'(z) = tf(qz)$, for a constant t with $t^n = 1$.

Case III. Suppose that $B = -1$. From (54) we have

$$G = \frac{(A+1)F - A}{F}\,. \tag{60}$$

If $A \ne -1$, we obtain from (60) that

$$\overline{N}\left(r,\frac{1}{F-\frac{A}{A+1}}\right) = \overline{N}\left(r,\frac{1}{G}\right)\,. \tag{61}$$

By the same reasoning discussed in Case II, we obtain a contradiction. Hence $A = -1$. From (60), we get $FG = 1$, that is

$$f^n(qz)[f'(z)]^n = 1\,. \tag{62}$$

Since $[f'(z)]^n$ and $f^n(qz)$ share $(\infty, 0)$, from (62) we get

$$N(r, f') = 0, \quad T(r, f') = T(r, f(qz)) + S(r, f)\,, \tag{63}$$

and

$$[f'(z)]^{2n} = \frac{[f'(z)]^n}{f^n(qz)} = \frac{\frac{[f'(z)]^n}{f^n(z)}}{\frac{f^n(qz)}{f^n(z)}}\,. \tag{64}$$

From Lemma 2.8 and the logarithmic derivative lemma, we get

$$m(r, f') = S(r, f)\,. \tag{65}$$

By (63) and (65), we know that

$$T(r, f(qz)) = T(r, f') = S(r, f)\,, \tag{66}$$

which is a contradiction with Lemma 2.7. The proof of Theorem 1.9 is completed.

7. Proof of Theorem 1.10

Let

$$F = f^n(qz), \quad G = [f'(z)]^n\,. \tag{67}$$

Then it is easy to verify F and G share $(1, 2)$ and (∞, ∞). Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.1 that

$$\begin{aligned} T(r, F) \le N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) \\ + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G)\,. \end{aligned} \tag{68}$$

According to Lemma 2.2 and Lemma 2.7, we have

$$T(r, F) = nT(r, f(qz)) + S(r, f) = nT(r, f) + S(r, f)\,, \tag{69}$$

$$\overline{N}(r, F) = \overline{N}(r, f(qz)) = \overline{N}(r, f(z)) + S(r, f) \le T(r, f) + S(r, f)\,, \tag{70}$$

$$N_2\left(r, \frac{1}{F}\right) = 2\overline{N}\left(r, \frac{1}{f(qz)}\right) \le 2T(r, f(qz)) = 2T(r, f) + S(r, f)\,. \tag{71}$$

It's obvious that

$$\overline{N}(r, G) = \overline{N}(r, f) \le T(r, f)\,. \tag{72}$$

$$\overline{N}_*(r, \infty; F, G) = 0\,. \tag{73}$$

Lemma 2.4 gives

$$\begin{aligned} N_2\left(r, \frac{1}{G}\right) = 2\overline{N}\left(r, \frac{1}{f'}\right) \le 2N_2\left(r, \frac{1}{f}\right) + 2\overline{N}(r, f) + S(r, f) \\ \le 4T(r, f) + S(r, f)\,. \end{aligned} \tag{74}$$

Combining (68), (69), (70), (71), (72), (73) and (74), we deduce

$$(n-8)T(r,f) \leq S(r,f)\,, \tag{75}$$

which contradicts with $n \geq 9$. Therefore $H \equiv 0$. Similar to the proof of Theorem 1.9, we can get the conclusion of Theorem 1.10.

8. Proof of Theorem 1.11

Let

$$F = f^n(qz), \quad G = [f'(z)]^n\,. \tag{76}$$

Then it is easy to verify F and G share $(1,0)$ and $(\infty,0)$. Let H be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.6 that

$$\begin{aligned} T(r,F)+T(r,G) \leq 3\overline{N}(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_E^{1)}\left(r,\frac{1}{F-1}\right) \\ +2N_E^{(2}\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{F-1}\right) + 3N_L\left(r,\frac{1}{G-1}\right) \\ +S(r,F)+S(r,G)\,. \end{aligned} \tag{77}$$

Since

$$\begin{aligned} N_E^{1)}\left(r,\frac{1}{F-1}\right) + 2N_E^{(2}\left(r,\frac{1}{F-1}\right) + N_L\left(r,\frac{1}{F-1}\right) + 2N_L\left(r,\frac{1}{G-1}\right) \\ \leq N\left(r,\frac{1}{G-1}\right) \leq T(r,G)+O(1)\,, \end{aligned} \tag{78}$$

we get from (77) and (78) that

$$\begin{aligned} T(r,F) \leq 3\overline{N}(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + 2N_L\left(r,\frac{1}{F-1}\right) \\ +N_L\left(r,\frac{1}{G-1}\right) + S(r,F)+S(r,G)\,. \end{aligned} \tag{79}$$

According to Lemma 2.2 and Lemma 2.7, we have

$$T(r,F) = nT(r,f(qz)) + S(r,f) = nT(r,f) + S(r,f)\,. \tag{80}$$

It's obvious that

$$\overline{N}(r,F) = \overline{N}(r,f(qz)) \leq T(r,f(qz)) = T(r,f)+S(r,f)\,, \tag{81}$$

$$N_2\left(r,\frac{1}{F}\right) = 2\overline{N}\left(r,\frac{1}{f(qz)}\right) \leq 2T(r,f(qz)) = 2T(r,f)+S(r,f)\,, \tag{82}$$

$$\begin{aligned} N_L\left(r,\frac{1}{F-1}\right) \leq N\left(r,\frac{F}{F'}\right) \leq N\left(r,\frac{F'}{F}\right) + S(r,f) \\ \leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \end{aligned}$$

$$\leq \overline{N}(r, f(qz)) + \overline{N}\left(r, \frac{1}{f(qz)}\right) + S(r,f)$$
$$\leq 2T(r,f) + S(r,f)\,. \tag{83}$$

Lemma 2.4 gives

$$N_2\left(r, \frac{1}{G}\right) = 2\overline{N}\left(r, \frac{1}{f'}\right) \leq 2N_2\left(r, \frac{1}{f}\right) + 2\overline{N}(r,f) + S(r,f)$$
$$\leq 4T(r,f) + S(r,f)\,, \tag{84}$$

$$N_L\left(r, \frac{1}{G-1}\right) \leq N\left(r, \frac{G}{G'}\right) \leq N\left(r, \frac{G'}{G}\right) + S(r,f)$$
$$\leq \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,f)$$
$$\leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f'}\right) + S(r,f)$$
$$\leq \overline{N}(r,f) + N_2\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + S(r,f)$$
$$\leq 3T(r,f) + S(r,f)\,. \tag{85}$$

Combining (79), (80), (81), (82), (83), (84) and (85), we deduce

$$(n-16)T(r,f) \leq S(r,f)\,, \tag{86}$$

which contradicts with $n \geq 17$. Therefore $H \equiv 0$. Similar to the proof of Theorem 1.9, we can get the conclusion of Theorem 1.11.

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