J. Appl. Math. & Informatics Vol. **37***(2019), No. 1 - 2, pp. 123 - 131*
https://doi.org/10.14317/jami.2019.123

UNIFORM MESH METHOD FOR A MAXWELL'S EQUATION WITH DISCONTINUOUS COEFFICIENTS†

JI HYUN KIM

Abstract. In this paper, we introduce a uniform mesh method for a Maxwell's equation with discontinuous coefficients. We observe optimal $O(h)$ order for the electric field and $O(h)$ order for the curl.

AMS Mathematics Subject Classification : 65N30, 65N25.
Key words and phrases : Maxwell's equation, uniform grid, refinement, edge finite element, optimal order convergence.

1. Introduction

The purpose of this paper is to develop edge finite element approximations [2], [3], [14], [17] to interface problem for a Maxwell's equation[1], [7], [5]. The main idea is to take a uniform grid and divide interface elements into subelements, which fit into the interface and satisfy the maximum angle condition. In our work we use edge elements, because it allows the sharp regularity of the solution and discontinuous electromagnetic properties. Numerical examples in two dimensions illustrate the accuracy of our method. Our method can be extended to time dependent Maxwell problems involving moving interfaces without losing optimal rate of convergence.

Our paper is organized as follows. In Section 2, we introduce our model problem. Next, we present finite element methods for a Maxwell's equation together with some function spaces. In Section 4, numerical experiments demonstrate the accuracy, robustness and reliability of our method.

Received September 23, 2018. Revised November 28, 2018. Accepted November 28, 2018.
†This work was supported by the National Research Foundation of Korea grant funded by the Ministry of Education $2017R1C1B5017646$.

2. Stationary model equations

We consider the following stationary Maxwell's equations in a dielectric medium:

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - \kappa^2 \varepsilon \mathbf{E} = \mathbf{f}, \quad \text{in } \Omega, \tag{1}$$

with the perfect conducting boundary condition

$$\mathbf{n} \times \mathbf{E} = 0, \quad \text{on } \partial\Omega.$$

Here $\Omega \subset R^d (d = 2, 3)$ is a simply-connected Lipschitz polyhedral domain with connected boundary which is occupied by the dielectric material. $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields. We assume that the permeability parameter μ and the permittivity parameter ε of the medium are discontinuous across an interface $\Gamma \subset \Omega$, where $\partial\Omega$ is the boundary of a simply connected Lipschitz polyhedral domain Ω^- with $\overline{\Omega^-} \subset \Omega$ and $\Omega^+ = \Omega \setminus \overline{\Omega^-}$. Ω^+ is also assumed to be simply-connected which, in turn, implies that Γ is connected. Without loss of generality, we consider only the case with ε and μ being two piecewise constant functions in the domain Ω, namely,

$$\begin{cases} \varepsilon^- & \text{in } \Omega^-, \\ \varepsilon^+ & \text{in } \Omega^+, \end{cases} \qquad \begin{cases} \mu^- & \text{in } \Omega^-, \\ \mu^+ & \text{in } \Omega^+, \end{cases} \tag{2}$$

and $\varepsilon^\pm, \mu^\pm$ are positive constants. It is known that the electric field $\mathbf{E}$ must satisfy the following jump conditions across the interface :

$$[\mathbf{n} \times \mathbf{E}] = 0, \quad \left[\mu^{-1}\nabla \times \mathbf{E}\right] = 0, \quad \left[\mathbf{n} \cdot \kappa^2 \varepsilon \mathbf{E}\right] = 0, \tag{3}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega^-$. Throughout the paper, the jump of any function U across the interface Γ is defined as

$$[U] := U^-|_\Gamma - U^+|_\Gamma$$

with $U^\star = U|_{\Omega^\star}$, $\star = +, -$.

3. Finite element methods

3.1. Function spaces. We first define the space of vector functions with **curl** in $H^s(\Omega)(s \geq 0)$ by

$$H^s(\mathbf{curl}; \Omega) = \left\{\boldsymbol{u} \in (H^s(\Omega))^d | \ \nabla \times \boldsymbol{u} \in (H^s(\Omega))^d\right\} \tag{4}$$

with the graph norm

$$\|\boldsymbol{u}\|_{\mathbf{curl},\Omega} = \left(\|\boldsymbol{u}\|^2_{s,\Omega} + \|\nabla \times \boldsymbol{u}\|^2_{s,\Omega}\right)^{1/2}. \tag{5}$$

For the convenience of presentation, we denote the notation $H(\mathbf{curl}; \Omega)$ by the space $H^0(\mathbf{curl}; \Omega)$. The space $H_0(\mathbf{curl}; \Omega)$ is defined by density as follows:

$$H_0(\mathbf{curl}; \Omega) = \text{closure of } (\mathcal{C}_0^\infty(\Omega))^d \text{ in } H(\mathbf{curl}; \Omega). \tag{6}$$

Here, the notations $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the norm and the seminorm of the Sobolev space $(H^2(\Omega))^d$.

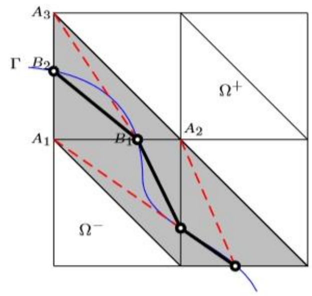

FIGURE 1. Interface elements and subtriangles.

3.2. Meshes and finite-element space.

Definition 3.1. We say that a triangle T_2 satisfies the *maximum(minimum) angle condition* with a constant $\omega < \pi$, or shortly $MAC(\omega)(mAC(\omega))$, if the angles of T_2 are less(greater) than or equal to ω, respectively. Let T_3 be a tetrahedron which the angles of each face of it are less(greater) than or equal to ω, respectively. We say that a tetrahedron T_3 satisfies $MAC(\omega)(mAC(\omega))$, respectively.

We consider meshes $\mathcal{T}_h$ that partition the domain into disjoint *regular* tetrahedral(triangular) elements $\{T\}$, such that $\overline{\Omega} = \cup_{T\in\mathcal{T}_h}\overline{T}$ in $\mathbb{R}^3(\mathbb{R}^2)$ and each T satisfies $mAC(\omega)$, respectively. So, we can assume that every element T of the $\mathcal{T}_h$ is affine equivalent to either a reference tetrahedron(triangle). We call an element $T \in \mathcal{T}_h$ an *interface element* if the interface Γ passes through the interior of T, otherwise we call T a *non-interface element*. We shall assume the interface Γ meets the edges of an interface element $T \in \mathcal{T}_h^I$ at no more than two intersections. Now, we introduce some symbols:

$$\mathcal{T}_h^I = \text{the set of all interface elements,}$$
$$\mathcal{T}_h^N = \text{the set of all non-interface elements}$$

For any interface element T, we will divide an element T into severance subelements and use subelements $\{T^s\}$ instead of the element T. Then, we define a new meshes $\mathcal{T}_h^s$ such that for any element $T \in \mathcal{T}_h^s$, T is a noninterface element or a subelement of an interface element. We will state details below.

3.2.1. Two dimensional case. Consider a triangle $\Delta A_1A_2A_3$ in the Figure 1. We will sever each interface element into subtriangles. This work consists of the following three steps.

(1) Divide a triangle $\Delta A_1A_2A_3$ into two subelements by an approximated local linear interface $\overline{B_1B_2}$.
(2) If all subelements are triangle(i.e. the interface $\overline{B_1B_2}$ cut through only one vertex of a triangle $\Delta A_1A_2A_3$), then we do not work any more.

(3) If one of two subelements is rectangle ($\square B_2B_1A_2A_3$), then we divide it two subtriangles. In this case, compare two angles $\angle A_3B_2B_1$, $\angle B_2B_1A_2$ and divide the bigger side.

Then, we can easily check that each triangle $T \in \mathcal{T}_h^s$ satisfies $MAC(\pi - \omega)$.

3.3. Local basis functions on an interface element. We consider the immersed finite element simlar to those in [9], [8], [13], [15], adapted to a Maxwell's equation. We construct a piecewise linear function of the form

$$\boldsymbol{\phi}(X) = \begin{cases} \boldsymbol{\phi}^+(X) = (a^+ - c^+y, b^+ + c^+x), & X = (x,y) \in T^+, \\ \boldsymbol{\phi}^-(X) = (a^- - c^-y, b^- + c^-x), & X = (x,y) \in T^-, \end{cases} \tag{7}$$

satisfying

$$\begin{aligned} \bar{\boldsymbol{\phi}} \cdot \mathbf{t}_i &= V_i,\ i = 1,2,3, & (8) \\ \boldsymbol{\phi}^+ \cdot \mathbf{t}_3(D) &= \boldsymbol{\phi}^- \cdot \mathbf{t}_3(D),\ \boldsymbol{\phi}^+ \cdot \mathbf{t}_2(E) = \boldsymbol{\phi}^- \cdot \mathbf{t}_2(E), & (9) \\ \beta^+ \mathrm{curl}\, \boldsymbol{\phi}^+|_\Gamma &= \beta^- \mathrm{curl}\, \boldsymbol{\phi}^-_\Gamma\,, & (10) \end{aligned}$$

where V_i, $i = 1,2,3$ are given values and $\mathbf{n}_{\overline{DE}}$ is the unit normal vector on the line segment $\overline{DE}$. This is a piecewise linear function on T that satisfies the homogeneous jump conditions along $\overline{DE}$.

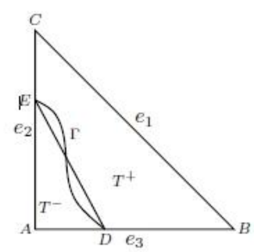

FIGURE 2. A reference interface triangle.

Suppose that a typical reference interface element T has vertices at $A(0,0)$, $B(1,0)$, $C(0,1)$. We assume that the interface meets with the edges at $D(x_0,0)$ and $E(0,y_0)$ where $0 < x_0, y_0 \le 1$. Then the normal vector to the interface is $\mathbf{n}_{\overline{DE}} = (y_0, x_0)/\sqrt{x_0^2 + y_0^2}$.

Theorem 3.2. *Given an reference interface triangle, the piecewise linear function $\boldsymbol{\phi}(x,y)$ defined by (7)-(10) is uniquely determined by three conditions*

$$\frac{1}{|e_i|}\int_{e_i} \boldsymbol{\phi} \cdot \mathbf{t}_i ds = V_i, \quad i = 1,2,3.$$

Proof. Let $X = (x, y)^T \in T$. Since $\boldsymbol{\phi}^+$ and $\boldsymbol{\phi}^-$ are linear functions, we have

$$\boldsymbol{\phi}(X) = \begin{cases} \boldsymbol{\phi}^+(X) = (a^+ - c^+ y, b^+ + c^+ x), & X = (x, y) \in T^+, \\ \boldsymbol{\phi}^-(X) = (a^- - c^- y, b^- + c^- x), & X = (x, y) \in T^- \end{cases} \tag{11}$$

The condition (8) gives the following three equations:

$$\begin{aligned} \frac{1}{\sqrt{2}} \int_{e_1} \boldsymbol{\phi} \cdot \mathbf{t}_1 \, ds &= \int_{e_1} (-a^+ + c^+ y + b^+ + c^+ x)/2 = V_1 \\ \int_{e_2} \boldsymbol{\phi} \cdot \mathbf{t}_2 \, dy &= \int_{y_0}^{1} (b^+ + c^+ x) \, dy + \int_0^{y_0} (b^- + c^- x) \, dy = V_2 \\ \int_{e_3} \boldsymbol{\phi} \cdot \mathbf{t}_3 \, dx &= \int_{x_0}^{1} (a^+ - c^+ y) \, dx + \int_0^{x_0} (a^- - c^- y) \, dx = V_3. \end{aligned}$$

So

$$\begin{aligned} \frac{1}{\sqrt{2}}(-a^+ + c^+ + b^+) &= V_1 \\ b^+(1 - y_0) + b^- y_0 &= V_2 \\ a^+(1 - x_0) + a^- x_0 &= V_3. \end{aligned}$$

Tangent continuity condition is

$$2\beta^+ c^+ = 2\beta^- c^-.$$

From the continuity condition at E and D of $\boldsymbol{\phi} \cdot \mathbf{t}_2$, and $\boldsymbol{\phi} \cdot \mathbf{t}_3$ we have

$$\begin{aligned} (b^+ + c^+ x)(E) &= (b^- + c^- x)(E) \to b^+ = b^-, \\ (a^+ - c^+ y)(D) &= (a^- - c^- y)(D) \to a^+ = a^-. \end{aligned}$$

Thus in this case(location of interface) the immersed basis is obtained just by $2\beta^+ c^+ = 2\beta^- c^-$, while a and b do not change. If $\beta^{\pm} = 1$, then

$$\begin{aligned} \boldsymbol{\phi}_1 &= \sqrt{2}(-y, x), \\ \boldsymbol{\phi}_2 &= (y, 1 - x), \\ \boldsymbol{\phi}_3 &= (1 - y, x). \end{aligned}$$

A tedious calculation shows that the determinant of the matrix is nonzero.

□

Remark 3.1. If $\bar{\boldsymbol{\phi}}_{e_1}, \bar{\boldsymbol{\phi}}_{e_2}$ and $\bar{\boldsymbol{\phi}}_{e_3}$ have the same value C, then the piecewise linear function $\boldsymbol{\phi}$ satisfying (8)-(10) reduces to a constant by uniqueness.

3.4. The variational formulation. Let $\mathbf{V}_h$ be the space of functions constructed above. For simplicity we write $\mathbf{u}$ for $\mathbf{E}$. Then the finite element formulation for Maxwell's equation is

$$\sum_K \int_K \operatorname{curl} \mathbf{u}_h \operatorname{curl} \mathbf{v}_h dx - \kappa^2 \varepsilon \mathbf{u} \cdot \mathbf{v}_h dx = \sum_K \int_K \mathbf{f} \cdot \mathbf{v}_h dx, \mathbf{v}_h \in \mathbf{V}_h. \tag{12}$$

4. Numerical experiments

In order to describe the interface, we consider the level-set function $\Phi(\mathbf{x})$ for the interface Γ which is assumed to be smooth. Let $\Phi : \Omega \to \mathbb{R}$ be a continuous function such that

$$\Phi(\mathbf{x}) = \begin{cases} < 0 & \mathbf{x} \text{ in } \Omega^-, \\ = 0 & \mathbf{x} \text{ on } \Gamma, \\ > 0 & \mathbf{x} \text{ in } \Omega^+. \end{cases} \tag{13}$$

Example 4.1. In this example, we set $\kappa = \varepsilon = 1$. Let the domain be $(-1, 1) \times (-1, 1)$ and triangularized by uniform triangle grids with $h_x = h_y = 1/2^{n-1}$ for $n = 3, \cdots, 8$. The level-set function $\Phi(\mathbf{x})$, the permeability coefficients $\mu^\pm$ and the solution $\mathbf{E}$ are given as follows:

$$\begin{aligned} \Phi(\mathbf{x}) &= \sqrt{x^2 + y^2} - 0.5002, \\ \mu &= \begin{cases} \mu^- = 1, & \mu^+ = 1, \\ \mu^- = 1, & \mu^+ = 1000, \\ \mu^- = 1000, & \mu^+ = 1, \end{cases} \\ \mathbf{E} &= ((y^2 - 1)(0.25001 - x^2 - y^2)\mu,\ (x^2 - 1)(0.25001 - x^2 - y^2)\mu). \end{aligned} \tag{14}$$

TABLE 1. Error for Example 4.1

	$N_x \times N_y$	$\|\|E - E_h\|\|_{L^2}$	Order	$\|\|\mathbf{curl}E - \mathbf{curl}E_h\|\|_{L^2}$	Order
case (a) $\mu^- = 1$, $\mu^+ = 1$	8×8	0.2802509	-	8.8795658	-
	16×16	0.1413693	0.98	4.4568688	0.96
	32×32	0.0708440	0.99	2.2290840	0.99
	64×64	0.0354420	0.99	1.1145724	0.99
	128×128	0.0177235	1.00	0.5572883	0.99
	256×256	0.0088620	1.00	0.2786444	1.00
case (b) $\mu^- = 1$, $\mu^+ = 1000$	8×8	279.11081	-	1393.4354	-
	16×16	142.36692	0.97	658.10831	1.08
	32×32	70.871506	1.00	235.09026	1.48
	64×64	35.448912	0.99	114.22412	1.04
	128×128	17.725278	1.00	56.366195	1.01
	256×256	8.8623676	1.00	27.966360	1.01
case (c) $\mu^- = 1000$, $\mu^+ = 1$	8×8	0.2802509	-	0.8645673	-
	16×16	0.1413693	0.98	0.4424536	0.96
	32×32	0.0708440	0.99	0.2224936	0.99
	64×64	0.0354420	0.99	0.1114050	0.99
	128×128	0.0177235	1.00	0.0557223	0.99
	256×256	0.0088620	1.00	0.0278636	1.00

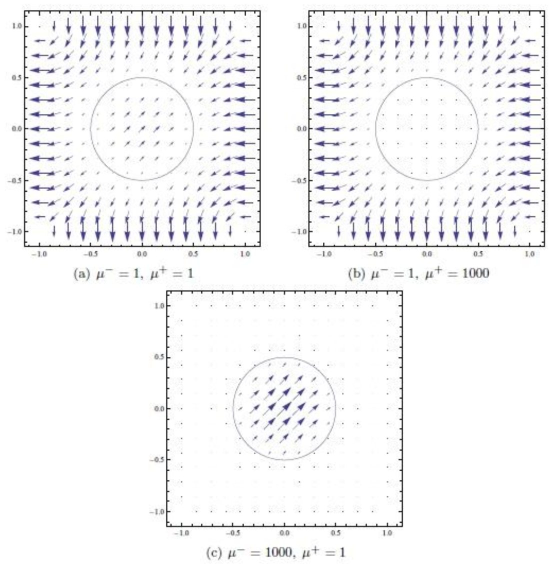

FIGURE 3. Numerical solution.

Example 4.2. In this example, we set $\kappa = 1$. Let the domain be $(-1,1) \times (-1,1)$ and triangularized by uniform triangle grids with $h_x = h_y = 1/2^{n-1}$ for $n = 3, \cdots, 8$. The level-set function $\Phi(\mathbf{x})$, the permeability coefficients $\mu^{\pm}$, the permittivity $\varepsilon^{\pm}$ and the solution $\mathbf{E}$ are given as follows:

$$
\begin{aligned}
\Phi(\mathbf{x}) &= \sqrt{x^2 + y^2} - 0.5002, \\
\mu &= \begin{cases} \mu^- = 1, & \mu^+ = 1, \\ \mu^- = 1, & \mu^+ = 10, \\ \mu^- = 1000, & \mu^+ = 1, \end{cases} \\
\varepsilon &= \begin{cases} \varepsilon^- = 1, & \varepsilon^+ = 100, \\ \varepsilon^- = 100, & \varepsilon^+ = 1, \end{cases}
\end{aligned}
\tag{15}
$$

$$
\mathbf{E} = ((y^2 - 1)(0.25001 - x^2 - y^2)\mu,\ (x^2 - 1)(0.25001 - x^2 - y^2)\mu).
$$

The results given table 1 and 2 strongly support our method that has the optimal convergence order.

TABLE 2. Error for Example 4.2

		$N_x \times N_y$	$\|\|E - E_h\|\|_{L^2}$	Order	$\|\|\mathbf{curl}E - \mathbf{curl}E_h\|\|_{L^2}$	Order
case (a1)	$\mu^- = 1,\ \mu^+ = 1$	8×8	0.2814080	-	0.9177317	-
	$\varepsilon^- = 1,\ \varepsilon^+ = 100$	16×16	0.1417586	0.98	0.4581779	1.00
		32×32	0.0709237	0.99	0.2255071	1.02
		64×64	0.0354577	1.00	0.1119654	1.01
		128×128	0.0177258	1.00	0.0558049	1.00
		256×256	0.0088623	1.00	0.0278744	1.00
case (a2)	$\mu^- = 1,\ \mu^+ = 1$	8×8	0.2802509	-	0.8645673	-
	$\varepsilon^- = 100,\ \varepsilon^+ = 1$	16×16	0.1413693	0.98	0.4424536	0.96
		32×32	0.0708440	0.99	0.2224936	0.99
		64×64	0.0354420	0.99	0.1114050	0.99
		128×128	0.0177235	1.00	0.0557223	0.99
		256×256	0.0088620	1.00	0.0278636	1.00
case (b1)	$\mu^- = 1,\ \mu^+ = 10$	8×8	2.7911081	-	13.934354	-
	$\varepsilon^- = 100,\ \varepsilon^+ = 1$	16×16	1.4236692	0.97	6.5810831	1.08
		32×32	0.7087150	1.00	2.3509026	1.48
		64×64	0.3544891	0.99	1.1422412	1.04
		128×128	0.1772527	1.00	0.5636619	1.01
		256×256	0.0886236	1.00	0.2796636	1.01
case (b2)	$\mu^- = 1,\ \mu^+ = 10$	8×8	2.8891306	-	8.8795658	-
	$\varepsilon^- = 100,\ \varepsilon^+ = 1$	16×16	1.4258004	1.01	4.4568688	0.99
		32×32	0.7099973	1.00	2.2290840	1.00
		64×64	0.3546165	1.00	1.1145724	1.00
		128×128	0.1772599	1.00	0.5572883	1.00
		256×256	0.0886238	1.00	0.2786444	1.00
case (c1)	$\mu^- = 1000,\ \mu^+ = 1$	8×8	0.2814080	-	0.9177317	-
	$\varepsilon^- = 100,\ \varepsilon^+ = 1$	16×16	0.1417586	0.98	0.4581779	1.00
		32×32	0.0709237	0.99	0.2255071	1.02
		64×64	0.0354577	1.00	0.1119654	1.01
		128×128	0.0177258	1.00	0.0558049	1.00
		256×256	0.0088623	1.00	0.0278744	1.00
case (c2)	$\mu^- = 1000,\ \mu^+ = 1$	8×8	0.2802509	-	0.8645673	-
	$\varepsilon^- = 100,\ \varepsilon^+ = 1$	16×16	0.1413693	0.98	0.4424536	0.96
		32×32	0.0708440	0.99	0.2224936	0.99
		64×64	0.0354420	0.99	0.1114050	0.99
		128×128	0.0177235	1.00	0.0557223	0.99
		256×256	0.0088620	1.00	0.0278636	1.00

References

1. P. Monk, *Finite Element Methods for Maxwells Equations*, Oxford University Press, New York, 2003.
2. J.C. Nedelec, *Mixed finite elements in R3, Numer. Math.* **35** (1980), 315-341.
3. J.C. Nedelec, *A new family of mixed finite elements in R3*, Numer. Math. **50** (1986), 57-81.
4. G.A. Baker, W.N. Jureidini, and O.A. Karakashian, *Piecewise solenoidal vector fields and the Stokes problem*, SIAM J. Numer. Anal. **27** (1990), 1466-1485.
5. M. Costabel, M. Dauge, and S. Nicaise, *Singularities of Maxwell interface problems*, M2AN Math. Model. Numer. Anal. **33** (1999), 627-649.
6. D. Boffi, *Fortin operator and discrete compactness for edge elements*, Numer. Math. **87** (2000), 229-246.
7. R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numer. **11** (2002), 237-339.
8. Z. Li, T. Lin, and X. Wu, *New Cartesian grid methods for interface problems using the finite element formulation*, Numer. Math. **96** (2003), 61-98.
9. Z. Li, T. Lin, Y. Lin, and R.C. Rogers, *An immersed finite element space and its approximation capability*, Numer. Methods. P.D.E. **20** (2004), 338-367.
10. A. Buffa, *Remarks on the discretization of some noncoercive operator with applications to heterogeneous Maxwell equations*, SIAM J. Numer. Anal. **43** (2005), 1-18.
11. P. Houston, I. Perugia, A. Schneebeli, and D. Schotzau, *Interior penalty method for the indefinite time-harmonic Maxwell equations*, Numer. Math. **100** (2005), 485-518.
12. D. Boffi, M. Costabel, M. Dauge, and L. Demkowicz, *Discrete compactness for the hp version of rectangular edge finite elements*, SIAM J. Numer. Anal. **44** (2006), 979-1004.
13. D.Y. Kwak, K.T. Wee, and K.S. Chang, *An analysis of a broken P_1-nonconforming finite element method for interface problems*, SIAM J. Numer. Aanal. **48** (2012), 2117-2134.
14. J.H. Kim, Do Y. Kwak, *New curl conforming finite elements on parallelepiped*, Numer. Math. **131** (2015), 473-488.
15. D.Y. Kwak, J. Lee, *A modified P_1-immersed finite element method*, Int. J. Pure Appl. Math. **104** (2015), 471-494.
16. D.Y. Kwak, S. Jin, and D. Kyeong , *A stabilized P_1-immersed finite element method for the interface for elasticity problems*, ESAIM: M^{2}AN **51** (2017) , 187-207.
17. J.H. Kim, *Discrete compactness property for Kim-Kwak finite elements*, J. Appl. Math. & Informatics **36** (2018), 411-418.

Ji Hyun KIM received M.Sc. and Ph.D at KAIST. She is currently an associate professor at Hannam University since 2011. Her research interests are numerical analysis and Maxwell's eigenvalue problem.

Department of Mathematics, Hannam University, 70 Hannamro, Daedeok-gu Daejeon 34430, Republic of Korea.

e-mail: kimjh@hnu.kr, lm97jh@naver.com