J. Appl. Math. & Informatics Vol. **37**(2019), No. 1 - 2, pp. 73 - 83
https://doi.org/10.14317/jami.2019.073

NEW RESULTS ON k-HYPONORMALITY OF BACKWARD EXTENSIONS OF SUBNORMAL WEIGHTED SHIFTS[†]

YANWU DONG, MI RYEONG LEE AND CHUNJI LI*

ABSTRACT. In this article, we introduce a new kind of subnormal weighted shifts, which is a generalized form of Bergman shift, and discuss the k-hyponormality of its backward extensions.

AMS Mathematics Subject Classification : 47B37, 47B20.
Key words and phrases : Subnormal, k-hyponormal, weighted shifts.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $K \supseteq \mathcal{H}$. For $A, B \in \mathcal{L}(\mathcal{H})$, let $[A, B] = AB - BA$. We say that an n-tuple $T = (T_1, \cdots, T_n)$ of operators in $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For a positive integer k, $T \in \mathcal{L}(\mathcal{H})$ is *k-hyponormal* if $(I, T, \cdots, T^k)$ is hyponormal. It is well known from Bram-Halmos criterion that T is subnormal if and only if T is k-hyponormal for all $k \in \mathbb{N}$ (cf. [1],[8],[10]). Thus the implications 'subnormal $\Rightarrow \cdots \Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal' hold, but each converse is not true in general. Since Curto in 1990 ([3]) introduced a bridge between the hyponormality and subnormality in the concept of k-hyponormality, many operator theorists have studied these classes of operators until now. In the study of these classes, the weighted shifts have played an important roles. ([3],[4],[7],[9],[11],[12], etc.)

Received July 19, 2018. Revised October 18, 2018. Accepted November 5, 2018.
*Corresponding author.
[†]The first author was supported by Key Scientific Research Projects in Zhanjiang Preschool Education College (ZJYZZD201801). The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2017R1D1A1A02017817).

Recall that let $\alpha := \{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence in the set $\mathbb{R}_+$. The (unilateral) *weighted shift* W_α acting on $\ell^2(\mathbb{N}_0)$, with an orthonormal basis $\{e_i\}_{i=0}^{\infty}$, is defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It follows straightforward that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}\,(n \geq 0)$. There are some problems to study the structure of weighted shifts W_α, such as flatness, completion of a finite initial weights to a weight sequence, backward extension of a weight sequence to a new one, etc. Stampfli in [14] proved that if a subnormal weighted shift W_α with $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \cdots$, i.e. W_α has flatness. Also in [3], Curto improved this result that 2-hyponormal of weighted shift W_α immediately forces the weight α to be flat.

If the weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ is given by $\alpha_n = \sqrt{\frac{n+1}{n+2}}\ (n \geq 0)$, then the corresponding weighted shift is called the *Bergman shift* ([2]). As an application of Bergman shift, there is the backward extension problem which was introduced by Curto and Fialkow ([5], [6]).

Problem. Let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots\ (x > 0)$ be an augmented weight sequence for the given $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ and let a weighted shift $W_{\alpha(x)}$ be a backward extension of W_α. Assume that W_α is k-hyponormal for $k \in \mathbb{N} \cup \{\infty\}$. Describe the sets

$$\mathbf{HE}(\alpha; n) = \{x \in \mathbb{R}_+ : W_{\alpha(x)} \text{ is } n\text{-hyponormal}\}\ \ (1 \leq n \leq k).$$

In [4], they showed the example that if $W_{\alpha(x)}$ is an one-step backward extension of the Bergman shift W_α, then there exists a sequence $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}_+$ with $\lim_{k\to\infty} \lambda_k = \sqrt{\frac{1}{2}}$ such that $\lambda_k > \lambda_{k+1}\ (k \geq 1)$ and $\mathbf{HE}(\alpha; k) = (0, \lambda_k]$, where $\lambda_1 = \sqrt{\frac{2}{3}}, \lambda_2 = \frac{3}{4}, \lambda_3 = \sqrt{\frac{8}{15}}, \lambda_4 = \sqrt{\frac{25}{48}}, \cdots$, and $\mathbf{HE}(\alpha, \infty) = \left(0, \sqrt{\frac{1}{2}}\right]$, which distinguishes the classes of k-hyponormal operators from one another. Furthermore, the authors obtained a formula in [11] that captured such examples. Also in [7], they considered an one-step backward extension of Bergman shift $W_{\alpha(x)}$ with weight sequence $\alpha(x) : \alpha_0 = \sqrt{x},\ \alpha_n = \sqrt{\frac{n+2}{n+3}}\ (n \geq 1)$ and obtained a formula for k-hyponormal of $W_{\alpha(x)}$ which contributed to the improvement of the study of relationships between subnormality and hyponormality. Moreover, authors in [9] considered a Bergman-like shift which is a generalization of Bergman shift and they proved that all Bergman-like shifts are subnormal.

In fact, the gaps among the classes of subnormal and hyponormal weighted shifts have been investigated markedly via some models of weighted shifts. In this point of view, it is worth studying some backward extensions of Bergman-type weighted shifts, that is, the corresponding weight sequence has Bergman tail. In this paper, we consider one-step [and two-step] backward extension of weighted shifts of Bergman-type and will present concrete formula of k-hyponormal for each $k \geq 1$.

This paper consists of four sections. In Section 2, we give some key lemmas on calculations of determinants beginning with so called as *Cauchy's double alternant*. In Section 3, we discuss the problem of k-hyponormality of one-step [and two-step] backward extension for Bergman-type shifts, which improves the results of [7], [11], [12], etc. In Section 4, we introduce a new kind of subnormal weighted shifts, which is a generalized form of Bergman shift and discuss the k-hyponormality of its one-step backward extension.

Throughout this note we denote $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{R}_+$ for the set of positive integer, nonnegative integer and nonnegative real numbers, respectively.

The calculations in this paper were obtained through computer experiments using the software tool *Scientific WorkPlace* [15].

2. Fundamental lemmas

In this section, we give essential lemmas to prove main results. First, we recall Cauchy's double alternant ([13, p.6]) that the determinant of the matrix with (i,j) entry $\frac{1}{X_i+Y_j}$ is

$$\det_{1\le i,j\le n}\left(\frac{1}{X_i+Y_j}\right)=\frac{\prod_{1\le i<j\le n}(X_i-X_j)(Y_i-Y_j)}{\prod_{1\le i,j\le n}(X_i+Y_j)}. \tag{2.1}$$

From (2.1) we have the following Lemma([7, p.460]).

Lemma 2.1 ([7]). *For $\omega\ge 0$, the determinant $A_n(\omega)$ of the matrix with (i,j) entry $\frac{1}{\omega+i+j-1}$ $(1\le i,j\le n)$ is* [1]

$$A_n(\omega)=(1!2!\cdots(n-1)!)^2\,\frac{\Gamma(\omega+1)\,\Gamma(\omega+2)\cdots\Gamma(\omega+n)}{\Gamma(n+\omega+1)\,\Gamma(n+\omega+2)\cdots\Gamma(2n+\omega)}.$$

By using Lemma 2.1, we can obtain some results for the following Hankel matrices, we omit the tedious calculations here.

Proposition 2.2. *For $x>0$, let $B_{n+1}(x):=(b_{i+j})_{0\le i,j\le n}$ be a $(n+1)\times(n+1)$ matrix, where*

$$b_0:=1,\ b_1:=\frac{1}{x},\ b_k:=\frac{1}{k+1}\ (2\le k\le 2n).$$

Then

$$\begin{aligned}\det(B_{n+1}(x))=&-\left(\frac{1}{x}-\frac{1}{2}\right)^2\left(\frac{(1!2!3!\cdots(n-2)!)^2\,(4!5!\cdots(n+2)!)}{(n+3)!\,(n+4)!\cdots(2n+1)!}\right)\\&-\left(\frac{1}{x}-\frac{1}{2}\right)\left(\frac{(1!2!3!\cdots(n-2)!)^3\,((n-1)!n!)^2}{(n+3)!\,(n+4)!\cdots(2n+1)!}\right)\\&+\frac{(1!2!3!\cdots n!)^3}{(n+1)!\,(n+2)!\cdots(2n+1)!}.\end{aligned}$$

[1] $\Gamma(\cdot)$ is the gamma function.

Proposition 2.3. *For $p>1$, the determinant of the matrix $C_n^{[1]}(p) := (c_{i+j})_{0\le i,j\le n-1}$, where*

$$c_k := \frac{1}{(k+1)p-k} \ (0 \le k \le 2n-2),$$

is

$$\det\left(C_n^{[1]}(p)\right) = \frac{1}{(p-1)^n}\frac{(1!2!\cdots(n-1)!)^2}{\left(\prod_{k=1}^n\prod_{s=1}^n\left(\frac{1}{p-1}+k+s-1\right)\right)}.$$

Proposition 2.4. *Let $p>1$ and $p\neq 2$. Consider the matrix $C_{n+1}^{[2]}(p) := (c_{i+j})_{0\le i,j\le n}$, where*

$$c_0 = \frac{1}{2-p},\ c_1 = 1,\ \ c_k = \frac{1}{(k-1)p-(k-2)} \ (2\le k\le 2n).$$

Then

$$\det\left(C_{n+1}^{[2]}(p)\right)$$
$$= \frac{1}{(p-1)^{n+1}}\frac{(1!2!\cdots n!)^2}{\left(\prod_{l=0}^n\left(\frac{1}{p-1}+l-1\right)^{l+1}\right)\left(\prod_{l=n+1}^{2n}\left(\frac{1}{p-1}+l-1\right)^{2n-l+1}\right)}.$$

3. k-hyponormality of backward extensions of Bergman shifts

Given weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ and the weighted shift W_α, we define the *moment sequence* $\{\gamma_i\}_{i=0}^\infty$ by $\gamma_0 := 1, \gamma_i := \alpha_0^2\cdots\alpha_{i-1}^2\,(i\ge 1)$. For $\ell\in\mathbb{N}$, we denote an augmented weight sequence of an increasing sequence α by

$$\alpha(x_1,...,x_\ell) : x_\ell, x_{\ell-1},...,x_1,\alpha_0,\alpha_1,\alpha_2,...$$

with $0<x_\ell\le\cdots\le x_2\le x_1\le\alpha_0$. Then the associated weighted shift $W_{\alpha(x_1,...,x_\ell)}$ with a sequence $\alpha(x_1,...,x_\ell)$ is an *ℓ-step backward extension* of W_α. Now we define a sequence $\widetilde{\gamma}(x_1,...,x_\ell) := \{\widetilde{\gamma}_i\}_{i\ge 0}$ as $\widetilde{\gamma}_0 := 1$ and

$$\widetilde{\gamma}_n := x_\ell^2\cdots x_{\ell-n+1}^2, \text{ for } 1\le n\le \ell;$$
$$\widetilde{\gamma}_n := x_\ell^2\cdots x_1^2\alpha_0^2\cdots\alpha_{n-\ell-1}^2, \text{ for } \ell+1\le n.$$

Then we call $\widetilde{\gamma}\equiv\widetilde{\gamma}(x_1,...,x_\ell)$, the *moment sequence* of $\alpha(x_1,...,x_\ell)$. Set

$$\mathbf{HE}_\ell(\alpha,n) := \{(x_1,...,x_\ell) : W_{\alpha(x_1,...,x_\ell)} \text{ is } n\text{-hyponormal}\}.$$

From now on we may assume that the subnormal weighted shift W_α satisfies $\alpha_0<\alpha_1<\alpha_2<\cdots<\alpha_n<\cdots$ to escape the trivial case, so called, flatness. The following Lemma is a modification of a 1-step backward extension in [6, Lemma 3.3].

Lemma 3.1 ([11, Lemma 2.1]). *Let W_α be a k-hyponormal weighted shift and $1 \le n \le k$. Then $W_{\alpha(x_1,\dots,x_\ell)}$ is n-hyponormal if and only if the Hankel matrix*

$$M_{n+1}(\ell,i) := \begin{bmatrix} \widetilde{\gamma}_i & \cdots & \widetilde{\gamma}_{i+n} \\ \vdots & \ddots & \vdots \\ \widetilde{\gamma}_{i+n} & \cdots & \widetilde{\gamma}_{i+2n} \end{bmatrix} \tag{3.1}$$

is positive for every i with $0 \le i \le \ell - 1$. Moreover, we have

$$\mathbf{HE}_\ell(\alpha,n) = \{(x_1,\dots,x_\ell) : M_{n+1}(\ell,i) \ge 0,\ 0 \le i \le \ell-1\}.$$

3.1. One-step backward extensions. For a positive integer m, we consider a weight sequence as follows:

$$\alpha\,(x;m) : \sqrt{\frac{x}{m}}, \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \sqrt{\frac{m+2}{m+3}}, \sqrt{\frac{m+3}{m+4}}, \cdots . \tag{3.2}$$

Then we obtain the following first main result.

Theorem 3.2. *Let $W_{\alpha(x;m)}$ be a weighted shift with weight $\alpha\,(x;m)$ in (3.2). Then the following assertions hold.*

(i) *For $m = 1$, $W_{\alpha(x;1)}$ is n-hyponormal if and only if $0 < x \le \lambda_n^{[1]}$, where*

$$\lambda_n^{[1]} := \frac{1}{2\sum_{i=1}^n \frac{1}{i}}.$$

(ii) *For $m \ge 2$, $W_{\alpha(x;m)}$ is n-hyponormal if and only if $0 < x \le \lambda_n^{[m]}$, where*

$$\lambda_n^{[m]} := \frac{(m-1)\left(\prod_{l=1}^{m-1}(n+l)\right)^2}{\left(\prod_{l=1}^{m-1}(n+l)\right)^2 - ((m-1)!)^2}.$$

Proof. To prove the result (i), we mimic the methods in the proof on [11, Theorem 2.1]. Denote $\lambda_n^{[1]}$ for the root of $\det(M_{n+1}(1,0)) = 0$. Then we can obtain that

$$\frac{1}{\lambda_{n+1}^{[1]}} = \frac{1}{\lambda_n^{[1]}} + \frac{A_{n+1}^2(0)}{A_n(1)\cdot A_{n+1}(1)}.$$

Using Lemma 2.1 for cases $\omega = 0, 1$, we have

$$\frac{1}{\lambda_{n+1}^{[1]}} = \frac{1}{\lambda_n^{[1]}} + \frac{2}{(n+1)},$$

which induces

$$\lambda_{n+1}^{[1]} = \frac{1}{2 + \sum_{i=1}^n \frac{2}{i+1}} \quad (n \ge 1).$$

(ii) For our convenience, consider $\Lambda_{n+1}(x) = \frac{m}{x}M_{n+1}(1,0)$ in (3.1). From Lemma 2.1, we have

$$\det(\Lambda_{n+1}(x)) = m^{n+1}\left(\left(\frac{1}{x}-\frac{1}{m-1}\right)A_n(m) + A_{n+1}(m-2)\right).$$

Write $\lambda_n^{[m]}$ for the root of $\det(\Lambda_{n+1}(x)) = 0$. From some computations, we can obtain that

$$\begin{aligned}\lambda_n^{[m]} &= \frac{(m-1)\left((m+n-1)(m+n-2)\cdots(n+1)\right)^2}{\left((m+n-1)(m+n-2)\cdots(n+1)\right)^2-\left((m-1)!\right)^2} \\ &= \frac{(m-1)\left(\prod_{l=1}^{m-1}(n+l)\right)^2}{\left(\prod_{l=1}^{m-1}(n+l)\right)^2-\left((m-1)!\right)^2}.\end{aligned}$$

The proof is complete. □

The following corollaries are direct consequences of Theorem 3.2.

Corollary 3.3. *Let* $\alpha(x;2) : \sqrt{\frac{x}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$. *Then* $W_{\alpha(x;2)}$ *is* n*-hyponormal if and only if* $0 < x \le \lambda_n^{[2]} := \frac{(n+1)^2}{n(n+2)}$.

Corollary 3.4 ([7, Theorem 4]). *Let* $\alpha(x;3) : \sqrt{\frac{x}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \cdots$. *Then* $W_{\alpha(x;3)}$ *is* n*-hyponormal if and only if* $0 < x \le \lambda_n^{[3]} := \frac{2(n+1)^2(n+2)^2}{n(n+3)(n^2+3n+4)}$.

Corollary 3.5. *Let* $\alpha(x;4) : \sqrt{\frac{x}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{6}{7}}, \cdots$. *Then* $W_{\alpha(x;4)}$ *is* n*-hyponormal if and only if* $0 < x \le \lambda_n^{[4]} := \frac{3(n+1)^2(n+2)^2(n+3)^2}{n(n+4)(n^2+2n+3)(n^2+6n+11)}$.

3.2. Two-step backward extensions. Let W_α be a subnormal weighted shift with a sequence $\alpha = \{\alpha_n\}_{n\ge 0}$. For $0 < y \le x$, consider a two-step backward extension weighted shift $W_{\alpha(x,y)}$ with a sequence $\alpha(x,y) : y, x, \alpha_0, \alpha_1, \cdots$. It is well known that $W_{\alpha(x,y)}$ is n-hyponormal if and only if two Hankel matrices $M_{n+1}(2,0)$ and $M_{n+1}(2,1)$ are positive semi-definite in (3.1) ([12, Lemma 3.1]). Now we introduce an example for two-step backward extension of Bergman shifts $W_{\alpha(x,y)}$ and show the concrete formula for n-hyponormality of $W_{\alpha(x,y)}$.

From now on, we consider a weight sequence as follows:

$$\alpha(x,y) : \sqrt{\frac{y}{2}}, \sqrt{\frac{x}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{6}{7}}, \cdots. \tag{3.3}$$

Theorem 3.6. *Let* $\alpha(x,y)$ *be given in* (3.3) *and let* $W_{\alpha(x,y)}$ *be the associated weighted shift. Then* $W_{\alpha(x,y)}$ *is* n*-hyponormal if and only if* $0 < x \le \frac{2(n+1)^2(n+2)^2}{n(n+3)(n^2+3n+4)}$ *and* $0 < y \le \frac{96(n+1)^2 x}{n(n+2)P_n(x)}$, *where*

$$\begin{aligned}P_n(x) := {}&(n-1)(n+3)\left(n^4+4n^3+9n^2+10n-8\right)x^2 \\ &-4(n-1)(n+3)\left(n^2+2n+4\right)(n+1)^2 x + 4n(n+2)(n+1)^4.\end{aligned}$$

Proof. Let $\alpha(x,y)$ be given in (3.3). Then the moments of $\widetilde{\gamma} := \widetilde{\gamma}(\alpha(x,y)) = \{\widetilde{\gamma}_i\}_{i\geq 0}$ as follows:

$$\widetilde{\gamma}_0 = 1,\ \widetilde{\gamma}_1 = \frac{y}{2},\ \widetilde{\gamma}_n = \frac{xy}{2(n+1)}\ (n \geq 2).$$

For simple computations, we consider two Hankel matrices

$$\Lambda_{n+1}(x,y;0) := \frac{2}{xy} M_{n+1}(2;0) = \begin{bmatrix} \frac{1}{\frac{1}{2}xy} & \frac{1}{x} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \frac{1}{x} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{bmatrix} \tag{3.4}$$

and

$$\Lambda_{n+1}(x,y;1) := \frac{2}{xy} M_{n+1}(2;1) = \begin{bmatrix} \frac{1}{x} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+3} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+2} \end{bmatrix}. \tag{3.5}$$

Then

$$\det \Lambda_{n+1}(x,y;0) = \left(\frac{2}{xy} - 1\right) A_n(2) + B_{n+1}(x),$$
$$\det \Lambda_{n+1}(x,y;1) = \left(\frac{1}{x} - \frac{1}{2}\right) A_n(3) + A_{n+1}(1).$$

If we apply Lemma 2.1 to (3.5), from computations we have

$$M_{n+1}(2;1) \geq 0 \Longleftrightarrow x \leq \frac{A_n(3)}{\frac{1}{2}A_n(3) - A_{n+1}(1)} = \frac{2(n+1)^2 (n+2)^2}{n(n+3)(n^2+3n+4)}.$$

Also we use Lemma 2.1 and Proposition 2.2 to (3.4), we can obtain

$$M_{n+1}(2;0) \geq 0 \Longleftrightarrow y \leq -\frac{2A_n(2)}{x(B_{n+1}(x) - A_n(2))} = \frac{96(n+1)^2 x}{n(n+2)P_n(x)}.$$

Hence we have proved the result. □

The following example([11, Example 3.4]) is a direct consequence of Theorem 3.6.

Example 3.7. Let a weight sequence $\alpha(x,y)$ be given in (3.3). For $k \in \mathbb{N} \cup \{\infty\}$, set

$$\mathbf{HE}_k(x,y) := \{(x,y) \in \mathbb{R}_+^2 : W_{\alpha(x,y)} \text{ is } k\text{-hyponormal}\}.$$

Then we obtain

$$\mathbf{HE}_2(x,y) = \left\{(x,y) : 0 < x \leq \frac{72}{35}, 0 < y \leq \frac{9x}{4(10x^2 - 45x + 54)}\right\},$$

$$\mathbf{HE}_3(x,y)=\left\{(x,y):0<x\le\frac{200}{99},0<y\le\frac{32x}{15(73x^2-304x+320)}\right\},$$
$$\mathbf{HE}_4(x,y)=\left\{(x,y):0<x\le\frac{225}{112},0<y\le\frac{25x}{12(301x^2-1225x+1250)}\right\},...,$$
$$\mathbf{HE}_\infty(x,y)=\{(x,y):x=2,0<y\le1\}.$$

It is easy to see that $\mathbf{HE}_2(x,y)\supsetneqq\mathbf{HE}_3(x,y)\supsetneqq\mathbf{HE}_4(x,y)\supsetneqq\cdots\supsetneqq\mathbf{HE}_\infty(x,y)$. See the following figure.

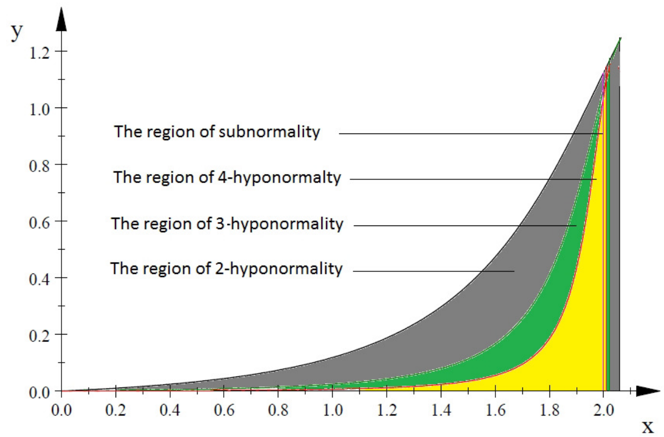

Figure 3.1: The regions of k-hyponormality for $k=2,3,4$ and subnormality.

4. k-hyponormality of Bergman-type shifts

In this section, we introduce new class of Bergman-type weighted shift operators. For a positive real number $p>1$, consider a weight sequence $\alpha^{[p]}:=\{\alpha_k^{[p]}\}_{k\ge0}$ as follows:

$$\alpha^{[p]}:\sqrt{\frac{1}{p}},\sqrt{\frac{p}{2p-1}},\cdots,\sqrt{\frac{(p-1)k+1}{(p-1)k+p}},...\ (k\ge2). \tag{4.1}$$

Then the corresponding weighted shift $W_{\alpha^{[p]}}$ is called a *Bergman-type shift*. In particular, if $p=2$, then $\alpha_k^{[2]}=\sqrt{\frac{k+1}{k+2}}$ for $k\ge0$, i.e. the Bergman-type shift $W_{\alpha^{[2]}}$ is the same as the Bergman shift W_α which was mentioned in Section 1. So we can see that the Bergman-type shift with weight $\alpha^{[p]}$ in (4.1) is a generalized form of Bergman shifts. Furthermore, it is obvious from [9, Theorem 2.7] that all Bergman-type shifts are subnormal.

Now we discuss k-hyponormality of backward extensions of Bergman-type shifts. Without loss of generality, we may assume that $p>1$ and $p\ne2$. For

$x>0$, we consider

$$\alpha^{[p]}(x):\sqrt{x},\sqrt{\frac{1}{p}},\sqrt{\frac{p}{2p-1}},\sqrt{\frac{2p-1}{3p-2}},\sqrt{\frac{3p-2}{4p-3}},\cdots. \tag{4.2}$$

Theorem 4.1. *For $p>1$ and $p\neq 2$, let $W_{\alpha^{[p]}(x)}$ be a Bergman-type shift with a weight $\alpha^{[p]}(x)$ in* (4.2). *Then $W_{\alpha^{[p]}(x)}$ is n-hyponormal if and only if $0<x\leq h_n^{[p]}$, where*

$$h_n^{[p]}:=\frac{(p-2)\left(1\cdot p\cdot(2p-1)\cdot\cdots\cdot((p-1)n-(p-2))\right)^2}{(p-1)^{2n}(n!)^2-\left(1\cdot p\cdot(2p-1)\cdot\cdots\cdot((p-1)n-(p-2))\right)^2}.$$

In particular, for the case $p=2$, the result is given (i) *in Theorem 3.2.*

Proof. For the weight sequence $\alpha^{[p]}(x)$ in (4.2), we have a moment sequence $\widetilde{\gamma}:=\widetilde{\gamma}(\alpha^{[p]}(x))\equiv\{\widetilde{\gamma}_n(\alpha^{[p]}(x))\}$ $(n\geq 0)$ of $\alpha^{[p]}$, as usual,

$$\widetilde{\gamma}_0:=1,\ \widetilde{\gamma}_1:=x,\ \widetilde{\gamma}_2:=\frac{x}{p},\ \cdots,\ \widetilde{\gamma}_n:=\frac{x}{(n-1)p-n+2}\ (n\geq 2). \tag{4.3}$$

Now we apply this moments $\widetilde{\gamma}$ to Lemma 3.1 and using Proposition 2.3 and Proposition 2.4, we can have

$$\begin{aligned}
&\det\left(\frac{1}{x}M_{k+1}(1,0)\right)\\
&=\begin{vmatrix}
\frac{1}{x} & 1 & \frac{1}{p} & \cdots & \frac{1}{(k-1)p-(k-2)}\\
1 & \frac{1}{p} & \frac{1}{2p-1} & \cdots & \frac{1}{(k)p-(k-1)}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{1}{(k-1)p-(k-2)} & \frac{1}{(k)p-(k-1)} & \frac{1}{(k+1)p-k} & \cdots & \frac{1}{(2k-1)p-(2k-2)}
\end{vmatrix}_{k+1}\\
&=\left(\frac{1}{x}-\frac{1}{2-p}\right)C_k^{[1]}(p)+C_{k+1}^{[2]}(p).
\end{aligned}$$

Hence

$$\begin{aligned}
W_{\alpha^{[p]}(x)}\text{ is }n\text{-hyponormal}&\Leftrightarrow M_{k+1}(1,0)\geq 0\\
&\Leftrightarrow x\leq\frac{p-2}{\frac{(n!)^2\left(\prod_{l=1}^{n}\prod_{s=1}^{n}\left(\frac{1}{p-1}+l+s-1\right)\right)}{\left(\prod_{l=0}^{n}\left(\frac{2-p}{p-1}+l\right)^{l+1}\right)\left(\prod_{l=n+1}^{2n}\left(\frac{2-p}{p-1}+l\right)^{2n-l+1}\right)}-1}\\
&\Leftrightarrow x\leq\frac{(p-2)\left(1\cdot p\cdot(2p-1)\cdot\cdots\cdot((p-1)n-(p-2))\right)^2}{(p-1)^{2n}(n!)^2-\left(1\cdot p\cdot(2p-1)\cdot\cdots\cdot((p-1)n-(p-2))\right)^2}.
\end{aligned}$$

Thus we have done. □

By Theorem 4.1, we obtain the following corollaries.

Corollary 4.2. *Let* $\alpha^{[3]}(x) : \sqrt{x}, \sqrt{\frac{1}{3}}, \sqrt{\frac{3}{5}}, \sqrt{\frac{5}{7}}, \cdots, \sqrt{\frac{2k+1}{2k+3}}, \cdots$. *Then* $W_{\alpha^{[3]}(x)}$ *is n-hyponormal if and only if* $0 < x \leq h_n^{[3]}$, *where*

$$h_n^{[3]} := \frac{\left((2n-1)!!\right)^2}{2^{2n}\left(n!\right)^2 - \left((2n-1)!!\right)^2}.$$

Corollary 4.3. *Let* $\alpha^{[4]}(x) : \sqrt{x}, \sqrt{\frac{1}{4}}, \sqrt{\frac{4}{7}}, \sqrt{\frac{7}{10}}, \cdots, \sqrt{\frac{3k+1}{3k+4}}, \cdots$. *Then* $W_{\alpha^{[4]}(x)}$ *is n-hyponormal if and only if* $0 < x \leq h_n^{[4]}$, *where*

$$h_n^{[4]} := \frac{2\left(1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)\right)^2}{3^{2n}\left(n!\right)^2 - \left(1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)\right)^2}.$$

References

1. J. Bram, *Subnormal operators*, Duke Math. J. **22**(1955), 75-94.
2. Y.B. Choi, J.K. Han and W.Y. Lee, *One-step extension of the Bergman shift*, Proc. Amer. Math. Soc. **128**(2000), 3639-3646.
3. R. Curto, *Joint hyponormality: A bridge between hyponormality and subnormality*, Proc. Sym. Math. **51**(1990), 69-91.
4. R. Curto, *Quadratically hyponormal weighted shifts,* Integral Equations Operator Theory **13**(1990), 49-66.
5. R. Curto and L. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equations Operator Theory **17**(1993), 202-246.
6. R. Curto and L. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem, II*, Integral Equations Operator Theory **18**(1994), 369-426.
7. R. Curto and S. Lee, *Quartically hyponormal weighted shifts need not be 3-hyponormal*, J. Math. Anal. Appl. **314**(2006), 455-463.
8. R. Curto, P. Muhly and J. Xia, *Hyponormal pairs of commuting operators*, Operator Theory:Adv. Appl. **35**(1988), 1-22.
9. R. Curto, Y. Poon and J. Yoon, *Subnormality of Bergman-like weighted shifts*, J. Math. Anal. Appl. **308**(2005), 334-342.
10. P. Halmos, *Normal dilations and extensions of operators*, Summer Bras. Math. **2**(1950), 124-134.
11. I.B. Jung and C. Li, *Backward extensions of hyponormal weighted shifts,* Math. Japonica **52**(2000), no.2, 267-278.
12. I.B. Jung and C. Li, *A formula for k-hyponormality of backstep extensions of subnormal weighted shifts,* Proc. Amer. Math. Soc. **129**(2001), 2343-2351.
13. C. Krattenthaler, *Advanced determinant calculus,* Séminaire Lotharingien Combin. **42**(1999).
14. J. Stampfli, *Which weighted shifts are subnormal*, Pacific J. Math. **17**(1966), 367-379.
15. MacKichan Software, Inc. *Scientific WorkPlace, Version 4.0*, MacKichan Software, Inc., 2002.

Yanwu Dong received M.Sc. from Northeastern University. His research interests include optimization and unilateral weighted shifts.

Department of Mathematics, Zhanjiang Preschool Education College (Fundamental Education College of Lingnan Normal University), Zhanjiang 524084, Guangdong, P. R. China.

e-mail: long__feng@sina.com

Mi Ryeong Lee received Ph.D. degree from Kyungpook National University. Her research interests are mainly on the gap theory of classes of bounded linear operators on Hilbert space.

Institute of Liberal Education, Catholic University of Daegu, Gyeongsan, Gyeongbuk 712-702, Korea.
e-mail: leemr@cu.ac.kr

Chunji Li received Ph.D. degree from Kyungpook National University, Korea. His research interests focus on the control theory, moment method, and unilateral weighted shifts.

Department of Mathematics, Northeastern University, Shenyang 110819, P. R. China.
e-mail: lichunji@mail.neu.edu.cn