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SOME PROPERTIES RELATED TO FUZZY FUNCTIONS ON COMPLETE RESIDUATED LATTICES†

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ABSTRACT. In this paper we give some properties related to fuzzy functions on complete residuated lattices.

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1. Introduction

A fuzzy function fuzzifies a concept of a function between two universes. This fuzzification has been researched by many authors (for examples, see [2, 3, 4, 5, 7, 8]). In this paper we give some properties related to fuzzy functions on complete residuated lattices.

Definition 1.1 ([1])**.** An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a *complete residuated lattice* if

(1) $(L, \wedge, \vee, 0, 1)$ is a complete lattice with the least element 0 and the greatest element 1;
(2) $(L, \odot, 1)$ is a commutative monoid (*i.e.,* $\odot$ is commutative, associated and $x \odot 1 = x$ for all $x \in L$);
(3) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in L$ (*i.e.,* $\odot$ and $\rightarrow$ form adjoint pair).

Throughout this paper we always assume that $L = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a complete residuated lattice.

Definition 1.2 ([7])**.** Let $R : X \times X \rightarrow L$ be a fuzzy relation on a set X.

(1) R is *reflexive* if $R(x, x) = 1$ for all $x \in X$.

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(2) R is *symmetric* if $R(x,y) = R(y,x)$ for all $x, y \in X$.
(3) R is *transitive* if $R(x,y) \odot R(y,z) \leq R(x,z)$ for all $x, y, z \in X$.
(4) R is an *indistinguishable operator on* X if R is reflexive, symmetric and transitive.

Definition 1.3 ([4])**.** Let E and F be two indistinguishable operators on X and Y respectively. A fuzzy relation $R : X \times Y \to L$ is *extensional with respect to* E *and* F if

$$R(x,y) \odot E(x,x') \odot F(y,y') \leq R(x',y').$$

Definition 1.4 ([4])**.** Let E and F be two indistinguishable operators on X and Y respectively. Let $R : X \times Y \to L$ be extensional with respect to E and F.

(1) R is a *partial fuzzy function* if $R(x,y) \odot R(x,y') \leq F(y,y')$ for all $x \in X$ and $y, y' \in Y$.
(2) R is *fully defined* if $\bigvee_{y \in Y} R(x,y) = 1$ for all $x \in X$.
(3) R is a *fuzzy function* if R is a partial fuzzy function and is fully defined.
(4) R is a *perfect fuzzy function* if (a) R is a partial fuzzy map and (b) for all $x \in X$, there exists $y \in Y$ such that $R(x,y) = 1$.

Definition 1.5 ([4])**.** Let E and F be two indistinguishable operators on X and Y respectively. A fuzzy relation $R : X \times Y \to L$ is a *strong fuzzy function with respect to* E *and* F if

(1) for all $x \in X$, there exists $y \in Y$ such that $R(x,y) = 1$, and
(2) $R(x,y) \odot R(x',y') \odot E(x,x') \leq F(y,y')$ for all $x, x' \in X$ and $y, y' \in Y$.

Definition 1.6 ([4])**.** Let E and F be two indistinguishable operators on X and Y respectively. Let $f : X \to Y$ be a crisp function. f is *extensional with respect to* E *and* F if

$$E(x,x') \leq F(f(x), f(x')) \qquad \text{for all } x, x' \in X.$$

Proposition 1.7 ([1])**.** *Let* $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ *be a complete residuated lattice. Then for all* $x, y, y_i \in L$*, the following hold.*

(1) $x \to x = 1$.
(2) $1 \to x = x$.
(3) $x \odot (x \to y) \leq y$.
(4) $x \odot \bigwedge_i y_i \leq \bigwedge_i (x \odot y_i)$.
(5) $y_1 \leq y_2$ *implies* $x \odot y_1 \leq x \odot y_2$ *(isotonicity of* $\odot$*).*

Definition 1.8 ([6])**.** Let $R : X \times Y \to L$ be a fuzzy relation from X to Y. Define $\sigma(R) : Y \times Y \to L$ by

$$\sigma(R)(y_1, y_2) = \bigwedge_{x \in X} [R(x, y_1) \to R(x, y_2)] \qquad \text{for all } y_1, y_2 \in Y.$$

Define $\rho(R) : X \times X \to L$ by

$$\rho(R)(x_1, x_2) = \bigwedge_{y \in Y} [R(x_2, y) \to R(x_1, y)] \qquad \text{for all } x_1, x_2 \in X.$$

2. Results

Lemma 2.1. *Let $R : X \times Y \to L$ be a fuzzy relation from a set X to a set Y.*

(1) *For all $y_1 \in Y$, there exists $y_2 \in Y$ such that $\sigma(R)(y_1, y_2) = 1$.*
(2) *For all $x_1 \in X$, there exists $x_2 \in X$ such that $\rho(R)(x_1, x_2) = 1$.*
(3) *$\bigvee_{y_2 \in Y} \sigma(R)(y_1, y_2) = 1$ for all $y_1 \in Y$, and $\bigvee_{x_2 \in X} \rho(R)(x_1, x_2) = 1$ for all $x_1 \in X$.*

Proof. (1) Let $y_1 \in Y$. Then

$$\begin{aligned}\sigma(R)(y_1, y_1) &= \bigwedge_{x \in X} [R(x, y_1) \to R(x, y_1)] \\ &= \bigwedge_{x \in X} 1 \qquad \text{by Proposition 1.7(1)} \\ &= 1\end{aligned}$$

(2) Let $x_1 \in X$. Then

$$\begin{aligned}\rho(R)(x_1, x_1) &= \bigwedge_{y \in Y} [R(x_1, y) \to R(x_1, y)] \\ &= \bigwedge_{y \in Y} 1 \qquad \text{by Proposition 1.7(1)} \\ &= 1\end{aligned}$$

(3) It follows from (1) and (2).

□

Lemma 2.2. *Let $R : X \times X \to L$ be a fuzzy relation from X to X. If R is reflexive, then $\sigma(R) \leq R$ and $\rho(R) \leq R$.*

Proof. Note that for all $y_1, y_2 \in X$, we have

$$\begin{aligned}\sigma(R)(y_1, y_2) &= \bigwedge_{x \in X} [R(x, y_1) \to R(x, y_2)] \\ &\leq R(y_1, y_1) \to R(y_1, y_2) \\ &= 1 \to R(y_1, y_2) \text{ since } R \text{ is reflexive} \\ &= R(y_1, y_2) \text{ by Proposition 1.7(2).}\end{aligned}$$

Hence $\sigma(R) \leq R$.

Similarly, for all $x_1, x_2 \in X$, we have

$$\begin{aligned}\rho(R)(x_1, x_2) &= \bigwedge_{y \in X} [R(x_2, y) \to R(x_1, y) \\ &\leq R(x_2, x_2) \to R(x_1, x_2) \\ &= 1 \to R(x_1, x_2) \text{ since } R \text{ is reflexive} \\ &= R(x_1, x_2).\end{aligned}$$

Hence $\rho(R) \leq R$. □

Theorem 2.3. *Let E be an indistinguishable operator on X. Let $R : X\times X \to L$ be a fuzzy relation such that*

$$R(x, y) \odot R(x', y') \odot E(x, x') \leq E(y, y') \text{ for all } x, x', y, y' \in X. \tag{1}$$

If R is reflexible, then $\sigma(R)$ and $\rho(R)$ are strong fuzzy functions with respect to E and E.

Proof. By Lemma 2.1 (1) and (2), both of $\sigma(R)$ and $\rho(R)$ satisfy the condition (1) in Definition 1.5.

Let $y_1, y_1', y_2, y_2' \in X$. Then

$$\begin{aligned}&\sigma(R)(y_1, y_2) \odot \sigma(R)(y_1', y_2') \odot E(y_1, y_1')\\ &\leq R(y_1, y_2) \odot R(y_1', y_2') \odot E(y_1, y_1') \qquad \text{by Lemma 2.2}\\ &\leq E(y_2, y_2') \qquad \text{by Eq. (1).}\end{aligned}$$

Therefore $\sigma(R)$ is a strong fuzzy function with respect to E and E.

Let $x_1, x_2, x_1', x_2' \in X$. Then

$$\begin{aligned}&\rho(R)(x_1, x_2) \odot \rho(R)(x_1', x_2') \odot E(x_1, x_1')\\ &\leq R(x_1, x_2) \odot R(x_1', x_2') \odot E(x_1, x_1') \qquad \text{by Lemma 2.2}\\ &\leq E(x_2, x_2') \qquad \text{by Eq. (1).}\end{aligned}$$

Therefore $\rho(R)$ is a strong fuzzy function with respect to E and E. □

By Theorem 2.3, we have the following.

Corollary 2.4. *Let E be an indistinguishable operator on X. Let $R : X\times X \to L$ be a strong fuzzy function with respect to E and E. If R is reflexible, then $\sigma(R)$ and $\rho(R)$ are strong fuzzy functions with respect to E and E.*

Theorem 2.5. *Let E be an indistinguishable operator on X. Let $R : X \times X \to L$ be extensional with respect to E and E. Then both of $\sigma(R)$ and $\rho(R)$ are extensional with respect to E and E.*

Proof. Let $y_1, y_1', y_2, y_2' \in X$. We must show that

$$\sigma(R)(y_1, y_2) \odot E(y_1, y_1') \odot E(y_2, y_2') \leq \sigma(R)(y_1', y_2').$$

Since

$$\begin{aligned}&\sigma(R)(y_1, y_2) \odot E(y_1, y_1') \odot E(y_2, y_2')\\ &= \bigwedge_{x\in X} [R(x, y_1) \to R(x, y_2)] \odot E(y_1, y_1') \odot E(y_2, y_2')\\ &\leq \bigwedge_{x\in X} \{[R(x, y_1) \to R(x, y_2)] \odot E(y_1, y_1') \odot E(y_2, y_2')\} \text{ by Proposition 1.7(4),}\end{aligned}$$

it is enough to show that for all $x \in X$,

$$[R(x, y_1) \to R(x, y_2)] \odot E(y_1, y_1') \odot E(y_2, y_2') \leq R(x, y_1') \to R(x, y_2'). \tag{2}$$

Note that Eq. (2) holds if and only if

$$[R(x,y_1) \to R(x,y_2)] \odot R(x,y_1') \odot E(y_1,y_1') \odot E(y_2,y_2') \leq R(x,y_2'). \qquad (3)$$

Note that

$$\begin{aligned}
&[R(x,y_1) \to R(x,y_2)] \odot R(x,y_1') \odot E(y_1,y_1') \odot E(y_2,y_2') \\
&= [R(x,y_1) \to R(x,y_2)] \odot R(x,y_1') \odot E(x,x) \odot E(y_1',y_1) \odot E(y_2,y_2') \\
&\leq R(x,y_1) \odot [R(x,y_1) \to R(x,y_2)] \odot E(y_2,y_2') \qquad \text{since } R \text{ is extensional} \\
&\leq R(x,y_2) \odot E(y_2,y_2') \qquad \text{by Propostion 1.7(3)} \\
&= R(x,y_2) \odot E(x,x) \odot E(y_2,y_2') \\
&\leq R(x,y_2') \qquad \text{since } R \text{ is extensional.}
\end{aligned}$$

Therefore $\sigma(R)$ is extensional with respect to E and E.

Let $x_1, x_1', x_2, x_2' \in X$. We must show that

$$\rho(R)(x_1,x_2) \odot E(x_1,x_1') \odot E(x_2,x_2') \leq \rho(R)(x_1',x_2').$$

Since

$$\begin{aligned}
&\rho(R)(x_1,x_2) \odot E(x_1,x_1') \odot E(x_2,x_2') \\
&= \bigwedge_{y\in X} [R(x_2,y) \to R(x_1,y)] \odot E(x_1,x_1') \odot E(x_2,x_2') \\
&\leq \bigwedge_{y\in X} \{[R(x_2,y) \to R(x_1,y)] \odot E(x_1,x_1') \odot E(x_2,x_2')\} \text{ by Proposition 1.7(4),}
\end{aligned}$$

it is enough to show that for any $y \in X$,

$$[R(x_2,y) \to R(x_1,y)] \odot E(x_1,x_1') \odot E(x_2,x_2') \leq R(x_2',y) \to R(x_1',y). \qquad (4)$$

Note that Eq. (4) holds if and only if

$$[R(x_2,y) \to R(x_1,y)] \odot R(x_2',y) \odot E(x_1,x_1') \odot E(x_2,x_2') \leq R(x_1',y).$$

Note that

$$\begin{aligned}
&R(x_2,y) \to R(x_1,y)] \odot R(x_2',y) \odot E(x_1,x_1') \odot E(x_2,x_2') \\
&= [R(x_2,y) \to R(x_1,y)] \odot R(x_2',y) \odot E(x_2',x_2) \odot E(y,y) \odot E(x_1,x_1') \\
&\leq [R(x_2,y) \to R(x_1,y)] \odot R(x_2,y) \odot E(x_1,x_1') \qquad \text{since } R \text{ is extensional} \\
&\leq R(x_1,y) \odot E(x_1,x_1') \qquad \text{by Proposition 1.7(4)} \\
&= R(x_1,y) \odot E(x_1,x_1') \odot E(y,y) \\
&\leq R(x_1',y) \qquad \text{since } R \text{ is extensional.}
\end{aligned}$$

Therefore $\rho(R)$ is extensional with respect to E and E. □

Theorem 2.6. *Let $R : X \times X \to L$ be a partial fuzzy function where E is an indistinguishable operator on X. If R is reflexive, then both of $\sigma(R)$ and $\rho(R)$ are partial fuzzy functions.*

Proof. We already know by Theorem 2.5 that both of $\sigma(R)$ and $\rho(R)$ are extensional with respect to E and E.

Let $y_1, y_2, y_2' \in X$. We must show that

$$\sigma(R)(y_1, y_2) \odot \sigma(R)(y_1, y_2') \le E(y_2, y_2').$$

Note that

$$\begin{aligned}\sigma(R)(y_1, y_2) \odot \sigma(R)(y_1, y_2') &\le R(y_1, y_2) \odot R(y_1, y_2') \qquad \text{by Lemma 2.2}\\ &\le E(y_2, y_2') \qquad \text{since } R \text{ is a partial fuzzy function.}\end{aligned}$$

Hence $\sigma(R)$ is a partial fuzzy function.

Let $x_1, x_2, x_2' \in X$. We must show that

$$\rho(R)(x_1, x_2) \odot \sigma(R)(x_1, x_2') \le E(x_2, x_2').$$

Note that

$$\begin{aligned}\rho(R)(x_1, x_2) \odot \rho(R)(x_1, x_2') &\le R(x_1, x_2) \odot R(x_1, x_2') \qquad \text{by Lemma 2.2}\\ &\le E(x_2, x_2') \qquad \text{since } R \text{ is a partial fuzzy function.}\end{aligned}$$

Hence $\rho(R)$ is a partial fuzzy function. □

Theorem 2.7. *If $R : X \times X \to L$ is fully defined where E is an indistinguishable operator on X, then $\sigma(R)$ and $\rho(R)$ is fully defined.*

Proof. Since R is fully defined, R is extensional with respect to E and E, and so by Theorem 2.5, both of $\sigma(R)$ and $\rho(R)$ are extensional with respect to E and E. Now, by Lemma 2.1(3), both of $\sigma(R)$ and $\rho(R)$ are fully defined. □

By Lemma 2.1 (1), (2), Theorems 2.5 and 2.6, we have the following.

Theorem 2.8. *Let $R : X \times X \to L$ be a partial fuzzy function where E is an indistinguishable operator on X. If R is reflexive, then both of $\sigma(R)$ and $\rho(R)$ are perfect fuzzy functions.*

As an immediate consequence of Theorem 2.8, we have the following.

Corollary 2.9. *If $R : X \times X \to L$ be a reflexive perfect fuzzy function where E is an indistinguishable operator on X, then both of $\sigma(R)$ and $\rho(R)$ are perfect fuzzy functions.*

By Theorems 2.6 and 2.7, we have the following.

Theorem 2.10. *Let $R : X \times X \to L$ be a partial fuzzy function where E is an indistinguishable operator on X. If R is reflexive, then both of $\sigma(R)$ and $\rho(R)$ are fuzzy functions.*

As an immediate consequence of Theorem 2.10, we have the following.

Corollary 2.11. *If $R : X \times X \to L$ be a reflexive fuzzy function where E is an indistinguishable operator on X, then both of $\sigma(R)$ and $\rho(R)$ are fuzzy functions.*

References

1. R. Bělohlávek, *Fuzzy Relational Systems, Foundations and Principles*, Kluwer Academic, Plenum Publishers, New York, 2002.
2. M. Demirci, *Fuzzy functions and their applications*, J. Math. Anal. Appl. **252** (2000), 495-517.
3. M. Demirci, *Fuzzy functions and their fundamental properties*, Fuzzy Sets and Systems **106** (1999), 239-246.
4. M. Demirci and J. Recasens, *Fuzzy groups, fuzzy functions and fuzzy equivalence relations*, Fuzzy Sets and Systems **144** (2004), 441-458.
5. F. Klawonn, *Fuzzy points, fuzzy relations and fuzzy functions* in “Discovering World with Fuzzy Logic”, Physica-Verlag, Heidelberg, 2000.
6. H. Lai and D. Zhang, *Good fuzzy preorders on fuzzy power structures*, Arch. Math. Log. **49** (2010), 469-489.
7. J. Recasens, *Indistinguishability operators*, Springer-Verlag, Berlin, 2010.
8. L. Valverde, *On the structure of F-indistinguishability operators*, Fuzzy sets and Systems **17** (1985), 313-328.

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