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COUPLED COINCIDENCE POINT RESULTS WITH MAPPINGS SATISFYING RATIONAL INEQUALITY IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper we prove certain coupled coincidence point and coupled common fixed point results in partially ordered metric spaces for a pair of compatible mappings which satisfy certain rational inequality. The results are supported with two examples.

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1. Introduction

Coupled fixed point results constitute a chapter in metric fixed point theory which has been in focus in recent times. Although the concept was introduced some time back in 1987 by Guo et al. [14], it was after the publication of the work of Bhaskar et al. [13] that a large number of papers have been written on this topic and on topics related to it. Particularly coupled coincidence point results appeared in works like [2, 3, 7, 8, 9, 11, 15, 17, 18, 20, 21]. The commutativity and compatibility conditions were defined in a separate way to suit the new situation [2, 3, 7, 8, 9, 17, 18, 20].

Here we address a problem of the existence of a coupled coincidence point between two functions under certain conditions. We assume that a particular rational inequality is satisfied by the two functions. The use of rational inequality in metric fixed point theory was initiated by Dass et al. in their work [12] in which they extended the Banach's contraction mapping principle by using a contractive rational inequality. After that the rational inequalities have been used in fixed point, coincidence point and proximity point problem in a large

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number of papers as for instances in [1, 3, 4, 5, 6, 10, 16]. Further we work out our result in partially ordered metric spaces. We use several partial order conditions in our results.

In this paper we establish a coupled coincidence point theorem for mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$, where $(X,\ d)$ is a metric space with a partial ordering. The uniqueness of the coupled common fixed point is ensured by imposing, amongst other conditions, the condition of coincidentally commuting, a concept which we introduce here. Our result extends some results in [13, 19]. Two supporting examples are given.

2. Mathematical Preliminaries

Let $(X,\ \preceq)$ be a partially ordered set and $F : X \longrightarrow X$. The mapping F is said to be nondecreasing if for all $x_1,\ x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and nonincreasing if for all $x_1,\ x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$.

Definition 2.1 ([13])**.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \longrightarrow X$. The mapping F is said to have the mixed monotone property if F is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument; that is, if

$x_1,\ x_2 \in X,\ \ x_1 \preceq x_2 \Longrightarrow F(x_1,\ y) \preceq F(x_2,\ y)$, for all $y \in X$

and

$y_1,\ y_2 \in X,\ \ y_1 \preceq y_2 \Longrightarrow F(x,\ y_1) \succeq F(x,\ y_2)$, for all $x \in X$.

Definition 2.2 ([18])**.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. We say F has the mixed g- monotone property if

$x_1,\ x_2 \in X,\ \ gx_1 \preceq gx_2 \Longrightarrow F(x_1,\ y) \preceq F(x_2,\ y)$, for all $y \in X$

and

$y_1,\ y_2 \in X,\ \ gy_1 \preceq gy_2 \Longrightarrow F(x,\ y_1) \succeq F(x,\ y_2)$, for all $x \in X$.

Definition 2.3 ([13])**.** Let X be a non-empty set and $F : X \times X \longrightarrow X$. An element $(x,\ y) \in X \times X$ is called a coupled fixed point of the mapping F if $x = F(x,\ y)$ and $y = F(y,\ x)$.

Definition 2.4 ([18])**.** Let X be a non-empty set and $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$. An element $(x,\ y) \in X \times X$ is called a coupled coincidence point of the mappings g and F if $gx = F(x,\ y)$ and $gy = F(y,\ x)$.

Definition 2.5 ([18])**.** Let X be a non-empty set and $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$. An element $(x,\ y) \in X \times X$ is called a coupled common fixed point of the mappings g and F if $x = gx = F(x,\ y)$ and $y = gy = F(y,\ x)$.

Definition 2.6 ([7])**.** Let X be a non-empty set and $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$. We say g and F are compatible if

$$\lim_{n\to\infty} d\Big(gF(x_n,\ y_n), F(gx_n,\ gy_n)\Big) = 0 \text{ and } \lim_{n\to\infty} d\Big(gF(y_n,\ x_n), F(gy_n,\ gx_n)\Big) = 0$$

, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim\limits_{n\to\infty} F(x_n,\ y_n) = \lim\limits_{n\to\infty} gx_n = x$ and $\lim\limits_{n\to\infty} F(y_n,\ x_n) = \lim\limits_{n\to\infty} gy_n = y$, for some $x,\ y \in X$ are satisfied.

We define coincidentally commuting mapping in the following.

Definition 2.7. The mappings g and F, where $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$, are said to be coincidentally commuting if they commute at their coupled coincidence points, that is, if $gx = F(x,\ y)$ and $gy = F(y,\ x)$, for some $(x,\ y) \in X \times X$, then $gF(x,\ y) = F(gx,\ gy)$ and $gF(y,\ x) = F(gy,\ gx)$.

3. Main results

Theorem 3.1. *Let $(X,\ \preceq)$ be a partially ordered set and d be a metric on X such that $(X,\ d)$ is a complete metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are two mappings such that F has the mixed g-monotone property on X. Suppose that F is continuous, $F(X \times X) \subseteq g(X)$, g is continuous non-decreasing and the pair $(F,\ g)$ is compatible. Further suppose that there exist non-negative real numbers α and L with $0 \leq \alpha < 1$ such that for all $x, y, u, v \in X$, with $gx \succeq gu$ and $gy \preceq gv$,*

$$\begin{aligned} d\Big(F(x,\ y),\ F(u,\ v)\Big) \leq \alpha\ max \Big\{ & d(gx,\ gu),\ d(gy,\ gv), \\ & \frac{d(gx,\ F(x,\ y))\Big(1 + d(gu,\ F(u,\ v))\Big)}{1 + d(gx,\ gu)}, \\ & \frac{d(gx,\ F(u,\ v))\Big(1 + d(gu,\ F(x,\ y))\Big)}{1 + d(gx,\ gu)}, \\ & \frac{d(gy,\ F(y,\ x))\Big(1 + d(gv,\ F(v,\ u))\Big)}{1 + d(gy,\ gv)}, \\ & \frac{d(gy,\ F(v,\ u))\Big(1 + d(gv,\ F(y,\ x))\Big)}{1 + d(gy,\ gv)} \Big\} \\ +L\ min \Big\{ d(F(x,\ y),\ gu),\ d(F(u,\ v),\ gx),\ & d(F(x,\ y),\ gx),\ d(F(u,\ v),\ gu) \Big\}. \end{aligned} \tag{1}$$

If there exist $x_0,\ y_0 \in X$ such that $gx_0 \preceq F(x_0,\ y_0)$ and $gy_0 \succeq F(y_0,\ x_0)$, then F and g have a coupled coincidence point in X, that is, there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Proof. Let $x_0,\ y_0 \in X$ be such that $gx_0 \preceq F(x_0,\ y_0)$ and $gy_0 \succeq F(y_0,\ x_0)$. Since $F(X \times X) \subseteq g(X)$, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n,\ y_n) \text{ and } gy_{n+1} = F(y_n,\ x_n),\ \text{ for all } n \geq 0. \tag{2}$$

We claim that for all $n \geq 0$,

$$gx_n \preceq gx_{n+1} \tag{3}$$

and

$$gy_n \succeq gy_{n+1}. \tag{4}$$

Since $gx_0 \preceq F(x_0,\ y_0)$ and $gy_0 \succeq F(y_0,\ x_0)$, it follows by (2) that $gx_0 \preceq F(x_0,\ y_0) = gx_1$ and $gy_0 \succeq F(y_0,\ x_0) = gy_1$, that is, (3) and (4) hold for $n = 0$. Suppose that (3) and (4) hold for some $n > 0$. As F has the mixed g-monotone property and $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$, from (2), we get

$$gx_{n+1} = F(x_n,\ y_n) \preceq F(x_{n+1},\ y_n) \preceq F(x_{n+1},\ y_{n+1}) = gx_{n+2} \tag{5}$$

and

$$gy_{n+1} = F(y_n,\ x_n) \succeq F(y_{n+1},\ x_n) \succeq F(y_{n+1},\ x_{n+1}) = gy_{n+2}. \tag{6}$$

Therefore, we obtain that $gx_{n+1} \preceq gx_{n+2}$ and $gy_{n+1} \succeq gy_{n+2}$. Thus by the mathematical induction, we conclude that (3) and (4) hold for all $n \geq 0$. Therefore,

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq ... \preceq gx_n \preceq gx_{n+1} \preceq .... \tag{7}$$

and

$$gy_0 \succeq gy_1 \succeq gy_2 \succeq ... \succeq gy_n \succeq gy_{n+1} \succeq .... \tag{8}$$

Since $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$ for all $n \geq 1$, applying (1) and using (2), we have

$$\begin{aligned}
d(gx_{n+1},\ gx_n) &= d\Big(F(x_n,\ y_n),\ F(x_{n-1},\ y_{n-1})\Big) \\
&\leq \alpha \max \Big\{d(gx_n,\ gx_{n-1}),\ d(gy_n,\ gy_{n-1}), \\
&\qquad \frac{d(gx_n,\ F(x_n,\ y_n))\Big(1 + d(gx_{n-1},\ F(x_{n-1},\ y_{n-1}))\Big)}{1 + d(gx_n,\ gx_{n-1})}, \\
&\qquad \frac{d(gx_n,\ F(x_{n-1},\ y_{n-1}))\Big(1 + d(gx_{n-1},\ F(x_n, y_n))\Big)}{1 + d(gx_n,\ gx_{n-1})}, \\
&\qquad \frac{d(gy_n,\ F(y_n,\ x_n))\Big(1 + d(gy_{n-1},\ F(y_{n-1},\ x_{n-1}))\Big)}{1 + d(gy_n,\ gy_{n-1})}, \\
&\qquad \frac{d(gy_n,\ F(y_{n-1},\ x_{n-1}))\Big(1 + d(gy_{n-1},\ F(y_n,\ x_n))\Big)}{1 + d(gy_n,\ gy_{n-1})}\Big\} \\
&\quad + L \min \Big\{d(F(x_n,\ y_n),\ gx_{n-1}),\ d(F(x_{n-1},\ y_{n-1}),\ gx_n), \\
&\qquad d(F(x_n,\ y_n),\ gx_n),\ d(F(x_{n-1},\ y_{n-1}),\ gx_{n-1})\Big\} \\
&\leq \alpha \max \Big\{d(gx_n,\ gx_{n-1}),\ d(gy_n,\ gy_{n-1}), \\
&\qquad \frac{d(gx_n,\ gx_{n+1})\Big(1 + d(gx_{n-1},\ gx_n)\Big)}{1 + d(gx_n,\ gx_{n-1})},
\end{aligned}$$

$$\frac{d(gx_n,\ gx_n)\Big(1+d(gx_{n-1},\ gx_{n+1})\Big)}{1+d(gx_n,\ gx_{n-1})},$$
$$\frac{d(gy_n,\ gy_{n+1})\Big(1+d(gy_{n-1},\ gy_n)\Big)}{1+d(gy_n,\ gy_{n-1})},$$
$$\frac{d(gy_n,\ gy_n)\Big(1+d(gy_{n-1},\ gy_{n+1})\Big)}{1+d(gy_n,\ gy_{n-1})}\Big\}$$
$$+L\min\Big\{d(gx_{n+1},\ gx_{n-1}),\ d(gx_n,\ gx_n),\ d(gx_{n+1},\ gx_n), d(gx_n,\ gx_{n-1})\Big\}$$
$$\leq \alpha\max\Big\{d(gx_n,\ gx_{n-1}),\ d(gy_n,\ gy_{n-1}),\ d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\Big\},$$

that is,

$$d(gx_{n+1}, gx_n)\leq \alpha\max\Big\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\Big\}.$$

Similarly, we can prove that

$$d(gy_{n+1}, gy_n)\leq \alpha\max\Big\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\Big\}.$$

Set $\rho_n=\max\Big\{d(gx_{n+1},\ gx_n),\ d(gy_{n+1},\ gy_n)\Big\}$.
So

$$\begin{aligned}\rho_n &= \max\Big\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\Big\}\\ &\leq \alpha\max\Big\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\Big\}\\ &= \alpha\ \rho_{n-1}.\end{aligned}$$

By mathematical induction, we have

$$\rho_n=\max\Big\{d(gx_{n+1},\ gx_n),\ d(gy_{n+1},\ gy_n)\Big\}\leq \alpha^n\rho_0,$$

which implies that

$$d(gx_{n+1},\ gx_n)\leq \alpha^n\rho_0 \ \text{ and } \ d(gy_{n+1},\ gy_n)\leq \alpha^n\rho_0.$$

Then for each $m,n\in N$ with $m<n$,

$$\begin{aligned}d(gx_m,\ gx_n) &\leq d(gx_m,\ gx_{m+1})+d(gx_{m+1},\ gx_{m+2})+...+d(gx_{n-1},\ gx_n)\\ &\leq (\alpha^m+\alpha^{m+1}+...+\alpha^{n-1})\ \rho_0\\ &\leq \frac{\alpha^m}{1-\alpha}\ \rho_0\to 0 \ \text{ as } \ m,n\to\infty,\end{aligned}$$

and

$$\begin{aligned}d(gy_m,\ gy_n) &\leq d(gy_m,\ gy_{m+1})+d(gy_{m+1},\ gy_{m+2})+...+d(gy_{n-1},\ gy_n)\\ &\leq (\alpha^m+\alpha^{m+1}+...+\alpha^{n-1})\ \rho_0\\ &\leq \frac{\alpha^m}{1-\alpha}\rho_0\to 0 \ \text{ as } \ m,n\to\infty.\end{aligned}$$

Hence, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Since X is a complete metric space, there exists $x,\ y \in X$ such that $gx_n \to x$ and $gy_n \to y$. Since g is continuous, we have

$$g(gx_n) \to gx \ \text{ and } \ g(gy_n) \to gy. \tag{9}$$

Since F and g are compatible mappings, we have

$$d\Big(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)\Big) = 0 \tag{10}$$

and

$$d\Big(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)\Big) = 0. \tag{11}$$

Next we prove that $gx = F(x,\ y)$ and $gy = F(y,\ x)$.
For all $n \geq 0$, we have

$$\begin{aligned} d\Big(gx,\ F(gx_n,\ gy_n)\Big) &\leq d\Big(gx,\ gF(x_n, y_n)\Big) + d\Big(gF(x_n, y_n),\ F(gx_n, gy_n)\Big) \\ &\leq d\Big(gx,\ g(gx_{n+1})\Big) + d\Big(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)\Big). \end{aligned} \tag{12}$$

Taking $n \to \infty$ in the above inequality, using (9), (10) and the continuities of F and g, we have $d(gx,\ F(x,\ y)) = 0$, that is, $gx = F(x,\ y)$. Similarly, we have $d(gy,\ F(y,\ x)) = 0$, that is, $gy = F(y,\ x)$. Hence $(x,\ y)$ is a coupled coincidence point of F and g. □

Now, we shall prove the existence and uniqueness of a coupled common fixed point. Note that if $(X,\ \preceq)$ is a partially ordered set, the product space $X \times X$ has the following partial order relation:

for $(x,\ y),\ (u,\ v) \in X \times X$, $(u,\ v) \succeq (x,\ y)$ which implies that $x \preceq u,\ y \succeq v$.

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, suppose that for every $(x,\ y),\ (x^*,\ y^*) \in X \times X$ there exists a $(u,\ v) \in X \times X$ such that $\Big(F(u,v), F(v,u)\Big)$ is comparable to $\Big(F(x,y), F(y,x)\Big)$ and $\Big(F(x^*,y^*), F(y^*,x^*)\Big)$ and also the pair functions $(g,\ F)$ is coincidentally commuting. Then F and g have a unique coupled common fixed point, that is, there exist a unique $(x,\ y) \in X \times X$ such that $x = gx = F(x,\ y)$ and $y = gy = F(y,\ x)$.*

Proof. From Theorem 3.1, the set of coupled coincidence points of F and g is non-empty. Suppose (x, y) and (x^*, y^*) are coupled coincidence points of F and g, that is, $gx = F(x,\ y)$, $gy = F(y,\ x)$ and $gx^* = F(x^*,\ y^*)$, $gy^* = F(y^*,\ x^*)$. Now, we show

$$gx = gx^* \text{ and } gy = gy^*. \tag{13}$$

By the assumption, there exists $(u,\ v) \in X \times X$ such that $\Big(F(u,\ v), F(v,\ u)\Big)$ is comparable with $\Big(F(x,\ y),\ F(y,\ x)\Big)$ and $\Big(F(x^*,\ y^*), F(y^*,\ x^*)\Big)$. Put $u_0 = u$, $v_0 = v$. Since $F(X \times X) \subseteq g(X)$, we choose $u_1,\ v_1 \in X$ so that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0,\ u_0)$. Similarly as in the proof of Theorem 3.1, we can inductively define two sequences $\{gu_n\}$ and $\{gv_n\}$ where $gu_{n+1} = F(u_n,\ v_n)$ and $gv_{n+1} = F(v_n,\ u_n)$, for all $n \geq 0$. Hence $\Big(F(x,\ y),\ F(y,\ x)\Big) = (gx,\ gy)$ and $\Big(F(u,\ v),\ F(v,\ u)\Big) = (gu_1,\ gv_1)$ are comparable. Suppose that $(gx,\ gy) \succeq (gu_1,\ gv_1)$ (the proof is similar in other cases).

We claim that $(gx,\ gy) \succeq (gu_n,\ gv_n)$, for each $n \in N$.
In fact, we will use mathematical induction. Since $(gx,\ gy) \succeq (gu_1,\ gv_1)$, our claim is true for $n = 1$. We assume that $(gx,\ gy) \succeq (gu_n,\ gv_n)$ holds for some $n > 1$. Then $gx \succeq gu_n$ and $gy \preceq gv_n$. Using the mixed g-monotone property of F, we get

$$gu_{n+1} = F(u_n,\ v_n) \preceq F(x,\ v_n) \preceq F(x,\ y) = gx$$

and

$$gv_{n+1} = F(v_n,\ u_n) \succeq F(y,\ u_n) \succeq F(y,\ x) = gy$$

and these proves our claim.
Since $gx \succeq gu_n$ and $gy \preceq gv_n$, applying (1), we have

$$\begin{aligned}
d(gx,\ gu_{n+1}) &= d\Big(F(x,\ y), F(u_n,\ v_n)\Big) \\
&\leq \alpha \max \Big\{d(gx,\ gu_n),\ d(gy,\ gv_n), \\
&\qquad \frac{d(gx,\ F(x,\ y))\Big(1 + d(gu_n,\ F(u_n,\ v_n))\Big)}{1 + d(gx,\ gu_n)}, \\
&\qquad \frac{d(gx,\ F(u_n,\ v_n))\Big(1 + d(gu_n,\ F(x,\ y))\Big)}{1 + d(gx,\ gu_n)}, \\
&\qquad \frac{d(gy,\ F(y,\ x))\Big(1 + d(gv_n,\ F(u_n,\ v_n))\Big)}{1 + d(gy,\ gv_n)}, \\
&\qquad \frac{d(gy,\ F(v_n,\ u_n))\Big(1 + d(gv_n,\ F(y,\ x))\Big)}{1 + d(gy,\ gv_n)}\Big\} \\
&\quad + L \min \Big\{d(F(x,\ y),\ gu_n),\ d(F(u_n,\ v_n),\ gx), \\
&\qquad d(F(x,\ y),\ gx),\ d(F(u_n,\ v_n),\ gu_n)\Big\} \\
&\leq \alpha \max \Big\{d(gx,\ gu_n),\ d(gy,\ gv_n), \\
&\qquad \frac{d(gx,\ gu_{n+1})\Big(1 + d(gu_n,\ gx)\Big)}{1 + d(gx,\ gu_n)}, \\
&\qquad \frac{d(gy,\ gv_{n+1})\Big(1 + d(gv_n,\ gy)\Big)}{1 + d(gy,\ gv_n)}\Big\} \\
&\leq \alpha \max \Big\{d(gx,\ gu_n),\ d(gy,\ gv_n),\ d(gx,\ gu_{n+1}),\ d(gy,\ gv_{n+1})\Big\}.
\end{aligned}$$

Similarly, we can prove that

$$d(gy,\ gv_{n+1}) \leq \alpha \max \Big\{d(gx,\ gu_n),\ d(gy,\ gv_n),\ d(gx,\ gu_{n+1}),\ d(gy,\ gv_{n+1})\Big\}.$$

Hence

$$\max\Big\{d(gx,\ gu_{n+1}),\ d(gy,\ gv_{n+1})\Big\} \le \alpha\ \max\Big\{d(gx,\ gu_n),\ d(gy,\ gv_n)\Big\}.$$

By mathematical induction, we have

$$\max\Big\{d(gx,\ gu_{n+1}),\ d(gy,\ gv_{n+1})\Big\} \le \alpha^n\ \max\Big\{d(gx,\ gu_1),\ d(gy,\ gv_1)\Big\},$$

that is,

$$d(gx,\ gu_{n+1}) \le \alpha^n\ \max\Big\{d(gx,\ gu_1),\ d(gy,\ gv_1)\Big\}$$

and

$$d(gy,\ gv_{n+1}) \le \alpha^n\ \max\Big\{d(gx,\ gu_1),\ d(gy,\ gv_1)\Big\}.$$

Taking the limit as $n \longrightarrow \infty$ in the above inequalities, we get

$$\lim_{n\to\infty} d(gx,\ gu_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(gy,\ gv_{n+1}) = 0. \tag{14}$$

Similarly, we show that

$$\lim_{n\to\infty} d(gx^*,\ gu_{n+1}) = \lim_{n\to\infty} d(gy^*,\ gv_{n+1}) = 0. \tag{15}$$

By the triangle inequality, (14) and (15), we have

$$d(gx,\ gx^*) \le \Big[d(gx,\ gu_{n+1}) + d(gu_{n+1},\ gx^*)\Big] \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

and

$$d(gy,\ gy^*) \le \Big[d(gy,\ gv_{n+1}) + d(gv_{n+1},\ gy^*)\Big] \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence $gx = gx^*$ and $gy = gy^*$. Thus we proved (13).
Since the pair $(g,\ F)$ is coincidentally commuting and $gx = F(x,\ y)$ and $gy = F(y,\ x)$, we have

$$ggx = gF(x,\ y) = F(gx,\ gy) \text{ and } ggy = gF(y,\ x) = F(gy,\ gx).$$

Denote $gx = z$ and $gy = w$. Then, we have

$$gz = F(z,\ w) \text{ and } gw = F(w,\ z). \tag{16}$$

Thus $(z,\ w)$ is a coupled coincidence point of F and g. Then from (13) with $x^* = z$ and $y^* = w$ it follows $gx = gz$ and $gy = gw$, that is,

$$gz = z \text{ and } gw = w. \tag{17}$$

From (16) and (17), we have that $z = gz = F(z,\ w)$ and $w = gw = F(w,\ z)$, that is, $(z,\ w)$ is a coupled common fixed point of F and g.

To prove the uniqueness, assume that $(r,\ s)$ is another coupled common fixed point of F and g, that is

$$r = gr = F(r,\ s) \text{ and } s = gs = F(s,\ r).$$

Then by (13), we have $r = gr = gz = z$ and $s = gs = gw = w$. Hence the coupled common fixed point of F and g is unique. □

Example 3.3. Let $X = [0,\ 1]$. Then $(X,\ \leq)$ is a partially ordered set with the natural ordering of real numbers. Let $d(x,\ y) = |x - y|$, for $x,\ y \in X$. Let $g : X \to X$ and $F : X \times X \to X$ be defined respectively as follows:

$$gx = x^2, \text{ for all } x \in X \text{ and } F(x,\ y) = \begin{cases} \dfrac{x^2 - y^2}{4}, & \text{if } x \geq y, \\ 0, & \text{if } x \leq y. \end{cases}$$

Let $x_0 = 0$ and $y_0 = c(> 0)$ be two points in X. Then

$$g(x_0) = g(0) = 0 = F(0,\ c) = F(x_0,\ y_0)$$

and

$$g(y_0) = g(c) = c^2 \geq \frac{c^2}{3} = F(c,\ 0) = F(y_0,\ x_0).$$

Let $\alpha = 0.97 \in [0,\ 1)$ and $L = 10$.

It is verified that all the conditions of Theorem 3.1 are satisfied and $(0,\ 0) \in X \times X$ is a coupled coincidence point of F and g. Further, $(0,\ 0) \in X \times X$ is the unique coupled common fixed point of F and g.

Example 3.4. Let $X = \mathbb{R}$. Then $(X,\ \leq)$ is a partially ordered set with the natural ordering of real numbers. Let $d(x,\ y) = |x - y|$, for $x,\ y \in X$. Let $g : X \to X$ and $F : X \times X \to X$ be defined respectively as follows:

$$gx = \frac{5}{6}x, \text{ for all } x \in X \text{ and } F(x,\ y) = \frac{x - 2y}{4}, \quad \text{for all } x,\ y \in X.$$

Let $x_0 = -3$ and $y_0 = 3$. Then $gx_0 \preceq F(x_0,\ y_0)$ and $gy_0 \succeq F(y_0,\ x_0)$. Let $\alpha = 0.7 \in [0,\ 1)$ and $L = 10$. It is verified that all the conditions of Theorem 3.1 are satisfied and $(0,\ 0) \in X \times X$ is a coupled coincidence point of F and g. Further, $(0,\ 0) \in X \times X$ is the unique coupled common fixed point of F and g.

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