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CURVATURE HOMOGENEITY AND BALL-HOMOGENEITY ON ALMOST COKÄHLER 3-MANIFOLDS

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Abstract. Let M be a curvature homogeneous or ball-homogeneous non-coKähler almost coKähler 3-manifold. In this paper, we prove that M is locally isometric to a unimodular Lie group if and only if the Reeb vector field ξ is an eigenvector field of the Ricci operator. To extend this result, we prove that M is homogeneous if and only if it satisfies $\nabla_\xi h = 2f\phi h$, $f \in \mathbb{R}$.

1. Introduction

In 1958, Boothby and Wang [3] introduced the notion of homogeneous contact manifolds. A contact manifold (M, ω) is said to be *homogeneous* if there exits a connected Lie group G acting transitively as a group of diffeomorphisms on M and leaving the contact form ω invariant. Later, such notion was extended on contact metric manifolds (see [4, 6]) and almost coKähler manifolds (see [15]), even on almost contact metric manifolds (see [5]). An almost contact metric manifold (M, η) is said to be *locally homogeneous* if the pseudogroup of local isometries acts transitively on M and leaves the almost contact form η invariant. This notion comes from the fact that *a Riemannian manifold equipped with a transitive pseudogroup of isometries is said to be locally homogeneous.*

A locally homogeneous Riemannian manifold has the property that the volume of any sufficiently small geodesic sphere or ball depends only on its radius but has nothing to do with its center. A Riemannian manifold satisfying such property is said to be *ball-homogeneous* (see [7, 10]). A Riemannian manifold is said to be *curvature homogeneous* if all eigenvalues of the Ricci operator are constants. In history, many authors have studied the problem whether a ball-homogeneous Riemannian manifold of dimension > 2 is necessarily a locally homogenous one or not. In general, this problem is very difficult to investigate. However, when a Riemannian manifold is equipped with some special

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geometric structures or conditions, such as contact metric and Sasakian structures or conditions related with Ricci tensor, this problem was solved partially in [4, 7, 9]. After the notion of curvature homogeneous Riemannian manifolds introduced by Singer [17], a natural question was proposed: is there a curvature homogeneous space which is not locally homogeneous? Such problem was studied in [4] on contact metric 3-manifolds and in [21] on general Riemannian 3-manifolds.

In this paper, we aim to explore the relationships among curvature homogeneity, ball-homogeneity, local homogeneity and homogeneity on an almost coKähler 3-manifold. We prove that a curvature homogeneous or ball-homogeneous non-coKähler almost coKähler 3-manifold with ξ an eigenvector field of the Ricci operator is locally isometric to a unimodular Lie group. We prove that a curvature homogeneous or ball-homogeneous non-coKähler almost coKähler 3-manifold is homogeneous if and only if there holds $\nabla_\xi h = 2f\phi h$ for $f \in \mathbb{R}$. We remark that some local classifications of almost coKähler 3-manifolds satisfying $Q\xi = S(\xi,\xi)\xi$ or $\nabla_\xi h = 2f\phi h$ can be seen in [18–20].

2. Almost coKähler manifolds

An *almost contact metric structure* defined on a smooth differentiable manifold M^{2n+1} of dimension $2n+1$ means a (ϕ,ξ,η,g)*-structure* satisfying

$$\phi^2 = -\mathrm{id} + \eta\otimes\xi,\ \eta(\xi) = 1, \tag{2.1}$$

$$\phi^* g = g - \eta\otimes\eta, \tag{2.2}$$

where ϕ is a $(1,1)$-type tensor field, ξ is a vector field called the *Reeb vector field* and η is a 1-form called the *almost contact* 1*-form* and g is a Riemannian metric called *compatible metric* with respect to the almost contact structure.

On an almost contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$, the *fundamental* 2*-form* Φ is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any vector fields X,Y. If an almost contact metric manifold M^{2n+1} satisfies $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 0$, it is called an *almost coKähler manifold*. We consider the product $M^{2n+1}\times\mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$ and define on it an almost complex structure J by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$-function on $M^{2n+1}\times\mathbb{R}$. We denote by $[\phi,\phi]$ the Nijenhuis tensor of ϕ. If

$$[\phi,\phi] = -2\mathrm{d}\eta\otimes\xi$$

holds, or equivalently, J is integrable, then the almost contact metric structure is said to be *normal*. A normal almost coKähler manifold is called a *coKähler manifold*.

An (almost) coKähler manifold is nothing but an (almost) cosymplectic manifold defined by Blair in [1] and studied in [2, 8, 13–16]. As pointed out in [11], a coKähler manifold is a really odd dimensional analog of a Kähler manifold since a coKähler manifold is a Kähler mapping torus.

The $(1,1)$-type tensor field $h := \frac{1}{2}\mathcal{L}_\xi \phi$ is important for the geometry of an almost contact metric manifold, where $\mathcal{L}$ is the Lie differentiation. The Jacobi operator generated by ξ is denoted by $l := R(\cdot, \xi)\xi$, where R is the curvature tensor. From [13, 14], we have the following equations:

$$h\xi = l\xi = 0, \mathrm{tr}h = \mathrm{tr}(h') = 0, \tag{2.3}$$

$$\nabla \xi = h', \tag{2.4}$$

where $h' := h \circ \phi$. Generally, an almost coKähler manifold is coKähler if and only if

$$\nabla \phi = 0 \ (\Leftrightarrow \nabla \Phi = 0). \tag{2.5}$$

In particular, an almost coKähler 3-manifold is coKähler if and only if h vanishes (see [13]). On a coKähler 3-manifold, from (2.4) we have $Q\xi = 0$, where Q denotes the Ricci operator associated with the Ricci tensor S.

All manifolds in this paper are assumed to be smooth and connected.

3. Homogeneity on almost coKähler 3-manifolds

By the definition of curvature homogeneity, we see that a curvature homogeneous Riemannian manifold has constant scalar curvature. Moreover, following [7] we know that the ball-homogeneity implies the constancy of an infinite number of scalar curvature invariants. In particular, on a ball-homogeneous space the scalar curvature r and the squared norm of the Ricci operator $\|Q\|^2$ are both constants. It was shown in [18] that a coKähler 3-manifold with constant scalar curvature is locally isometric to the product $\mathbb{R} \times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature c. Therefore, next we study only curvature and ball-homogeneity on strictly almost coKähler 3-manifolds.

From now on, let M be a non-coKähler almost coKähler 3-manifold. Since $h \neq 0$, there exists a local orthonormal ϕ-basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for any point of M. More precisely, we set $he = \lambda e$ and hence $h\phi e = -\lambda \phi e$, where λ is assumed to be a positive and continuous function.

Lemma 3.1 ([16, Lemma 2.1])**.** *The Levi-Civita connection of M is given by*

$$\nabla_\xi \xi = 0,\ \nabla_\xi e = f\phi e,\ \nabla_\xi \phi e = -fe,\ \nabla_e \xi = -\lambda \phi e,\ \nabla_{\phi e}\xi = -\lambda e,$$
$$\nabla_e e = \frac{1}{2\lambda}\Big(\phi e(\lambda) + \sigma(e)\Big)\phi e,\ \nabla_{\phi e}\phi e = \frac{1}{2\lambda}\Big(e(\lambda) + \sigma(\phi e)\Big)e,$$
$$\nabla_{\phi e} e = \lambda\xi - \frac{1}{2\lambda}\Big(e(\lambda) + \sigma(\phi e)\Big)\phi e,\ \nabla_e \phi e = \lambda \xi - \frac{1}{2\lambda}\Big(\phi e(\lambda) + \sigma(e)\Big)e,$$

where f is a smooth function and σ is the 1-form defined by $\sigma(\cdot) = S(\cdot, \xi)$.

Applying Lemma 3.1, the Ricci operator Q can be written as the following:

$$
\begin{aligned}
Q\xi &= -2\lambda^2\xi + \sigma(e)e + \sigma(\phi e)\phi e,\\
Qe &= \sigma(e)\xi + \frac{1}{2}\Big(r + 2\lambda^2 - 4\lambda f\Big)e + \xi(\lambda)\phi e,\\
Q\phi e &= \sigma(\phi e)\xi + \xi(\lambda)e + \frac{1}{2}\Big(r + 2\lambda^2 + 4\lambda f\Big)\phi e,
\end{aligned}
\tag{3.1}
$$

with respect to the local basis $\{\xi, e, \phi e\}$, where r denotes the scalar curvature.

Theorem 3.1. *A non-coKähler almost coKähler 3-manifold with ξ an eigenvector field of the Ricci operator is curvature homogeneous or ball-homogeneous if and only if it is locally isometric to one of the following unimodular Lie group*:

- *the group $E(1,1)$ of rigid motions of the Minkowski 2-space*;
- *the universal covering $\tilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space*;
- *the Heisenberg group H^3.*

Proof. As shown at the beginning of this section, a curvature homogeneous or ball-homogeneous Riemannian manifold has the following property:

$$
r \text{ and } \|Q\|^2 \text{ are constants.} \tag{3.2}
$$

Let M be a non-cokähler almost coKähler 3-manifold such that $Q\xi = S(\xi,\xi)\xi$. From (3.1), the Ricci operator is given by

$$
\begin{aligned}
Q\xi &= -2\lambda^2\xi,\\
Qe &= \frac{1}{2}\Big(r + 2\lambda^2 - 4\lambda f\Big)e + \xi(\lambda)\phi e,\\
Q\phi e &= \xi(\lambda)e + \frac{1}{2}\Big(r + 2\lambda^2 + 4\lambda f\Big)\phi e.
\end{aligned}
\tag{3.3}
$$

Applying Lemma 3.1, in view of $r =$ constant, from (3.3) we get

$$
(\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi,
$$

$$
(\nabla_e Q)e = \lambda\xi(\lambda)\xi + (e(\xi(\lambda)) - 2f\phi e(\lambda))\phi e + \left(e\left(\lambda^2 - 2\lambda f\right) - \frac{\xi(\lambda)}{\lambda}\phi e(\lambda)\right)e,
$$

$$
(\nabla_{\phi e} Q)\phi e = \lambda\xi(\lambda)\xi + (\phi e(\xi(\lambda)) + 2fe(\lambda))e + \left(\phi e\left(\lambda^2 + 2\lambda f\right) - \frac{\xi(\lambda)}{\lambda}e(\lambda)\right)\phi e.
$$

In view of $\lambda > 0$, putting the previous three equations into the well known formula $\mathrm{div}Q = \frac{1}{2}\mathrm{grad}r$, we obtain

$$
\xi(\lambda) = 0,\ e(\lambda) - e(f) = 0,\ \phi e(\lambda) + \phi e(f) = 0. \tag{3.4}
$$

On the other hand, from (3.3) we have $\|Q\|^2 = \frac{1}{2}r^2 + 2\lambda^2 r + 6\lambda^4 + 8\lambda^2 f^2$. Since r is a constant and $\xi(\lambda) = 0$, the action of ξ on $\|Q\|^2$ gives

$$
f\xi(f) = 0. \tag{3.5}
$$

Case 1: $f = 0$. In this case, from (3.4) we see that λ is a positive constant. From Lemma 3.1, we have

$$[\xi, e] = \lambda\phi e, \ [e, \phi e] = 0, \ [\phi e, \xi] = -\lambda e. \tag{3.6}$$

Following Milnor [12] we state that M is locally isometric to the unimodular Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space. We refer the reader to [5, 15, 16] for constructions of left invariant almost coKähler structures on three-dimensional metric Lie groups.

Case 2: $f \neq 0$ holds on some open subset. In this case, from (3.5) we have $\xi(f) = 0$. In view of $r =$ constant and the second term of (3.4), the action of e on $\|Q\|^2$ gives

$$2\lambda e(\lambda)(r + 6\lambda^2 + 4f^2 + 4\lambda f) = 0. \tag{3.7}$$

We now suppose that $e(\lambda) \neq 0$ holds on some open subset. Then, it follows directly that $r + 6\lambda^2 + 4f^2 + 4\lambda f = 0$. In view of $r =$ constant, the action of e on this relation implies $3\lambda e(\lambda) + 2fe(f) + \lambda e(f) + fe(\lambda) = 0$. Putting the second term of (3.4) into this relation gives $4\lambda + 3f = 0$. It follows that $4\phi e(\lambda) + 3\phi e(f) = 0$. Consequently, taking into account the third term of (3.4) we have $\phi e(\lambda) = 0$. From Lemma 3.1, the Lie bracket is given as the following:

$$\begin{aligned} [\xi, e] &= (f + \lambda)\phi e, \\ [\phi e, \xi] &= (f - \lambda)e; \\ [e, \phi e] &= -\frac{1}{2\lambda}\phi e(\lambda)e + \frac{1}{2\lambda}e(\lambda)\phi e. \end{aligned} \tag{3.8}$$

The second term of (3.8) implies $\phi e(\xi(\lambda)) - \xi(\phi e(\lambda)) = (f - \lambda)e(\lambda)$. Taking into account $\phi e(\lambda) = 0$, the first term of (3.4), $4\lambda + 3f = 0$ and the assumption $\lambda > 0$, we obtain $e(\lambda) = 0$, a contradiction. Therefore, it follows from (3.7) that $e(\lambda) = 0$.

Similarly, according to the first term of (3.8) we obtain $\xi(e(\lambda)) - e(\xi(\lambda)) = (f + \lambda)\phi e(\lambda)$. In this context, using $e(\lambda) = 0$ and $\xi(\lambda) = 0$ we have

$$(f + \lambda)\phi e(\lambda) = 0. \tag{3.9}$$

Assume that $\phi e(\lambda) \neq 0$ holds on some open subset. Then it follows that $f = -\lambda$. Making use of this, we obtain $\|Q\|^2 = \frac{1}{2}r^2 + 2\lambda^2 r + 14\lambda^4$. As the scalar curvature r and $\|Q\|^2$ both are constants, it follows that λ is also a constant. This contradicts the assumption $\phi e(\lambda) \neq 0$. Thus, we obtain from (3.9) that $\phi e(\lambda) = 0$ and hence λ is a constant. Now, (3.8) becomes

$$[\xi, e] = (f + \lambda)\phi e, \ [e, \phi e] = 0, \ [\phi e, \xi] = (f - \lambda)e. \tag{3.10}$$

Following Milnor [12] (see also D. Perrone [15,16]), we state that M is locally isometric to a unimodular Lie group G. More precisely, G is the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $-\lambda < f < \lambda$. G is the universal covering $\tilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if either

$f > \lambda$ or $f < -\lambda$. G is the Heisenberg group H^3 if either $f = \lambda$ or $f = -\lambda$. This completes the proof. □

We remark that Theorem 3.1 is still true even when the curvature homogeneity or ball-homogeneity are replaced by the weaker condition (3.2).

Theorem 3.2. *Let M be a non-coKähler almost coKähler 3-manifold. Then, the following statements are equivalently.*

- *M is homogeneous.*
- *M is curvature homogeneous or ball-homogeneous and satisfies $\nabla_\xi h = 2f\phi h$, $f \in \mathbb{R}$.*
- *M is locally isometric to one of the three unimodular Lie groups $E(1,1)$, $\tilde{E}(2)$, H^3 or a non-unimodular Lie group G_1 whose Lie algebra is given by $[\xi, e] = \gamma\phi e$, $[e, \phi e] = \alpha\phi e$, $[\phi e, \xi] = 0$, where both α and γ are nonzero constants.*

Proof. It was proved in [15] that a homogeneous non-coKähler almost coKähler manifold is locally isometric to $E(1,1)$, $\tilde{E}(2)$, H^3 or G_1. On the other hand, the homogeneity implies curvature homogeneity and curvature homogeneity. Also, one can check that $\nabla_\xi h = 2f\phi h$, $f \in \mathbb{R}$, holds for almost coKähler structures on the above four Lie groups. Therefore, next we need only to prove that the second statement implies the third one.

As M is assumed to be curvature homogeneous or ball-homogeneous, then (3.2) is true. For a general non-coKähler almost coKähler 3-manifold, following Lemma 3.1 we have

$$\nabla_\xi h = \frac{1}{\lambda}\xi(\lambda)h + 2f\phi h.$$

In view of our assumption $\nabla_\xi h = 2f\phi h$, we obtain $\xi(\lambda) = 0$, $\lambda > 0$ and f is a constant. Moreover, it follows from (3.1) that

$$\|Q\|^2 = 4\lambda^4 + 2(\sigma(e))^2 + 2(\sigma(\phi e))^2 + \frac{1}{2}(r + 2\lambda^2)^2 + 8\lambda^2 f^2. \tag{3.11}$$

According to Lemma 3.1, the Lie bracket on M is given as the following:

$$\begin{aligned} [\xi, e] &= (f + \lambda)\phi e, \\ [\phi e, \xi] &= (f - \lambda)e, \\ [e, \phi e] &= -\frac{1}{2\lambda}(\phi e(\lambda) + \sigma(e))e + \frac{1}{2\lambda}(e(\lambda) + \sigma(\phi e))\phi e. \end{aligned} \tag{3.12}$$

In view of $r =$ constant, from Lemma 3.1 and (3.1) we obtain

$$(\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi + \{\xi(\sigma(e)) - f\sigma(\phi e)\}e + \{f\sigma(e) + \xi(\sigma(\phi e))\}\phi e,$$

$$\begin{aligned} (\nabla_e Q)e = {} & \{e(\sigma(e)) - \frac{1}{2\lambda}\sigma(\phi e)(\phi e(\lambda) + \sigma(e))\}\xi \\ & + e(\lambda^2 - 2\lambda f)e - \{\lambda\sigma(e) + 2f\phi e(\lambda) + 2f\sigma(e)\}\phi e, \end{aligned}$$

$$
\begin{aligned}
(\nabla_{\phi e}Q)\phi e = & \{\phi e(\sigma(\phi e)) - \frac{1}{2\lambda}\sigma(e)(e(\lambda)+\sigma(\phi e))\}\xi \\
& - \{\lambda\sigma(\phi e) - 2fe(\lambda) - 2f\sigma(\phi e)\}e + \phi e(\lambda^2+2\lambda f)\phi e.
\end{aligned}
$$

Putting the previous three equations into the well known formula $\mathrm{div}Q = \frac{1}{2}\mathrm{grad}r$ and using $f = $ constant, we obtain

$$
\begin{aligned}
e(\sigma(e)) + \phi e(\sigma(\phi e)) - \frac{1}{2\lambda}(\sigma(\phi e)\phi e(\lambda) + 2\sigma(e)\sigma(\phi e) + \sigma(e)e(\lambda)) &= 0, \\
\xi(\sigma(e)) + (f-\lambda)\sigma(\phi e) + 2\lambda e(\lambda) &= 0, \\
\xi(\sigma(\phi e)) - (f+\lambda)\sigma(e) + 2\lambda\phi e(\lambda) &= 0.
\end{aligned} \tag{3.13}
$$

Since the scalar curvature r, f and $\|Q\|^2$ are all constants, in view of $\xi(\lambda) = 0$, the action of ξ on $\|Q\|^2$ gives

$$\sigma(e)\xi(\sigma(e)) + \sigma(\phi e)\xi(\sigma(\phi e)) = 0. \tag{3.14}$$

Adding the second term of (3.13) multiplied by $\sigma(e)$ to the third term of (3.13) multiplied by $\sigma(\phi e)$ gives an equation. Comparing the resulting equation with (3.14) we obtain

$$e(\lambda)\sigma(e) - \sigma(e)\sigma(\phi e) + \phi e(\lambda)\sigma(\phi e) = 0, \tag{3.15}$$

where we have used $\lambda > 0$. Using $\xi(\lambda) = 0$, the first two terms of (3.12) gives

$$\xi(e(\lambda)) = (f+\lambda)\phi e(\lambda),\ \xi(\phi e(\lambda)) = (\lambda - f)e(\lambda). \tag{3.16}$$

Taking into account (3.16) and the last two terms of (3.13), the action of ξ on (3.15) gives

$$
\begin{aligned}
& 2(f+2\lambda)\sigma(e)\phi e(\lambda) + 2(2\lambda - f)\sigma(\phi e)e(\lambda) - 2\lambda(e(\lambda))^2 \\
& + (f-\lambda)(\sigma(\phi e))^2 - (f+\lambda)(\sigma(e))^2 - 2\lambda(\phi e(\lambda))^2 = 0.
\end{aligned} \tag{3.17}
$$

Similarly, in view of (3.16) and the last two terms of (3.13), the action of ξ on (3.17) gives

$$
\begin{aligned}
& (3\lambda^2 - \lambda f - f^2)\sigma(\phi e)\phi e(\lambda) + (3\lambda^2 + \lambda f - f^2)\sigma(e)e(\lambda) \\
& - 6\lambda^2 e(\lambda)\phi e(\lambda) + (f^2 - \lambda^2)\sigma(e)\sigma(\phi e) = 0.
\end{aligned} \tag{3.18}
$$

In view of $\lambda > 0$, adding (3.15) multiplied by $f^2 - \lambda^2$ to (3.18) implies

$$(2\lambda + f)\sigma(e)e(\lambda) + (2\lambda - f)\sigma(\phi e)\phi e(\lambda) - 6\lambda e(\lambda)\phi e(\lambda) = 0. \tag{3.19}$$

In view of (3.16) and the last two terms of (3.13), the action of ξ on (3.19) gives

$$
\begin{aligned}
& 2(\lambda - f)\sigma(\phi e)e(\lambda) + 2(\lambda + f)\sigma(e)\phi e(\lambda) \\
& - (5\lambda - 2f)(e(\lambda))^2 - (5\lambda + 2f)(\phi e(\lambda))^2 = 0.
\end{aligned} \tag{3.20}
$$

In view of (3.16) and the last two terms of (3.13), the action of ξ on (3.20) gives

$$(\lambda^2 - f^2)\sigma(e)e(\lambda) + (\lambda^2 - f^2)\sigma(\phi e)\phi e(\lambda) + (2f^2 - 7\lambda^2)e(\lambda)\phi e(\lambda) = 0. \tag{3.21}$$

For simplicity, we continuous our discussions by the following several cases:

Case 1: $f = 0$. In this case, taking into account $\lambda > 0$ and adding (3.21) to (3.15) multiplied by $-\lambda^2$ we obtain $e(\lambda)\phi e(\lambda) = 0$. We now assume that $e(\lambda) \neq 0$ holds on some open subset and hence we obtain $\phi e(\lambda) = 0$. However, in view of $\lambda > 0$ and $f =$ constant, from the second term of (3.16) we have $e(\lambda) = 0$, a contradiction. Similarly, if we assume $\phi e(\lambda) \neq 0$ holds on some open subset, then from the first term of (3.16) we arrive at a contradiction. Consequently, it follows that $e(\lambda) = \phi e(\lambda) = 0$ and hence by $\xi(\lambda) = 0$ we see that λ is a constant. Therefore, from (3.17) we have $\sigma(e) = \sigma(\phi e) = 0$. In this context, (3.12) becomes

$$[\xi, e] = \lambda\phi e,\ [e, \phi e] = 0,\ [\phi e, \xi] = -\lambda e,$$

where λ is a positive constant. We state that M is locally isometric to a unimodular Lie group $E(1, 1)$ of rigid motions of the Minkowski 2-space.

Case 2: $f \neq 0$. Subtracting (3.19) multiplied by $\lambda^2 - f^2$ from (3.21) multiplied by $2\lambda - f$ we have

$$(2f(\lambda^2 - f^2)\sigma(e) + (8\lambda^3 - 7\lambda^2 f + 2\lambda f^2 + 2f^3)\phi e(\lambda))e(\lambda) = 0. \tag{3.22}$$

Next, we consider two subcases corresponding to (3.22).

Case 2.1: $e(\lambda) \neq 0$ holds on some open subset. In this case, from (3.22) we have

$$2f(\lambda^2 - f^2)\sigma(e) + (8\lambda^3 - 7\lambda^2 f + 2\lambda f^2 + 2f^3)\phi e(\lambda) = 0. \tag{3.23}$$

In view of the second terms of (3.16) and (3.13), the action of ξ on (3.23) gives

$$2f(\lambda^2 - f^2)(\lambda - f)\sigma(\phi e) + (\lambda - f)(8\lambda^3 - 11\lambda^2 f - 2\lambda f^2 + 2f^3)e(\lambda) = 0. \tag{3.24}$$

Using the third term of (3.16) and the first term of (3.13), the action of ξ on (3.24) gives

$$2f(\lambda^2 - f^2)^2\sigma(e) + (\lambda^2 - f^2)(8\lambda^3 - 15\lambda^2 f + 2\lambda f^2 + 2f^3)\phi e(\lambda) = 0. \tag{3.25}$$

Subtracting (3.25) from (3.23) multiplied by $\lambda^2 - f^2$ gives

$$\lambda^2 f(\lambda^2 - f^2)\phi e(\lambda) = 0. \tag{3.26}$$

Case 2.1.1: $\lambda^2 - f^2 \neq 0$ holds on some open subset. In view of $f \neq 0$ and $\lambda > 0$, from (3.26) we have $\phi e(\lambda) = 0$. Using this in (3.25) we have $\sigma(e) = 0$. Now, (3.20) becomes $2(\lambda - f)\sigma(\phi e) - (5\lambda - 2f)e(\lambda) = 0$. On the other hand, the second term of (3.13) becomes $(f - \lambda)\sigma(\phi e) + 2\lambda e(\lambda) = 0$. Comparing this with previous relation gives $\lambda = 2f$ and hence $\sigma(\phi e) = 4e(\lambda)$, where we have used still $f \neq 0$ and the assumption $e(\lambda) \neq 0$. In this context, (3.17) becomes $\lambda(e(\lambda))^2 = 0$. This contradicts the assumption $e(\lambda) \neq 0$.

Case 2.1.2: $\phi e(\lambda) \neq 0$. In view of $f \neq 0$, from (3.26) we have $\lambda^2 - f^2 = 0$. This implies that λ is a positive constant and then we have $\phi e(\lambda) = 0$, a contradiction.

According to the above two subcases we conclude that (3.26) contradicts the assumption $e(\lambda) \neq 0$. In other words, it follows from (3.22) that $e(\lambda) = 0$. Next we focus on the study of this subcase.

Case 2.2: $e(\lambda) = 0$. Putting this into (3.21) gives

$$(\lambda^2 - f^2)\sigma(\phi e)\phi e(\lambda) = 0. \tag{3.27}$$

Similarly, next we discuss two subcases corresponding to (3.27).

Case 2.2.1: $\sigma(\phi e) \neq 0$ holds on some open subset. In this case, it follows from (3.27) that $(\lambda^2 - f^2)\phi e(\lambda) = 0$. If $\phi e(\lambda) \neq 0$ holds on some open subset, it follows that λ is a constant, a contradiction. Thus, we have $\phi e(\lambda) = 0$ and in view of $\xi(\lambda) = 0$ we observe that λ is a positive constant. Applying this in (3.15) we have $\sigma(e)\sigma(\phi e) = 0$ and hence $\sigma(e) = 0$. Applying this in (3.17) we obtain $\lambda = f$. Finally, from (3.11) we say that $\sigma(\phi e)$ is a nonzero constant. In this context, (3.12) becomes

$$[\xi, e] = 2f\phi e,\ [e, \phi e] = \frac{1}{2f}\sigma(\phi e)\phi e,\ [\phi e, \xi] = 0, \tag{3.28}$$

where f is a positive constant. One can check that a Lie group whose Lie algebra is given by (3.28) is non-unimodular because of

$$\mathrm{tr}(\mathrm{ad}_\xi) = 0,\ \mathrm{tr}(\mathrm{ad}_e) = \frac{1}{2f}\sigma(\phi e) \neq 0,\ \mathrm{tr}(\mathrm{ad}_{\phi e}) = 0.$$

Here we remark that such Lie group corresponds to the (NC_2) case shown in [15, Theorem 4.1].

Case 2.2.2: $\sigma(\phi e) = 0$. Putting $e(\lambda) = 0$ into the first term of (3.16) we obtain $(f + \lambda)\phi e(\lambda) = 0$. In this relation, $\phi e(\lambda) \neq 0$ reduces to $\lambda =$ constant, a contradiction. Then, it follows directly that $\phi e(\lambda) = 0$ and hence λ is a positive constant because of $\xi(\lambda) = 0$. Applying this (3.17) we have

$$(f + \lambda)(\sigma(e))^2 = 0. \tag{3.29}$$

According to (3.11), we see that $\sigma(e)$ is a constant. Next, we need to consider the last two subcases as follows.

Case 2.2.2.1: $\sigma(e) \neq 0$. In this case, it follows from (3.29) that $\lambda = -f$, a positive constant. Therefore, (3.12) becomes

$$[\xi, e] = 0,\ [e, \phi e] = \frac{1}{2f}\sigma(e)e,\ [\phi e, \xi] = 2fe, \tag{3.30}$$

where f is a negative constant. By (3.30) and a simple calculation we obtain

$$\mathrm{tr}(\mathrm{ad}_\xi) = 0,\ \mathrm{tr}(\mathrm{ad}_e) = 0,\ \mathrm{tr}(\mathrm{ad}_{\phi e}) = -\frac{1}{2f}\sigma(e).$$

According to the above relation, we state that M is locally isometric to a non-unimodular Lie group. We remark that such Lie group corresponds still to the (NC_2) case shown in [15, Theorem 4.1].

Case 2.2.2.2: $\sigma(e) = 0$. In this case, (3.12) becomes

$$[\xi, e] = (f + \lambda)\phi e,\ [e, \phi e] = 0,\ [\phi e, \xi] = (f - \lambda)e. \tag{3.31}$$

As discussed in proof of Theorem 3.1, now M is locally isometric to the unimodular Lie group G. Moreover, G is the group $E(1, 1)$ of rigid motions of the Minkowski 2-space if $-\lambda < f < \lambda$. G is the universal covering $\tilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if either $f > \lambda$ or $f < -\lambda$. G is the Heisenberg group H^3 if either $f = \lambda$ or $f = -\lambda$. This completes the proof. □

The Reeb vector field of the almost coKähler structure defined on Lie group G_1 is not an eigenvector field of the Ricci operator.

Corollary 3.1. *A non-coKähler almost coKähler* 3*-manifold satisfying* (3.2) *and* $\nabla_\xi h = 2f\phi h$, $f \in \mathbb{R}$, *is locally homogeneous.*

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