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TRUNCATED HANKEL OPERATORS AND THEIR MATRICES

Bartosz Łanucha and Małgorzata Michalska

Abstract. Truncated Hankel operators are compressions of classical Hankel operators to model spaces. In this paper we describe matrix representations of truncated Hankel operators on finite-dimensional model spaces. We then show that the obtained descriptions hold also for some infinite-dimensional cases.

1. Introduction

Let H^2 denote the Hardy space in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. The space H^2 consists of functions analytic in $\mathbb{D}$ with square summable Maclaurin coefficients. It can also be identified with a closed subspace of $L^2 := L^2(\partial\mathbb{D})$, namely, the closed linear span of analytic polynomials. Denote by P the orthogonal projection from L^2 onto H^2.

The Toeplitz operator T_φ with symbol $\varphi \in L^\infty(\partial\mathbb{D})$ is defined on H^2 by

$$T_\varphi f = P(\varphi f), \qquad f \in H^2.$$

The Hankel operator H_φ with symbol $\varphi \in L^\infty(\partial\mathbb{D})$ can be defined on H^2 by

$$H_\varphi f = J(I - P)(\varphi f), \qquad f \in H^2,$$

where $J : L^2 \to L^2$ is the "flip" operator given by

$$Jf(z) = \overline{z} f(\overline{z}), \quad |z| = 1. \tag{1.1}$$

Note that if $\varphi \in L^2$, then both T_φ and H_φ are densely defined on $H^\infty \subset H^2$, the algebra of bounded analytic functions in $\mathbb{D}$.

Toeplitz and Hankel operators have been long and intensely studied (see, e.g. [2, 15, 17]). Recently, research into properties of the compressions of Toeplitz and Hankel operators to model spaces has gained more and more interest and has resulted in deep and relevant discoveries [1, 3, 4, 7, 9–11, 14, 18].

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Let $\alpha \in H^\infty$ be such that $|\alpha| = 1$ a.e. on $\partial\mathbb{D}$. Then α is called an inner function. The corresponding model space K_α is the orthogonal complement of αH^2 in H^2,

$$K_\alpha = H^2 \ominus \alpha H^2.$$

The model space K_α is invariant under the backward shift $S^* = T_{\overline{z}}$. It is a reproducing kernel Hilbert space with the kernel function

$$k_w^\alpha(z) = \frac{1 - \overline{\alpha(w)}\alpha(z)}{1 - \overline{w}z}, \quad w, z \in \mathbb{D}.$$

Note that since k_w^α is bounded, the set $K_\alpha^\infty = K_\alpha \cap H^\infty$ is dense in K_α.

The truncated Toeplitz operator (TTO) A_φ, $\varphi \in L^2$, is densely defined on K_α by

$$A_\varphi f = P_\alpha(\varphi f),$$

where P_α is the orthogonal projection from L^2 onto K_α. The operator A_φ can be seen as the compression of T_φ to the model space K_α, that is, $A_\varphi = P_\alpha T_{\varphi|K_\alpha}$. The study of these operators began with D. Sarason's paper [18].

Truncated Hankel operators where introduced by C. Gu in [12]. The truncated Hankel operator (THO) B_φ, $\varphi \in L^2$, is the compression of H_φ to K_α, $B_\varphi = P_\alpha H_{\varphi|K_\alpha}$. More precisely, B_φ is defined on K_α by

$$B_\varphi = P_\alpha J(I - P)(\varphi f),$$

where J is given by (1.1).

For an inner function α let

$$\mathscr{T}(\alpha) = \{A_\varphi : \varphi \in L^2 \text{ and } A_\varphi \text{ is bounded}\}$$

and

$$\mathscr{H}(\alpha) = \{B_\varphi : \varphi \in L^2 \text{ and } B_\varphi \text{ is bounded}\}.$$

The purpose of this paper is to investigate matrix representations of operators from $\mathscr{H}(\alpha)$.

Recall that classical Toeplitz and Hankel operators can be characterized in terms of their matrix representations with respect to the monomial basis $\{z^k : k \geq 0\}$ of H^2. A bounded linear operator $T : H^2 \to H^2$ is a Toeplitz operator if and only if its matrix is a Toeplitz matrix, that is, it has constant diagonals. Similarly, $T : H^2 \to H^2$ is a Hankel operator if and only if its matrix is a Hankel matrix, that is, its entries are constant along each skew-diagonal.

Since for $\alpha(z) = z^n$, n positive integer, the model space K_α is the set of all polynomials of degree less than n, the above gives a matrix characterization of operators from $\mathscr{T}(z^n)$ and $\mathscr{H}(z^n)$.

Matrix representations of TTO's on finite-dimensional model spaces were considered in [4]. Some infinite-dimensional cases were considered in [14]. Authors in [4] and [14] provided characterizations of operators from $\mathscr{T}(\alpha)$ in terms of matrix representations with respect to some natural bases of K_α, namely, kernel bases, conjugate kernel bases, Clark bases (see Section 3 for definitions).

Here we give similar characterizations for operators from $\mathscr{H}(\alpha)$. In Section 2 we present some basic properties of operators from $\mathscr{H}(\alpha)$. In Section 3 we consider the finite-dimensional cases. We show for example that if α is a finite Blaschke product with distinct zeros $a_1, \ldots, a_n$, then the matrix of a THO with respect to the kernel basis $\{k^\alpha_{a_1}, \ldots, k^\alpha_{a_n}\}$ is determined by its entries from the first row and the first column. In Section 4 we investigate some infinite-dimensional cases.

2. Basic properties of THO's

We begin with some basic properties of truncated Hankel operators (see [12, 13]).

Let α be an inner function. Then $\alpha^{\#}(z) = \overline{\alpha(\overline{z})}$ is also an inner function. It is easy to verify that the map $J^{\#} : L^2 \to L^2$ defined by

$$J^{\#} f(z) = f^{\#}(z) = \overline{f(\overline{z})}, \quad |z| = 1,$$

is an anti-linear, isometric involution on L^2 (such a map is called a conjugation). The authors of [3] proved that $f \in K_\alpha$ if and only if $f^{\#} \in K_{\alpha^{\#}}$, that is, the conjugation $J^{\#}$ transforms K_α onto $K_{\alpha^{\#}}$.

Another conjugation on L^2, this one associated with the inner function α, is the operator C_α defined on L^2 by

$$C_\alpha f(z) = \widetilde{f}(z) = \alpha(z)\overline{z f(z)}, \quad |z| = 1. \tag{2.1}$$

The conjugation C_α transforms K_α onto K_α.

By the conjugate kernel function we mean the function $\widetilde{k}^\alpha_w = C_\alpha k^\alpha_w$, $w \in \mathbb{D}$. A simple computation gives

$$\widetilde{k}^\alpha_w(z) = \frac{\alpha(z) - \alpha(w)}{z - w}, \quad w, z \in \mathbb{D}.$$

Clearly, $B_\varphi = 0$ whenever $\varphi \in H^2$. Moreover, if $\varphi \in \overline{\alpha\alpha^{\#} H^2}$, then $B_\varphi = 0$. Indeed, let $\varphi = \overline{\alpha\alpha^{\#}\psi}$, $\psi \in H^2$. Then, for $f, g \in K^\infty_\alpha$,

$$\begin{aligned}
\langle B_\varphi f, g\rangle &= \left\langle B_{\overline{\alpha\alpha^{\#}\psi}} f, g\right\rangle = \left\langle \overline{\alpha\alpha^{\#}\psi} f, Jg\right\rangle = \left\langle J\left(\overline{\alpha\alpha^{\#}\psi} f\right), g\right\rangle \\
&= \left\langle \overline{z}\alpha^{\#}\alpha\psi^{\#}\overline{f^{\#}}, g\right\rangle = \left\langle \alpha\overline{zg}\cdot\alpha^{\#}\psi^{\#}, f^{\#}\right\rangle \\
&= \left\langle \alpha^{\#}\psi^{\#}\cdot C_\alpha g, f^{\#}\right\rangle = 0,
\end{aligned}$$

since $f^{\#} \in K_{\alpha^{\#}}$. Hence, $B_\varphi = 0$ for $\varphi \in H^2 + \overline{\alpha\alpha^{\#} H^2}$. More can be proved.

Proposition 2.1 ([12], Rem. 4.3)**.** *Let $\varphi \in L^2(\partial\mathbb{D})$. Then $B_\varphi = 0$ if and only if $\varphi \in H^2 + \overline{\alpha\alpha^{\#} H^2}$.*

Corollary 2.2. *For each $B \in \mathscr{H}(\alpha)$ there exists $\psi \in K_{\alpha\alpha^{\#}}$ such that $B = B_{\overline{\psi}}$.*

In [18] D. Sarason characterized the operators from $\mathscr{T}(\alpha)$ using the compressed shift $S_\alpha = A_z$. A similar characterization was given by C. Gu in [12] (see also [13]).

Theorem 2.3 ([12], Thm. 3.1)**.** *A bounded linear operator B on K_α is a truncated Hankel operator if and only if there exist functions $\psi, \chi \in K_\alpha$ such that*

$$B - S_\alpha^* B S_\alpha^* = \psi \otimes k_0^\alpha + \widetilde{k}_0^\alpha \otimes \chi.$$

In what follows we will need some examples of rank-one truncated Hankel operators. These are described in Proposition 2.4 (see [12] for proofs). We first recall the notion of the angular derivative in the sense of Carathéodory (ADC). We say that α has an ADC at $\lambda \in \partial\mathbb{D}$ if there exist finite nontangential limits $\alpha(\lambda)$ and $\alpha'(\lambda)$, that is, $\alpha(z)$ tends to $\alpha(\lambda)$ and $\alpha'(z)$ tends to $\alpha'(\lambda)$ whenever $z \in \mathbb{D}$ tends to λ nontangentially (with $\frac{|z-\lambda|}{1-|z|}$ bounded) and $|\alpha(\lambda)| = 1$.

It is known that if α has an ADC at $\lambda \in \partial\mathbb{D}$, then $k_\lambda^\alpha = \frac{1-\overline{\alpha(\lambda)}\alpha(z)}{1-\overline{\lambda}z}$ belongs to K_α and $k_z^\alpha \to k_\lambda^\alpha$ in norm whenever $z \to \lambda$ nontangentially.

Proposition 2.4 ([12], Thm. 7.4)**.**

(a) *For every w in $\mathbb{D}$ the operator $k_{\overline{w}}^\alpha \otimes k_w^\alpha$ belongs to $\mathscr{H}(\alpha)$ and*

$$k_{\overline{w}}^\alpha \otimes k_w^\alpha = B_{\overline{z k_w}} = B_{\overline{z k_w^\alpha \alpha^\#}}.$$

(b) *For every w in $\mathbb{D}$ the operator $\widetilde{k}_{\overline{w}}^\alpha \otimes \widetilde{k}_w^\alpha$ belongs to $\mathscr{H}(\alpha)$ and*

$$\widetilde{k}_{\overline{w}}^\alpha \otimes \widetilde{k}_w^\alpha = B_{\overline{\frac{\alpha\alpha^\#}{z-\overline{w}}}} = B_{\overline{\widetilde{k}_w^\alpha \alpha^\#}}.$$

(c) *If α has an ADC at both $\lambda, \overline{\lambda} \in \partial\mathbb{D}$, then the operators $k_{\overline{\lambda}}^\alpha \otimes k_\lambda^\alpha$ and $\widetilde{k}_{\overline{\lambda}}^\alpha \otimes \widetilde{k}_\lambda^\alpha$ belong to $\mathscr{H}(\alpha)$.*

Actually, C. Gu [12] proved that nonzero scalar multiples of the operators from Proposition 2.4 are the only rank-one THO's.

It is known that the model space K_α is finite-dimensional if and only if α is a finite Blaschke product. More precisely, $\dim K_\alpha = n$ if and only if α is a Blaschke product with n (not necessarily distinct) zeros.

D. Sarason proved that if $\dim K_\alpha = n$, then $\dim \mathscr{T}(\alpha) = 2n - 1$. He also provided a basis for $\mathscr{T}(\alpha)$ in that case (see [18, Thm. 7.1]). Proposition 2.5 gives an analogous result for $\mathscr{H}(\alpha)$.

Note that if α is a finite Blaschke product, then α is analytic on a domain containing the closed unit disk $\overline{\mathbb{D}}$ and has an ADC at every $\lambda \in \partial\mathbb{D}$. Hence, for each $\lambda \in \overline{\mathbb{D}}$, $k_\lambda^\alpha \in K_\alpha$ and the operators $k_{\overline{\lambda}}^\alpha \otimes k_\lambda^\alpha$, $\widetilde{k}_{\overline{\lambda}}^\alpha \otimes \widetilde{k}_\lambda^\alpha$ belong to $\mathscr{H}(\alpha)$.

Proposition 2.5 ([12], Thm. 7.9)**.** *Let α be a finite Blaschke product with $n > 0$ zeros.*

(a) *The dimension of $\mathscr{H}(\alpha)$ is $2n - 1$.*

(b) *If $\lambda_1, \ldots, \lambda_{2n-1}$ are distinct points from $\mathbb{D}$, then the operators $k_{\overline{\lambda_j}}^\alpha \otimes k_{\lambda_j}^\alpha$, $j = 1, \ldots, 2n-1$, form a basis for $\mathscr{H}(\alpha)$.*

(c) *If $\lambda_1, \ldots, \lambda_{2n-1}$ are distinct points from $\mathbb{D}$, then the operators $\widetilde{k}_{\overline{\lambda_j}}^\alpha \otimes \widetilde{k}_{\lambda_j}^\alpha$, $j = 1, \ldots, 2n-1$, form a basis for $\mathscr{H}(\alpha)$.*

3. Matrix representations of THO's on finite-dimensional model spaces

In this section we describe matrix representations of operators from $\mathscr{H}(\alpha)$.

3.1. Kernel basis and conjugate kernel basis

Let α be a finite Blaschke product with distinct zeros $a_1, \ldots, a_n$. Then the kernel functions $\{k^\alpha_{a_1}, \ldots, k^\alpha_{a_n}\}$ form a basis for K_α and so do the conjugate kernel functions $\{\widetilde{k}^\alpha_{a_1}, \ldots, \widetilde{k}^\alpha_{a_n}\}$.

Let B be any linear operator on K_α. Using

$$\langle k^\alpha_{a_j}, \widetilde{k}^\alpha_{a_s}\rangle = \begin{cases} \overline{\alpha'(a_s)} & \text{for } j = s, \\ 0 & \text{for } j \neq s, \end{cases}$$

one can verify that the matrix representation $M_B = (r_{s,p})$ of B with respect to the kernel basis $\{k^\alpha_{a_1}, \ldots, k^\alpha_{a_n}\}$ is given by

$$r_{s,p} = \left(\overline{\alpha'(a_s)}\right)^{-1} \langle Bk^\alpha_{a_p}, \widetilde{k}^\alpha_{a_s}\rangle, \tag{3.1}$$

and the matrix representation $\widetilde{M}_B = (t_{s,p})$ of B with respect to the conjugate kernel basis $\{\widetilde{k}^\alpha_{a_1}, \ldots, \widetilde{k}^\alpha_{a_n}\}$ is given by

$$t_{s,p} = \alpha'(a_s)^{-1}\langle B\widetilde{k}^\alpha_{a_p}, k^\alpha_{a_s}\rangle. \tag{3.2}$$

Theorem 3.1. *Let α be a finite Blaschke product with n distinct zeros $a_1, \ldots,$ a_n. Let B be any linear operator on K_α. If $M_B = (r_{s,p})$ is the matrix representation of B with respect to the basis $\{k^\alpha_{a_1}, \ldots, k^\alpha_{a_n}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if*

$$r_{s,p} = \frac{\overline{\alpha'(a_s)(1 - a_s a_1)} r_{s,1} - \overline{\alpha'(a_1)(1 - a_1^2)} r_{1,1} + \overline{\alpha'(a_1)(1 - a_1 a_p)} r_{1,p}}{\overline{\alpha'(a_s)(1 - a_s a_p)}} \tag{3.3}$$

for all $1 \leq p, s \leq n$.

Proof. Let α be a finite Blaschke product with n distinct zeros $a_1, \ldots, a_n$ and let $B \in \mathscr{H}(\alpha)$. Fix $2n-1$ distinct points $\lambda_1, \ldots, \lambda_{2n-1}$ from $\mathbb{D} \setminus \{\overline{a_1}, \ldots, \overline{a_n}\}$. By Proposition 2.5 the operators $k^\alpha_{\overline{\lambda_j}} \otimes k^\alpha_{\lambda_j}$, $j = 1, 2, \ldots, 2n-1$, form a basis for $\mathscr{H}(\alpha)$ and there exist complex numbers $c_1, \ldots, c_{2n-1}$ such that

$$B = \sum_{j=1}^{2n-1} c_j k^\alpha_{\overline{\lambda_j}} \otimes k^\alpha_{\lambda_j}.$$

Hence

$$Bk^\alpha_{a_p} = \sum_{j=1}^{2n-1} c_j \langle k^\alpha_{a_p}, k^\alpha_{\lambda_j}\rangle k^\alpha_{\overline{\lambda_j}} = \sum_{j=1}^{2n-1} c_j k^\alpha_{a_p}(\lambda_j) k^\alpha_{\overline{\lambda_j}},$$

and, by (3.1), we have

$$\begin{aligned} r_{s,p} &= \left(\overline{\alpha'(a_s)}\right)^{-1} \langle Bk^{\alpha}_{a_p}, \widetilde{k}^{\alpha}_{a_s}\rangle = \left(\overline{\alpha'(a_s)}\right)^{-1} \sum_{j=1}^{2n-1} c_j k^{\alpha}_{a_p}(\lambda_j) \left\langle k^{\alpha}_{\overline{\lambda_j}}, \widetilde{k}^{\alpha}_{a_s}\right\rangle \\ &= \left(\overline{\alpha'(a_s)}\right)^{-1} \sum_{j=1}^{2n-1} c_j k^{\alpha}_{a_p}(\lambda_j) \overline{\widetilde{k}^{\alpha}_{a_s}(\overline{\lambda_j})} \\ &= \left(\overline{\alpha'(a_s)}\right)^{-1} \sum_{j=1}^{2n-1} \frac{d_j}{(1-\lambda_j\overline{a_p})(\lambda_j - \overline{a_s})}, \end{aligned}$$

where $d_j = c_j\overline{\alpha(\overline{\lambda_j})}$ does not depend on s and p. Observe that ($\lambda_j \neq 1$)

$$\frac{1-\overline{a_s a_p}}{(1-\lambda_j\overline{a_p})(\lambda_j-\overline{a_s})} = \frac{1}{1-\lambda_j}\left(\frac{1-\overline{a_s}}{\lambda_j-\overline{a_s}} - \frac{1-\overline{a_p}}{1-\lambda_j\overline{a_p}}\right),$$

which gives

$$\begin{aligned} &r_{s,p} \\ &= \left(\overline{\alpha'(a_s)}\right)^{-1} (1-\overline{a_s a_p})^{-1} \sum_{j=1}^{2n-1} d_j \frac{1-\overline{a_s a_p}}{(1-\lambda_j\overline{a_p})(\lambda_j-\overline{a_s})} \\ &= \left(\overline{\alpha'(a_s)}\right)^{-1} (1-\overline{a_s a_p})^{-1} \sum_{j=1}^{2n-1} d_j \left\{ \frac{1-\overline{a_s a_1}}{(1-\lambda_j\overline{a_1})(\lambda_j-\overline{a_s})} - \frac{1-\overline{a_1}^2}{(1-\lambda_j\overline{a_1})(\lambda_j-\overline{a_1})} \right. \\ &\quad \left. + \frac{1-\overline{a_1 a_p}}{(1-\lambda_j\overline{a_p})(\lambda_j-\overline{a_1})} \right\} \\ &= \overline{\left(\frac{1-a_s a_1}{1-a_s a_p}\right)} r_{s,1} - \overline{\left(\frac{\alpha'(a_1)}{\alpha'(a_s)}\frac{1-a_1^2}{1-a_s a_p}\right)} r_{1,1} + \overline{\left(\frac{\alpha'(a_1)}{\alpha'(a_s)}\frac{1-a_1 a_p}{1-a_s a_p}\right)} r_{1,p}. \end{aligned}$$

This means that the matrix representation of every $B \in \mathscr{H}(\alpha)$ satisfies (3.3).

To prove the converse observe that V, the space of all $n \times n$ matrices satisfying (3.3), has dimension $2n-1$ and contains $V_0 = \{M_B : B \in \mathscr{H}(\alpha)\}$. Since, by Proposition 2.5(a), the dimension of V_0 is also equal to $2n-1$, we have $V = V_0$. □

Theorem 3.2. *Let α be a finite Blaschke product with n distinct zeros $a_1, \ldots,$ a_n. Let B be any linear operator on K_α. If $\widetilde{M}_B = (t_{s,p})$ is the matrix representation of B with respect to the basis $\{\widetilde{k}^{\alpha}_{a_1}, \ldots, \widetilde{k}^{\alpha}_{a_n}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if*

$$t_{s,p} = \frac{\alpha'(a_s)(1-a_s a_1)t_{s,1} - \alpha'(a_1)(1-a_1^2)t_{1,1} + \alpha'(a_1)(1-a_1 a_p)t_{1,p}}{\alpha'(a_s)(1-a_s a_p)} \tag{3.4}$$

for all $1 \le p, s \le n$.

Proof. Let α be a finite Blaschke product with n distinct zeros $a_1, \dots, a_n$ and let $\widetilde{M}_B = (t_{s,p})$ be the matrix representation of B with respect to the basis $\{\widetilde{k}^\alpha_{a_1}, \dots, \widetilde{k}^\alpha_{a_n}\}$. By (3.2) we get

$$t_{s,p} = \alpha'(a_s)^{-1}\overline{\langle B^* k^\alpha_{a_s}, \widetilde{k}^\alpha_{a_p}\rangle} = \frac{\alpha'(a_p)}{\alpha'(a_s)}\overline{r_{p,s}},$$

where $(r_{p,s})$ is the matrix representation of B^* with respect to $\{k^\alpha_{a_1}, \dots, k^\alpha_{a_n}\}$. It is now easy to verify that $(t_{s,p})$ satisfies (3.4) if and only if $(r_{p,s})$ satisfies (3.3). Since $B \in \mathscr{H}(\alpha)$ if and only if $B^* \in \mathscr{H}(\alpha)$ (note that $B^*_\varphi = B_{\varphi^\#}$), this completes the proof. $\square$

3.2. Clark basis and modified Clark basis

For any $\lambda \in \partial\mathbb{D}$ the so-called Clark operator U^α_λ is the operator from K_α onto K_α defined by

$$U^\alpha_\lambda = S_\alpha + \frac{\alpha(0) + \lambda}{1 - |\alpha(0)|^2} k^\alpha_0 \otimes \widetilde{k}^\alpha_0. \tag{3.5}$$

The operator U^α_λ is unitary and the countable set of its eigenvalues consists of $\eta_m \in \partial\mathbb{D}$ such that α has an ADC at η_m with

$$\alpha(\eta_m) = \alpha_\lambda = \frac{\alpha(0) + \lambda}{1 + \overline{\alpha(0)}\lambda}. \tag{3.6}$$

If η_m is an eigenvalue of U^α_λ, then the corresponding eigenvector is

$$k^\alpha_{\eta_m}(z) = \frac{1 - \overline{\alpha_\lambda}\alpha(z)}{1 - \overline{\eta_m} z}$$

(for proofs and details see [5]).

If U^α_λ has a pure point spectrum (which happens for example when α is a Blaschke product with a countable set of limit points of its zeros, see [5, p. 185]), then the normalized eigenvectors

$$v^\alpha_{\eta_m} = \|k^\alpha_{\eta_m}\|^{-1} k^\alpha_{\eta_m}$$

form an orthonormal basis for K_α. The basis $\{v^\alpha_{\eta_m}\}$ is called the Clark basis corresponding to $\lambda \in \partial\mathbb{D}$ (see [5] and [9]). The modified Clark basis is defined by

$$e^\alpha_{\eta_m} = \omega^\alpha_m v^\alpha_{\eta_m},$$

where

$$\omega^\alpha_m = e^{-\frac{i}{2}(\arg \eta_m - \arg \lambda)}.$$

The basis $\{e^\alpha_{\eta_m}\}$ has the property $C_\alpha e^\alpha_{\eta_m} = e^\alpha_{\eta_m}$, where C_α is the conjugation given by (2.1). It is easy to verify that if α is a finite Blaschke product with n zeros, then (3.6) has precisely n distinct solutions $\eta_1, \dots, \eta_n \in \partial\mathbb{D}$.

Theorem 3.3. *Let α be a finite Blaschke product with $n>0$ (not necessarily distinct) zeros and let $\{v^\alpha_{\eta_1},\dots,v^\alpha_{\eta_n}\}$ be the Clark basis for K_α corresponding to $\lambda\in\partial\mathbb{D}$ with $\overline{\eta_p}\neq\eta_s$ for all $1\le p,s\le n$. Let B be any linear operator on K_α. If $M_B=(r_{s,p})$ is the matrix representation of B with respect to the basis $\{v^\alpha_{\eta_1},\dots,v^\alpha_{\eta_n}\}$, then $B\in\mathscr{H}(\alpha)$ if and only if*

$$\begin{aligned} r_{s,p} = {} & \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}}\frac{\eta_s-\overline{\eta_1}}{\eta_s-\overline{\eta_p}}r_{s,1} - \frac{|\alpha'(\eta_1)|}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\alpha'(\eta_s)|}}\frac{\eta_1-\overline{\eta_1}}{\eta_s-\overline{\eta_p}}r_{1,1} \\ & + \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_s)|}}\frac{\eta_1-\overline{\eta_p}}{\eta_s-\overline{\eta_p}}r_{1,p} \end{aligned}$$

for all $1\le p,s\le n$.

Proof. The proof is analogous to the proof of Theorem 3.1. The details are left to the reader. ☐

As a corollary we obtain the matrix representation of B with respect to the modified Clark basis.

Theorem 3.4. *Let α be a finite Blaschke product with $n>0$ (not necessarily distinct) zeros and let $\{e^\alpha_{\eta_1},\dots,e^\alpha_{\eta_n}\}$ be the modified Clark basis for K_α corresponding to $\lambda\in\partial\mathbb{D}$ with $\overline{\eta_p}\neq\eta_s$ for all $1\le p,s\le n$. Let B be any linear operator on K_α. If $\widetilde{M}_B=(t_{s,p})$ is the matrix representation of B with respect to the basis $\{e^\alpha_{\eta_1},\dots,e^\alpha_{\eta_n}\}$, then $B\in\mathscr{H}(\alpha)$ if and only if*

$$\begin{aligned} t_{s,p} = {} & \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}}\frac{\omega^\alpha_p}{\omega^\alpha_1}\frac{\eta_s-\overline{\eta_1}}{\eta_s-\overline{\eta_p}}t_{s,1} - \frac{|\alpha'(\eta_1)|}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\alpha'(\eta_s)|}}\frac{\omega^\alpha_p}{\omega^\alpha_s}\frac{\eta_1-\overline{\eta_1}}{\eta_s-\overline{\eta_p}}t_{1,1} \\ & + \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_s)|}}\frac{\omega^\alpha_1}{\omega^\alpha_s}\frac{\eta_1-\overline{\eta_p}}{\eta_s-\overline{\eta_p}}t_{1,p} \end{aligned}$$

for all $1\le p,s\le n$.

Proof. Let B be any bounded linear operator on K_α. The proof follows from Theorem 3.3 and the comparison of the matrix representations of B with respect to the Clark basis and the modified Clark basis. ☐

Remark 3.5. By Theorem 3.1 the matrix representing a THO is determined by the entries in the first row and the first column. A simple modification of the proof shows that one can take any other row and any other column instead. The same is true for Theorems 3.2–3.4.

4. Matrix representations of THO's on some infinite-dimensional model spaces

In this section we generalize the results from Section 3 to some infinite-dimensional cases.

4.1. Kernel and conjugate kernel bases

Let α be an infinite Blaschke product with uniformly separated zeros $\{a_m\}$, that is,

$$\inf_k \prod_{m\neq k} \left| \frac{a_m - a_k}{1 - \overline{a_m} a_k} \right| \geq \delta$$

for some $\delta > 0$.

It is known (see [6, p. 151]) that a sequence of complex numbers $\{a_m\}$ is uniformly separated if and only if the transformation

$$f \mapsto \{f(a_m)\sqrt{1 - |a_m|^2}\} \tag{4.1}$$

maps H^2 boundedly onto l^2, the space of all complex square summable sequences. So if $\{a_m\}$ satisfies (4.1), then there exists $c > 0$ such that

$$\sum_{m=1}^{\infty} |f(a_m)|^2 (1 - |a_m|^2) < c\|f\|^2$$

for every $f \in H^2$. Moreover, if $\{f_m\}$ is a sequence of complex numbers satisfying $\sum_{m=1}^{\infty} |f_m|^2(1-|a_m|^2) < \infty$, then there exists $f \in H^2$ such that $f(a_m) = f_m$ for each $m \geq 1$. The most general form of f is given by

$$f = \sum_{m=1}^{\infty} \frac{f_m}{\alpha'(a_m)} \widetilde{k}^{\alpha}_{a_m} + \alpha h, \quad h \in H^2,$$

where the series converges in norm. In particular, the interpolation problem $f(a_m) = f_m$, $m \geq 1$, has a unique solution from K_α given by

$$f = \sum_{m=1}^{\infty} \frac{f_m}{\alpha'(a_m)} \widetilde{k}^{\alpha}_{a_m}.$$

Every $f \in K_\alpha$ can thus be written as

$$f = \sum_{m=1}^{\infty} \frac{f(a_m)}{\alpha'(a_m)} \widetilde{k}^{\alpha}_{a_m} = \sum_{m=1}^{\infty} \frac{\langle f, k^{\alpha}_{a_m}\rangle}{\alpha'(a_m)} \widetilde{k}^{\alpha}_{a_m}, \tag{4.2}$$

and the family of conjugate kernel functions $\{\widetilde{k}^{\alpha}_{a_m}\}$ forms a basis for K_α (see [8, 16] for proofs and details). Similarly, $\{k^{\alpha}_{a_m}\}$ is a basis for K_α and each $f \in K_\alpha$ can be written as

$$f = \sum_{m=1}^{\infty} \overline{\left(\frac{\widetilde{f}(a_m)}{\alpha'(a_m)} \right)} k^{\alpha}_{a_m} = \sum_{m=1}^{\infty} \frac{\langle f, \widetilde{k}^{\alpha}_{a_m}\rangle}{\overline{\alpha'(a_m)}} k^{\alpha}_{a_m}.$$

To see this, substitute f with $\widetilde{f} = C_\alpha f$ in (4.2) and apply C_α to both sides of the resulting equation.

Theorem 4.1. *Let α be an infinite Blaschke product with uniformly separated zeros $\{a_m\}$ and let B be a bounded linear operator on K_α. If $M_B = (r_{s,p})$ is the matrix representation of B with respect to the basis $\{k^\alpha_{a_m}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if*

$$r_{s,p} = \frac{\overline{\alpha'(a_s)(1-a_s a_1)} r_{s,1} - \overline{\alpha'(a_1)(1-a_1^2)} r_{1,1} + \overline{\alpha'(a_1)(1-a_1 a_p)} r_{1,p}}{\overline{\alpha'(a_s)(1-a_s a_p)}} \tag{4.3}$$

for all $p, s \geq 1$.

Theorem 4.2. *Let α be an infinite Blaschke product with uniformly separated zeros $\{a_m\}$ and let B be a bounded linear operator on K_α. If $\widetilde{M}_B = (t_{s,p})$ is the matrix representation of B with respect to the basis $\{\widetilde{k}^\alpha_{a_m}\}$, then $B \in \mathscr{H}(\alpha)$ if and only if*

$$t_{s,p} = \frac{\alpha'(a_s)(1-a_s a_1) t_{s,1} - \alpha'(a_1)(1-a_1^2) t_{1,1} + \alpha'(a_1)(1-a_1 a_p) t_{1,p}}{\alpha'(a_s)(1-a_s a_p)}$$

for all $p, s \geq 1$.

Since the proof of Theorem 4.2 is the same as the proof of Theorem 3.2, we only prove Theorem 4.1.

Proof of Theorem 4.1. Here the proof is based on Theorem 2.3. Let B be a bounded linear operator on K_α. Recall that the matrix representation $M_B = (r_{s,p})$ with respect to $\{k^\alpha_{a_m}\}$ is given by

$$r_{s,p} = \left(\overline{\alpha'(a_s)}\right)^{-1} \langle B k^\alpha_{a_p}, \widetilde{k}^\alpha_{a_s}\rangle.$$

By Theorem 2.3, B belongs to $\mathscr{H}(\alpha)$ if and only if there exist functions $\psi, \chi \in K_\alpha$ such that

$$B - S^*_\alpha B S^*_\alpha = \psi \otimes k^\alpha_0 + \widetilde{k}^\alpha_0 \otimes \chi. \tag{4.4}$$

The formula (4.4) can be expressed in terms of matrix representations with respect to $\{k^\alpha_{a_m}\}$ as

$$\begin{aligned}&\left(\overline{\alpha'(a_s)}\right)^{-1} \langle (B - S^*_\alpha B S^*_\alpha) k^\alpha_{a_p}, \widetilde{k}^\alpha_{a_s}\rangle \\ &= \left(\overline{\alpha'(a_s)}\right)^{-1} \langle (\psi \otimes k^\alpha_0 + \widetilde{k}^\alpha_0 \otimes \chi) k^\alpha_{a_p}, \widetilde{k}^\alpha_{a_s}\rangle, \quad s, p \geq 1.\end{aligned}$$

Since

$$S^*_\alpha k^\alpha_{a_p} = \overline{a_p} k^\alpha_{a_p} \quad \text{and} \quad S_\alpha \widetilde{k}^\alpha_{a_s} = a_s \widetilde{k}^\alpha_{a_s}$$

(see [18, Lemma 2.2]), we have

$$\begin{aligned}\langle (B - S^*_\alpha B S^*_\alpha) k^\alpha_{a_p}, \widetilde{k}^\alpha_{a_s}\rangle &= \langle B k^\alpha_{a_p}, \widetilde{k}^\alpha_{a_s}\rangle - \langle B S^*_\alpha k^\alpha_{a_p}, S_\alpha \widetilde{k}^\alpha_{a_s}\rangle \\ &= (1 - \overline{a_s a_p}) \langle B k^\alpha_{a_p}, \widetilde{k}^\alpha_{a_s}\rangle = \overline{(1 - a_s a_p)\alpha'(a_s)}\, r_{s,p}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle(\psi\otimes k_0^\alpha+\widetilde{k}_0^\alpha\otimes\chi)k_{a_p}^\alpha,\widetilde{k}_{a_s}^\alpha\rangle &= \langle k_{a_p}^\alpha,k_0^\alpha\rangle\langle\psi,\widetilde{k}_{a_s}^\alpha\rangle+\langle k_{a_p}^\alpha,\chi\rangle\langle\widetilde{k}_0^\alpha,\widetilde{k}_{a_s}^\alpha\rangle\\ &= \overline{\widetilde{\psi}(a_s)+\chi(a_p)}.\end{aligned}$$

Therefore, (4.4) holds if and only if there exist infinite sequences of complex numbers $\{\widetilde{\psi}_s\}$ and $\{\chi_p\}$ such that

$$\overline{(1-a_sa_p)\alpha'(a_s)}\,r_{s,p}=\overline{\widetilde{\psi}_s+\chi_p}\quad\text{for all } s,p\geq 1, \tag{4.5}$$

and

$$\sum_{s=1}^{\infty}|\widetilde{\psi}_s|^2(1-|a_s|^2)<\infty,\qquad \sum_{p=1}^{\infty}|\chi_p|^2(1-|a_p|^2)<\infty. \tag{4.6}$$

The functions ψ,χ from (4.4) are then given by

$$\psi=\sum_{s=1}^{\infty}\overline{\left(\frac{\widetilde{\psi}_s}{\alpha'(a_s)}\right)}k_{a_s}^\alpha,\qquad\qquad \chi=\sum_{p=1}^{\infty}\frac{\chi_p}{\alpha'(a_p)}\,\widetilde{k}_{a_p}^\alpha.$$

If B belongs to $\mathscr{H}(\alpha)$, then $M_B=(r_{s,p})$ satisfies (4.5) and a simple computation shows that $(r_{s,p})$ also satisfies (4.3).

To complete the proof we need to show that if $(r_{s,p})$ satisfies (4.3), then $B\in\mathscr{H}(\alpha)$. To this end, it is enough to find sequences of complex numbers $\{\widetilde{\psi}_s\}$, $\{\chi_p\}$ such that (4.5) and (4.6) hold.

Using (4.3) it can be verified that the sequences $\{\widetilde{\psi}_s\}$, $\{\chi_p\}$ satisfy (4.5) if and only if they satisfy

$$\begin{cases}\overline{(1-a_sa_1)\alpha'(a_s)}\,r_{s,1}=\overline{\widetilde{\psi}_s+\chi_1}, & s\geq 1,\\ \overline{(1-a_1a_p)\alpha'(a_1)}\,r_{1,p}=\overline{\widetilde{\psi}_1+\chi_p}, & p>1.\end{cases} \tag{4.7}$$

Fix arbitrary χ_1. Then the sequences $\{\widetilde{\psi}_s\}$, $\{\chi_p\}$ satisfying (4.7), and so (4.5), are clearly given by

$$\begin{cases}\widetilde{\psi}_s=(1-a_sa_1)\alpha'(a_s)\,\overline{r_{s,1}}-\chi_1, & s\geq 1,\\ \chi_p=(1-a_1a_p)\alpha'(a_1)\,\overline{r_{1,p}}-\widetilde{\psi}_1, & p>1.\end{cases}$$

Moreover, such $\{\widetilde{\psi}_s\}$, $\{\chi_p\}$ also satisfy (4.6). Indeed,

$$\begin{aligned}\sum_{s=1}^{\infty}|\widetilde{\psi}_s|^2(1-|a_s|^2) &\leq C\cdot\sum_{s=1}^{\infty}|\alpha'(a_s)\overline{r_{s,1}}|^2(1-|a_s|^2)+C\cdot|\chi_1|^2\sum_{s=1}^{\infty}(1-|a_s|^2)\\ &= C\cdot\sum_{s=1}^{\infty}|C_\alpha Bk_{a_1}^\alpha(a_s)|^2(1-|a_s|^2)+C'<\infty,\end{aligned}$$

and similarly for $\{\chi_p\}$. Hence $B\in\mathscr{H}(\alpha)$. □

4.2. Clark and modified Clark bases

To give an infinite-dimensional analogs of Theorems 3.3 and 3.4 we need the following generalization of Theorem 2.3.

Theorem 4.3. *Let c be a complex number. A bounded linear operator B belongs to $\mathscr{H}(\alpha)$ if and only if there exist functions $\psi, \chi \in K_\alpha$ such that*

$$B - S_{\alpha,c}^* B S_{\alpha,c}^* = \psi \otimes k_0^\alpha + \widetilde{k}_0^\alpha \otimes \chi,$$

where $S_{\alpha,c} = S_\alpha + c(k_0^\alpha \otimes \widetilde{k}_0^\alpha)$ is the modified compressed shift.

Proof. The proof is analogous to the proof of [18, Theorem 10.1] and we leave the details to the reader. □

Recall that the Clark unitary operator U_λ^α, $\lambda \in \partial\mathbb{D}$, given by (3.5) is a modified compressed shift, $U_\lambda^\alpha = S_{\alpha,c_\lambda}$ with $c_\lambda = (\alpha(0) + \lambda)/(1 - |\alpha(0)|^2)$. Therefore, by Theorem 4.3, a bounded linear operator B on K_α belongs to $\mathscr{H}(\alpha)$ if and only if for every $\lambda \in \partial\mathbb{D}$ there exist functions $\psi, \chi \in K_\alpha$ such that

$$B - (U_\lambda^\alpha)^* B (U_\lambda^\alpha)^* = \psi \otimes k_0^\alpha + \widetilde{k}_0^\alpha \otimes \chi. \tag{4.8}$$

Equation (4.8) can now be expressed in terms of matrix representations with respect to the Clark basis corresponding to λ and a proof similar to the proof of Theorem 4.1 can be given to show the following.

Theorem 4.4. *Let α be an inner function such that K_α has a Clark basis $\{v_{\eta_m}^\alpha\}$ with $\overline{\eta_p} \neq \eta_s$ for all $p, s \geq 1$. If $M_B = (r_{s,p})$ is the matrix representation of a bounded linear operator B on K_α with respect to the basis $\{v_{\eta_m}^\alpha\}$, then $B \in \mathscr{H}(\alpha)$ if and only if*

$$\begin{aligned} r_{s,p} = {} & \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\eta_s - \overline{\eta_1}}{\eta_s - \overline{\eta_p}} r_{s,1} - \frac{|\alpha'(\eta_1)|}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\alpha'(\eta_s)|}} \frac{\eta_1 - \overline{\eta_1}}{\eta_s - \overline{\eta_p}} r_{1,1} \\ & + \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_s)|}} \frac{\eta_1 - \overline{\eta_p}}{\eta_s - \overline{\eta_p}} r_{1,p} \end{aligned}$$

for all $p, s \geq 1$.

A generalized version of Theorem 3.4 easily follows.

Theorem 4.5. *Let α be an inner function such that K_α has a modified Clark basis $\{e_{\eta_m}^\alpha\}$ with $\overline{\eta_p} \neq \eta_s$ for all $p, s \geq 1$. If $\widetilde{M}_B = (t_{s,p})$ is the matrix representation of a bounded linear operator B on K_α with respect to the basis $\{e_{\eta_m}^\alpha\}$, then $B \in \mathscr{H}(\alpha)$ if and only if*

$$\begin{aligned} t_{s,p} = {} & \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_p)|}} \frac{\omega_p^\alpha}{\omega_1^\alpha} \frac{\eta_s - \overline{\eta_1}}{\eta_s - \overline{\eta_p}} t_{s,1} - \frac{|\alpha'(\eta_1)|}{\sqrt{|\alpha'(\eta_p)|}\sqrt{|\alpha'(\eta_s)|}} \frac{\omega_p^\alpha}{\omega_s^\alpha} \frac{\eta_1 - \overline{\eta_1}}{\eta_s - \overline{\eta_p}} t_{1,1} \\ & + \frac{\sqrt{|\alpha'(\eta_1)|}}{\sqrt{|\alpha'(\eta_s)|}} \frac{\omega_1^\alpha}{\omega_s^\alpha} \frac{\eta_1 - \overline{\eta_p}}{\eta_s - \overline{\eta_p}} t_{1,p} \end{aligned}$$

for all $p, s \geq 1$.

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References

[1] A. Baranov, I. Chalendar, E. Fricain, J. E. Mashreghi, and D. Timotin *Bounded symbols and reproducing kernel thesis for truncated Toeplitz operators*, J. Funct. Anal. **259** (2010), no. 10, 2673–2701.

[2] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, second edition, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006.

[3] J. A. Cima, S. R. Garcia, W. T. Ross, and W. R. Wogen, *Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity*, Indiana Univ. Math. J. **59** (2010), no. 2, 595–620.

[4] J. A. Cima, W. T. Ross, and W. R. Wogen, *Truncated Toeplitz operators on finite dimensional spaces*, Oper. Matrices **2** (2008), no. 3, 357–369.

[5] D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Anal. Math. **25** (1972), 169–191.

[6] P. L. Duren, *Theory of H^p Spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York, 1970.

[7] S. R. Garcia, J. Mashreghi, and W. T. Ross, *Introduction to Model Spaces and Their Operators*, Cambridge Studies in Advanced Mathematics, **148**, Cambridge University Press, Cambridge, 2016.

[8] S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315.

[9] S. R. Garcia and W. T. Ross, *A non-linear extremal problem on the Hardy space*, Comput. Methods Funct. Theory **9** (2009), no. 2, 485–524.

[10] ———, *The norm of a truncated Toeplitz operator*, in Hilbert spaces of analytic functions, 59–64, CRM Proc. Lecture Notes, **51**, Amer. Math. Soc., Providence, RI, 2010.

[11] ———, *Recent progress on truncated Toeplitz operators*, in Blaschke products and their applications, 275–319, Fields Inst. Commun., **65**, Springer, New York, 2013.

[12] C. Gu, *Algebraic properties of truncated Hankel operators*, preprint.

[13] D.-O. Kang and H. J. Kim, *Products of truncated Hankel operators*, J. Math. Anal. Appl. **435** (2016), no. 2, 1804–1811.

[14] B. Łanucha, *Matrix representations of truncated Toeplitz operators*, J. Math. Anal. Appl. **413** (2014), no. 1, 430–437.

[15] R. A. Martinez-Avendano and P. Rosenthal, *An Introduction to Operators on the Hardy-Hilbert Space*, Graduate Texts in Mathematics, **237**, Springer, New York, 2007.

[16] N. K. Nikolski, *Treatise on the Shift Operator*, translated from the Russian by Jaak Peetre, Grundlehren der Mathematischen Wissenschaften, **273**, Springer-Verlag, Berlin, 1986.

[17] V. V. Peller, *Hankel Operators and Their Applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[18] D. Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices **1** (2007), no. 4, 491–526.

Bartosz Łanucha
Institute of Mathematics
Maria Curie-Skłodowska University
pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
Email address: bartosz.lanucha@poczta.umcs.lublin.pl

Małgorzata Michalska
Institute of Mathematics
Maria Curie-Skłodowska University
pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
Email address: malgorzata.michalska@poczta.umcs.lublin.pl