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A REMARK ON QF RINGS

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Abstract. This article mainly concentrates on the open question whether a right self-injective ring R is necessary QF if R/S_l is left Goldie. It is answered affirmatively under the condition $S_l \subseteq S_r$, where S_l and S_r denote the left socle and right socle of R respectively. And the original condition "right self-injective" can be weakened to "right CS and right P-injective". It is also proved that a semiperfect, left and right mininjective ring R is QF if $S_r \subseteq^{ess} R_R$ and R/S_l is left Goldie.

1. Introduction

Let R be an associative ring with identity. We use S_l, S_r, J, Z_l, Z_r to denote the left socle, right socle, Jacobson radical, left singular ideal, right singular ideal of R respectively. Let N be a submodule of a module M, write $N \subseteq^{ess} M$ by showing that N is an essential submodule of M. We use $Soc(M)$ to denote the socle of M. Let X be a subset of a ring R, $\mathbf{l}(X)$ means the left annihilator of X in R. The right annihilator of X can be defined similarly. A left ideal I of R is called a left annihilator ideal if $I = \mathbf{l}(X)$ for some subset X of R. Right annihilator ideals can be obtained similarly.

Quasi-Frobenius (QF) rings were firstly introduced by Nakayama [12]. It is defined to be an one-sided artinian ring such that for any basic set of primitive idempotents $\{e_1, e_2, \ldots, e_n\}$ of R, there exists a permutation σ of $\{1, 2, \ldots, n\}$ satisfying $Soc(Re_k) \cong Re_{\sigma(k)}/Je_{\sigma(k)}$ and $Soc(e_{\sigma(k)}R) \cong e_kR/e_kJ$. It is always an interesting topic to characterize QF rings through various chain conditions. Much work have been done by many algebraists. It is proved that a ring R is QF if and only if R is right self-injective and satisfies any of the following chain conditions:

(1) R is right (or left) artinian [7];
(2) R is right (or left) noetherian [7, 8];
(3) R satisfies ACC on right (or left) annihilators [8];
(4) R satisfies DCC on essential right (or left) ideals [2];
(5) R satisfies ACC on essential right (or left) ideals [6];

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(6) R/S_r is right Goldie [3, 11].

Recall that a ring R is called a *right Goldie* ring if R satisfies ACC on right annihilators and the right R-module R_R has finite uniform dimension. Left Goldie rings can be defined similarly. It is obvious that if R is right noetherian, then R must be right Goldie. It is proved in [6, Lemma 2] that a module M satisfies ACC on essential submodules if and only if $M/Soc(M)$ is noetherian. Hence, if R satisfies ACC on essential right ideals, then R/S_r is right Goldie. According to the condition (6), it is natural to ask the following question [4, Question 2.8]:

Is a right self-injective ring R necessarily QF if R/S_l is left Goldie?

In this article, given that R is left mininjective or $S_l \subseteq S_r$, the question is answered affirmatively in Theorem 3.6. And the condition "right self-injective" is weakened to "right CS and right P-injective". Recall that a ring R is called *left mininjective* if every homomorphism from a minimal left ideal of R to ${}_RR$ can be extended from ${}_RR$ to ${}_RR$. A ring R is called *left P-injective* if every homomorphism from a principal left ideal of R to ${}_RR$ can be extended to one from ${}_RR$ to ${}_RR$. It is clear that a left P-injective ring must be left mininjective. A ring R is called a *left C1* (*left CS*) ring if every left ideal is essential in a direct summand of ${}_RR$. R is called *left C2* if every left ideal that is isomorphic to a direct summand of ${}_RR$ is also a direct summand of ${}_RR$. R is called *left continuous* if R is both left C1 and left C2. The right sides of these definitions can be defined similarly. Before we obtain the main result, some preparation work on annihilators of R and its quotient rings are discussed. It is also proved in Theorem 3.11 that a semiperfect, left and right mininjective ring is QF if $S_r \subseteq^{ess} R_R$ and R/S_l is left Goldie.

2. Annihilators

Let R be a ring and I be a two-sided ideal of R. First we look at some results on annihilators of R and its quotient ring $\overline{R} = R/I$. For any two nonempty subsets X and Y of R. We use XY and $\overline{X}$ to denote the sets $\{\sum_{i=1}^{n} x_i y_i \mid x_i \in X, y_i \in Y, n \geq 1\}$ and $\{x + I \mid x \in X\}$, respectively.

The following lemma appeared in [9], but it was unproved. We will show the proof and generalize it to Theorem 2.2.

Lemma 2.1 ([9, Sublemma]). *Let R be a ring and $I = \mathbf{l}(X)$ be a two-sided ideal of R, where $X \subseteq R$. Set $\overline{R} = R/I$. Then for any left annihilator $\mathbf{l}_{\overline{R}}(\overline{Y})$ of $\overline{R}$, where $Y \subseteq R$. We have*

$$\mathbf{l}_{\overline{R}}(\overline{Y}) = \mathbf{l}_R(YX)/I.$$

Proof. Firstly, we show that $I \subseteq \mathbf{l}_R(YX)$. If $r \in I$, then $rX = \{0\}$. Since I is a right ideal, $rY \subseteq I$. So $rYX = \{0\}$. Hence $r \in \mathbf{l}_R(YX)$. Now for any $\overline{r} = r + I \in \mathbf{l}_{\overline{R}}(\overline{Y})$, we have $\overline{r}\overline{Y} = \{\overline{0}\}$. This means $rY \subseteq I$. So $rYX = \{0\}$. Thus $r \in \mathbf{l}_R(YX)$. Therefore, $\mathbf{l}_{\overline{R}}(\overline{Y}) \subseteq \mathbf{l}_R(YX)/I$. The converse can also be verified directly. $\square$

Let L be a left ideal of a ring R. We denote $L^0 = R$ and $L^k = L^{k-1}L$, $k \geq 1$. Then for any integer $m \geq 0$, $\mathbf{l}(L^m)$ is a two-sided ideal of R. Set $R_m = R/\mathbf{l}(L^m)$. Then for any $0 \leq k < m$, it is clear that $\mathbf{l}(L^k) \subseteq \mathbf{l}(L^m)$. So there is a natural ring homomorphism $f_{km} : R_k \to R_m$ such that for any $x + \mathbf{l}(L^k) \in R_k$, $f_{km}(x + \mathbf{l}(L^k)) = x + \mathbf{l}(L^m)$. Using these symbols, we have:

Theorem 2.2. *Let L be a left ideal of a ring R. For any $0 \leq k < m$, if T is a left annihilator ideal in R_m, then $f_{km}^{\leftarrow}(T)$ is also a left annihilator ideal in R_k.*

Proof. Assume T is a left annihilator ideal in R_m. Then there exists a left ideal $M \supseteq \mathbf{l}(L^m)$ of R such $T = M/\mathbf{l}(L^m)$. Thus $f_{km}^{\leftarrow}(T) = M/\mathbf{l}(L^k)$. Next we show that $M/\mathbf{l}(L^k)$ is also a left annihilator in R_k. Since T is a left annihilator ideal in R_m, there is a subset A in R such that $M/\mathbf{l}(L^m) = \mathbf{l}_{R_m}(\overline{A})$, where $\overline{A} = \{a + \mathbf{l}(I^m) \mid a \in A\}$. We only need to show that

$$M/\mathbf{l}(L^k) = \mathbf{l}_{R_k}(\widehat{AL^{m-k}}), \text{ where } \widehat{AL^{m-k}} = \{b + \mathbf{l}(L^k) \mid b \in AL^{m-k}\}.$$

By Lemma 2.1, $M = \mathbf{l}_R(AL^m)$. Then for any $x \in M$, we have

$$xAL^m = xAL^{m-k}L^k = \{0\}.$$

Hence, again by Lemma 2.1, $x + \mathbf{l}(L^k) \in \mathbf{l}_{R_k}(\widehat{AL^{m-k}})$. This informs

$$M/\mathbf{l}(L^k) \subseteq \mathbf{l}_{R_k}(\widehat{AL^{m-k}}).$$

Conversely, let $x + \mathbf{l}(L^k) \in \mathbf{l}_{R_k}(\widehat{AL^{m-k}})$. By Lemma 2.1,

$$xAL^{m-k}L^k = xAL^m = \{0\}.$$

So $x \in M$. Therefore,

$$\mathbf{l}_{R_k}(\widehat{AL^{m-k}}) \subseteq M/\mathbf{l}(L^k).$$

Thus $M/\mathbf{l}(L^k) = \mathbf{l}_{R_k}(\widehat{AL^{m-k}})$. □

By Theorem 2.2, we can easily obtain several results on chain condition of annihilators of a ring R and its quotient rings R_k, $k \geq 1$.

Proposition 2.3. *Let L be a left ideal of a ring R. For any $0 \leq k < m$, if R_k satisfies ACC (DCC) on left annihilators, then R_m also satisfies ACC (DCC) on left annihilators. In particular, if $R/\mathbf{l}(L)$ satisfies ACC on left annihilators, so is $R/\mathbf{l}(L^m)$ for any $m > 1$.*

Proof. We only prove the case of ACC, the other case can be obtained by a similar proof. Let $T_1 \subseteq T_2 \subseteq \cdots$ be an ascending chain of left annihilators in R_m. By Theorem 2.2, $f_{km}^{\leftarrow}(T_1) \subseteq f_{km}^{\leftarrow}(T_2) \subseteq \cdots$ is also an ascending chain of left annihilators in R_k. Since R_k satisfies ACC on left annihilators, there exists $n \geq 1$ such that for any $i \geq n$, $f_{km}^{\leftarrow}(T_i) = f_{km}^{\leftarrow}(T_n)$. Thus

$$T_i = f_{km}(f_{km}^{\leftarrow}(T_i)) = f_{km}(f_{km}^{\leftarrow}(T_n)) = T_n.$$

So R_m satisfies ACC on left annihilators. □

Since a ring satisfies ACC on left annihilators if and only if it satisfies DCC on right annihilators, by taking $k = 0$ and $m = 1$ in the right side of Proposition 2.3, we have:

Corollary 2.4 ([10, Lemma 2]). *Let R satisfy the ascending chain condition on left annihilators, and suppose that $\mathbf{r}(S)$ is a two-sided ideal of R. Then $R/r(S)$ has the ascending chain condition on left annihilators.*

Proposition 2.5. *Let L be a left ideal of a ring R. If R_k satisfies ACC on left annihilators for some $k \geq 0$, then there exists a positive integer n such that $\mathbf{l}(L^n) = \mathbf{l}(L^{n+1})$.*

Proof. Considering the descending chain $f_{0k}(L) \supseteq f_{0k}(L^2) \supseteq \cdots$ in R_k, it is clear that $\mathbf{l}_{R_k}(f_{0k}(L)) \subseteq \mathbf{l}_{R_k}(f_{0k}(L^2)) \subseteq \cdots$ is an ascending chain of left annihilators in R_k. Since R_k satisfies ACC on left annihilators, we have some positive integer m such that

$$\mathbf{l}_{R_k}(f_{0k}(L^m)) = \mathbf{l}_{R_k}(f_{0k}(L^{m+1})).$$

By Lemma 2.1,

$$\mathbf{l}_{R_k}(f_{0k}(L^m)) = \mathbf{l}(L^{m+k})/\mathbf{l}(L^k) \text{ and } \mathbf{l}_{R_k}(f_{0k}(L^{m+1})) = \mathbf{l}(L^{m+1+k})/\mathbf{l}(L^k).$$

We are done by taking $n = m + k$. □

Proposition 2.6. *Let T be a right ideal of a ring R. If $R/\mathbf{r}(T)$ satisfies ACC on left annihilators and $T \subseteq Z_r$, then for any $k \geq 2$, $\mathbf{r}(T^k) \subseteq^{ess} R_R$.*

Proof. We only need to prove that $\mathbf{r}(T^2) \subseteq^{ess} R_R$. Since $\overline{R} = R/\mathbf{r}(T)$ satisfies ACC on left annihilators, it satisfies DCC on right annihilators. If $T^2 = 0$, then $\mathbf{r}(T^2) = R$. If $T^2 \neq 0$, $\overline{T}$ is a nonzero subset of $\overline{R}$. Since $\overline{R}$ satisfies DCC on right annihilators, there exist nonzero elements $a_1, a_2, \ldots, a_n \in T$ such that

$$\mathbf{r}_{\overline{R}}(\overline{T}) = \mathbf{r}_{\overline{R}}(\overline{Ra_1 + Ra_2 + \cdots + Ra_n}).$$

By the right side of Lemma 2.1, $\mathbf{r}_{\overline{R}}(\overline{T}) = \mathbf{r}_R(T^2)/\mathbf{r}_R(T)$ and

$$\mathbf{r}_{\overline{R}}(\overline{Ra_1 + Ra_2 + \cdots + Ra_n}) = \mathbf{r}_R(T(Ra_1 + Ra_2 + \cdots + Ra_n))/\mathbf{r}_R(T).$$

Thus

$$\mathbf{r}_R(T^2) = \mathbf{r}_R(T(Ra_1 + Ra_2 + \cdots + Ra_n)) = \mathbf{r}_R(Ta_1 + Ta_2 + \cdots + Ta_n).$$

So $\cap_{i=1}^n \mathbf{r}_R(a_i) \subseteq \mathbf{r}_R(Ta_1 + Ta_2 + \cdots + Ta_n)$. Since $a_i \in T \subseteq Z_r$, $i = 1, \ldots, n$,

$$\cap_{i=1}^n \mathbf{r}_R(a_i) \subseteq^{ess} R_R.$$

We have $\mathbf{r}_R(Ta_1 + Ta_2 + \cdots + Ta_n) \subseteq^{ess} R_R$. Hence $\mathbf{r}_R(T^2) \subseteq^{ess} R_R$. □

The socle series of an R-module M is defined inductively by:

$$Soc_1(M) = Soc(M),\ Soc_{n+1}(M) = Soc(M/Soc_n(M)).$$

Lemma 2.7 ([13, Lemma 3.36]). *If R is a semilocal ring for which $S_r = S_l$, then*

$$Soc_n(R_R) = Soc_n({}_RR) = \mathbf{l}(J^n) = \mathbf{r}(J^n) \text{ for all } n \geq 1.$$

If a ring R satisfies the condition of the above lemma, we briefly write

$$S_n = Soc_n(R_R) = Soc_n({}_RR),\ \forall\ n \geq 1.$$

3. On QF rings

The following lemmas are needed to establish the main result of our paper.

Lemma 3.1 ([13, Theorem 2.21(a)]). *Let R be a right mininjective ring. If kR is a simple right ideal, then Rk is a simple left ideal.*

Lemma 3.2 ([13, Proposition 5.10, Theorem 5.14]). *If R is a right P-injective ring, then*

(1) *R is right C2*;
(2) *$J = Z_r$.*

Recall that a ring R is called *orthogonally finite* if R has no infinite sets of orthogonal idempotents.

Lemma 3.3 ([1, Theorem 1.1]). *If R is a left continuous ring such that R/S_l is orthogonally finite, then S_l is left artinian and R is semiperfect.*

Lemma 3.4 ([13, Theorem 3.7(1)]). *Let R be a semiperfect and right mininjective ring. Then S_r is semisimple and artinian as a left R-module.*

Lemma 3.5 ([13, Lemma 3.37]). *Let R be a semiprimary ring with $S_r = S_l$. If S_1 is right artinian and S_2 is left artinian, then R is left and right artinian.*

In the next theorem, we provide the main result of our paper.

Theorem 3.6. *Let R be a right CS, right P-injective ring and R/S_l is left Goldie. Then the following are equivalent*:

(1) *R is QF.*
(2) *R is left mininjective.*
(3) *$S_l \subseteq S_r$.*

Proof. It is clear (1) $\Rightarrow$ (2). By the left side of Lemma 3.1, (2) $\Rightarrow$ (3). For (3) $\Rightarrow$ (1), since R is right P-injective, by Lemma 3.2, R is right C2 and $J = Z_r$. As R is a right CS ring, R is right continuous. Again since R is right P-injective, R is right mininjective, by Lemma 3.1, $S_r \subseteq S_l$. Thus, $S_l = S_r = S_1$. According to the assumption, R/S_1 is an orthogonally finite ring. By the right side of Lemma 3.3, R is a semiperfect ring and S_1 is artinian as a right R-module. Then by Lemma 2.7, $S_n = \mathbf{l}(J^n) = \mathbf{r}(J^n)$ for all $n \geq 1$. Next we show that J is nilpotent. Since $R/\mathbf{l}(J) = R/S_1$ satisfies ACC on left annihilators, by Proposition 2.5, there exists a positive integer $n \geq 2$ such that $\mathbf{l}(J^n) = \mathbf{l}(J^{2n})$. Let

$$\overline{R} = R/\mathbf{l}(J^n) = R/\mathbf{l}(J^{2n}).$$

Assume that J is not nilpotent, then $J^n \neq 0$. So $\overline{R}$ is nonzero. By Proposition 2.3, $\overline{R}$ satisfies ACC on left annihilators. Thus the nonempty set $\{\mathbf{l}_{\overline{R}}(\overline{a}) \mid 0 \neq \overline{a} \in \overline{R}\}$ has a maximal element $\mathbf{l}_{\overline{R}}(\overline{x})$. Since

$$\mathbf{r}(J^n) = \mathbf{l}(J^n) = \mathbf{l}(J^{2n}) = \mathbf{r}(J^{2n}),$$

$0 \neq J^n x \notin \mathbf{r}(J^n)$. Thus there exists $b \in J^n$ such that $bx \notin \mathbf{r}(J^n)$. So $\overline{b} \notin \mathbf{l}_{\overline{R}}(\overline{x})$. Since $R/\mathbf{l}(J)$ satisfies ACC on left annihilators and $\mathbf{l}(J) = \mathbf{r}(J)$, $R/\mathbf{r}(J)$ satisfies ACC on left annihilators, by Proposition 2.6, $\mathbf{r}(J^2) \subseteq^{ess} R_R$, so is $\mathbf{r}(J^n)$. As $bx \neq 0$, $bxR \cap \mathbf{r}(J^n) \neq 0$. Hence there exists $y \in R$ such that $0 \neq bxy \in \mathbf{r}(J^n)$. Now let $x' = xy$. Then $\overline{x'} \neq \overline{0}$, if not, $x' \in \mathbf{r}(J^n)$. So $bx' = 0$. It is a contradiction. Hence we have $\mathbf{l}_{\overline{R}}(\overline{x'}) \supseteq \mathbf{l}_{\overline{R}}(\overline{x})$ and $\overline{b} \in \mathbf{l}_{\overline{R}}(\overline{x'}) \backslash \mathbf{l}_{\overline{R}}(\overline{x})$. This is a contradiction to the fact that $\mathbf{l}_{\overline{R}}(\overline{x})$ is a maximal element. Therefore, J is nilpotent. So R is a semiprimary ring. Since R is right mininjective, by Lemma 3.4, S_1 is artinian as a left R-module. As R/S_1 is left Goldie, S_2/S_1 is left artinian. So S_2 is artinian as a left R-module. Since we have showed that S_1 is artinian as a right R-module, by Lemma 3.5, R is two-sided artinian. Followed by [5, Theorem 10], R is QF. □

With an argument similar to the one used in the proof of Theorem 3.6, we can establish the next proposition.

Proposition 3.7. *Let R be a semilocal ring with $S_l = S_r$ and $J \subseteq Z_r$. If R/S_l satisfies ACC on left annihilators, then R is a semiprimary ring.*

Corollary 3.8. *Let R be a right self-injective ring and R/S_l be left Goldie. Then the following are equivalent*:

(1) *R is QF.*
(2) *R is left mininjective.*
(3) *$S_l \subseteq S_r$.*

Since a von Neumann regular ring is P-injective, the following example shows that the condition " right CS and right P-injective" is weaker than the condition "right self-injective".

Example 3.9 ([13, Lemma 1.34(4)]). If F_i is a field and $K_i \subseteq F_i$ is a proper subfield for $i \geq 1$, let R denote the set of all sequences in $\prod F_i$ with almost all entries in K_i. Then R is a regular continuous ring that is not self-injective.

Recall that a ring R is called *semiregular* if R/J is von Neumann regular and idempotents lift modulo J. It is clear that a semiperfect ring must be semiregular.

Lemma 3.10 ([13, Proposition 2.27]). *If R is a right mininjective, semiregular ring in which $S_r \subseteq^{ess} R_R$, then $J = Z_r$.*

Theorem 3.11. *Let R be a semiperfect, left and right mininjective ring, $S_r \subseteq^{ess} R_R$ and R/S_l is left Goldie. Then R is QF.*

Proof. Since R is mininjective, by Lemma 3.1, $S_l = S_r = S_1$. We obtain $J = Z_r$ from Lemma 3.10. And by Lemma 3.4, S_1 is artinian as a left and right R-module. Since S_1 is artinian as a left R-module and S_2/S_1 is left artinian, we have that S_2 is artinian as a left R-module. By Proposition 3.7, R is a semiprimary ring. Then Lemma 3.5 implies that R is two-sided artinian. Hence R is QF by [14, Theorem 2.5]. □

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