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NULLITY OF THE LEVI-FORM AND THE ASSOCIATED SUBVARIETIES FOR PSEUDO-CONVEX CR STRUCTURES OF HYPERSURFACE TYPE

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Abstract. Let M^{2n+1}, $n \geq 1$, be a smooth manifold with a pseudo-convex integrable CR structure of hypersurface type. We consider a sequence of CR invariant subsets $M = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_n$, where $\mathcal{S}_q$ is the set of points where the Levi-form has nullity $\geq q$. We prove that $\mathcal{S}_q$'s are locally given as common zero sets of the coefficients A_j, $j = 0, 1, \ldots, q-1$, of the characteristic polynomial of the Levi-form. Some sufficient conditions for local existence of complex submanifolds are presented in terms of the coefficients A_j.

Introduction

Let M, or to specify the dimension M^{2n+1}, $n \geq 1$, be a smooth (C^∞) manifold equipped with a pseudo-convex integrable CR structure $(H(M), J)$ of hypersurface type. We consider a sequence of CR-invariant subsets

$$M = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_n,$$

where $\mathcal{S}_q$ is the set of those points where the Levi-form has nullity greater than or equal to q. In §2 we prove that $\mathcal{S}_q$ is locally given as a common zero set of the coefficients of the characteristic polynomial of the matrix representation of the Levi-form. A complex submanifold of M of complex dimension q, $1 \leq q \leq n$, is a local embedding f of an open subset $\mathcal{O} \subset \mathbb{C}^q$ that satisfies

$$\begin{aligned} &df(T_x(\mathcal{O})) \subset H_{f(x)}(M), \ \forall x \in \mathcal{O} \quad \text{and} \\ &J \circ df = df \circ J_{st}, \end{aligned} \tag{0.1}$$

where J_{st} is the standard complex structure tensor of $\mathbb{C}^q$. Solving (0.1) for f is an over-determined problem and there is no solution generically. If such f

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exists, then its image must be contained in $\mathcal{S}_q$. Hence a necessary condition for the existence of a complex manifold of dimension q is that $\mathcal{S}_q$ has real dimension $\geq 2q$. In §3 we present some sufficient conditions for the existence of real defining functions for a complex submanifold of dimension q. These are conditions on the coefficients of the characteristic polynomial of the levi-form that define $\mathcal{S}_q$. In several complex variables the function theory of a domain often depends on the geometry of the boundary, for which we refer the readers to [3] and [8]. In particular, at a boundary point x of a pseudo-convex domain with smooth real-analytic boundary the subelliptic estimate for the $\bar{\partial}$-Neumann problem holds for (p, q)-forms if there is no germ at x of a complex variety of dimension greater than or equal to q in the boundary (see [5] and [9]). For real hypersurfaces in general, without assuming pseudo-convexity, the existence of a germ of complex hypersurface is an obstruction to the extension of holomorphic functions of one side of M to the other side (see [11]). The existence of complex submanifolds in real hypersurfaces has been studied in [1], [6] and [7] without assuming the pseudo-convexity. The present paper makes essential use of the fact that the eigen-values of the Levi-form are non-negative for pseudo-convex CR manifolds. The coefficients A_j of the characteristic polynomial of the Levi-form depends on the choice of local basis of $(1, 0)$ vectors, they are invariant only under unitary change of basis. However, $\mathcal{S}_q$ the common zero set of A_j, $j = 0, 1, \ldots, q-1$, is a CR invariant, which is interesting from the viewpoint of the invariant theory. We work in C^∞ category. All the manifolds in this paper shall be assumed to be connected and oriented, or to be a connected submanifold sitting in a small neighborhood $U \subset M$. The authors thank Dmitri Zaitsev for the discussions we had while he was visiting KIAS. We thank Sungyeon Kim for the valuable comments and suggestions that she gave us.

1. Pseudo-convexity and the nullity of the Levi-form

Let M be a smooth (C^∞) manifold of dimension $2n + 1$, $n \geq 1$. A CR structure of hypersurface type, or simply a CR structure, on M is a pair $(H(M), J)$, where $H(M)$ is a smooth subbundle of codimension 1 of the tangent bundle $T(M)$ and J is an almost complex structure on $H(M)$, that is, $J : H(M) \longrightarrow H(M)$ is a smooth bundle isomorphism for which

$$\begin{array}{ccc} H(M) & \xrightarrow{J} & H(M) \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

commutes and

$$J^2 = -I_{2n}.$$

Then the complexification $\mathbb{C} \otimes H(M)$ has a decomposition

$$\mathbb{C} \otimes H(M) = H^{1,0}(M) \oplus H^{0,1}(M),$$

where $H^{1,0}(M)$ (resp. $H^{0,1}(M)$) is the set of eigen-vectors of J corresponding to the eigen-value $\sqrt{-1}$ (resp. $-\sqrt{-1}$). Locally, there exist real vector fields $X_1, \ldots, X_n$ so that X_j, JX_j, $j = 1, \ldots, n$, span $H(M)$. Then

$$L_j := X_j - \sqrt{-1}JX_j, \ j = 1, \ldots, n \tag{1.1}$$

span $H^{1,0}(M)$ and

$$\bar{L}_j := X_j + \sqrt{-1}JX_j, \ j = 1, \ldots, n \tag{1.2}$$

span $H^{0,1}(M)$, respectively. A diffeomorphism F of M onto a CR manifold of the same dimension $\widetilde{M}$ is called a CR mapping if F preserves the CR structure, namely, dF maps $H(M)$ onto $H(\widetilde{M})$ and

$$dF \circ J = J \circ dF \tag{1.3}$$

on $H(M)$. The CR structure bundle $H(M)$ is locally given by a non-vanishing real 1-form that annihilates $H(M)$: any point $x \in M$ has a neighborhood U on which there is a smooth non-vanishing real 1-form θ such that

$$H(U) = \theta^{\perp} := \{V \in T(U) : \theta(V) = 0\}. \tag{1.4}$$

We fix U for our local arguments assuming the local bases (1.1) and (1.2) are defined on U. By $\Omega^0(U) = C^\infty(U)$ we denote the ring of smooth complex valued functions on U and by $\Omega^p(U)$ the module over $\Omega^0(U)$ of smooth p-forms and by

$$\Omega^*(U) = \bigoplus_{p=0}^{2n+1} \Omega^p(U)$$

the exterior algebra of smooth differential forms on U with complex coefficients. A subalgebra $\mathcal{I} \subset \Omega^*(U)$ is called an ideal if the following conditions hold:

i) $\mathcal{I} \wedge \Omega^*(U) \subset \mathcal{I}$,
ii) if $\phi = \sum_{p=0}^{2n+1} \phi_p \in \mathcal{I}$, $\phi_p \in \Omega^p(U)$, then each $\phi_p \in \mathcal{I}$ (homogeneity).

Because of the homogeneity condition $\mathcal{I}$ is two-sided, that is,

$$\Omega^*(U) \wedge \mathcal{I} \subset \mathcal{I}.$$

We consider in this paper only those ideals that are generated by 0-forms (functions) and 1-forms: for a finite set of smooth functions $\mathbf{r} = (r^1, \ldots, r^d)$ and a finite set of smooth 1-forms $\mathbf{\Theta} = (\theta^1, \ldots, \theta^s)$ let $\mathcal{I}(\mathbf{r}, \mathbf{\Theta})$ be the ideal generated by $\mathbf{r}$ and $\mathbf{\Theta}$. For two elements $\phi, \psi \in \Omega^*(U)$ we write

$$\phi \equiv \psi, \quad \text{mod } (\mathbf{r}, \mathbf{\Theta})$$

or

$$\phi \overset{(\mathbf{r},\mathbf{\Theta})}{\equiv} \psi,$$

if and only if

$$\phi - \psi \in \mathcal{I}(\mathbf{r}, \mathbf{\Theta}).$$

The system $\mathbf{r}$ is said to be *non-degenerate* if $dr^1 \wedge \cdots \wedge dr^d \neq 0$. For other definitions and notations concerning the exterior algebra we refer the readers to [2]. The CR structure bundle $H(M)$ is *integrable* in the sense of Frobenius if

$$d\theta \equiv 0, \quad \text{mod } (\theta), \tag{1.5}$$

where θ is as in (1.4). By the Frobenius theorem if (1.5) holds M is locally foliated by integral manifolds of $H(M)$. If this is the case J is an almost complex structure on each leaf.

Definition 1.1. The *torsion tensor* of the CR structure is

$$d\theta, \text{ mod } (\theta). \tag{1.6}$$

We regard (1.6) as an element of the quotient module $\Omega^2(U)/(\theta)$, where (θ) is the set of all 2-forms

$$\{\theta \wedge \omega : \ \omega \in \Omega^1(U)\}.$$

From the viewpoint of (1.5), the torsion tensor is the obstruction to the local integrability of $H(M)$. The torsion tensor of the CR structure defines a hermitian form $\mathcal{L}$, which is called the *Levi-form,* on $H^{1,0}(U)$ by

$$\mathcal{L}(L, L') := \frac{1}{\sqrt{-1}} d\theta(L \wedge \bar{L}'). \tag{1.7}$$

If $\mathcal{L}$ is semi-definite at x, then the CR structure is said to be *pseudo-convex* at x. If $\mathcal{L}$ is semi-definite at every point of M, then the CR structure is said to be pseudo-convex.

Definition 1.2. At a point $x \in M$ the *null space* of $\mathcal{L}$ is

$$\mathcal{N}_x := \{L \in H_x^{1,0}(M) : \mathcal{L}(L, L) = 0\}. \tag{1.8}$$

Notice that if M is pseudo-convex, then $\mathcal{N}_x$ is a subspace. The complex dimension of $\mathcal{N}_x$ is called the *nullity* of the Levi-form at x.

If F is a CR mapping of M onto a CR manifold of same dimension $\widetilde{M}$, then by (1.3) dF preserves the type of vectors, that is, dF extends to the isomorphisms

$$H^{1,0}(M) \xrightarrow{dF} H^{1,0}(\widetilde{M}),$$
$$H^{0,1}(M) \xrightarrow{dF} H^{0,1}(\widetilde{M}).$$

Thus we have:

Proposition 1.3. *Let* $(H(M), J)$ *be a smooth pseudo-convex CR structure on a smooth manifold* M^{2n+1}, $n \geq 1$. *Then the notions of the null space and the nullity of the Levi-form defined locally by* (1.7)-(1.8) *do not depend on the choice of* θ, *and therefore, well defined globally. The nullity of the Levi-form is invariant under CR mappings.*

Now for each $q = 0, 1, \ldots, n$, let $\mathcal{S}_q$ be the set of points where the nullity of the Levi-form is greater than or equal to q. Then we have a sequence of CR-invariant subsets

$$M = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_n.$$

2. Invariant subvarieties of pseudo-convex CR manifolds

Consider the matrix representation of the Levi-form $\mathcal{L}$ with respect to the basis (1.1): let

$$T := [T_{jk}]_{n\times n}\,, \text{ where } T_{jk} := \mathcal{L}(L_j, L_k). \tag{2.1}$$

T is a hermitian matrix and assumed to be positive semi-definite.

Theorem 2.1. *Suppose that a smooth real manifold M^{2n+1} admits a pseudo-convex CR structure $(H(M), J)$. Let T be a matrix representation of the Levi-form defined locally as in* (2.1) *and*

$$det\ (T - \lambda I) := \sum_{k=0}^{n-1} A_k(-\lambda)^k + (-\lambda)^n$$

be the characteristic polynomial of T. Then for each $q = 1, \ldots, n$, $\mathcal{S}_q$ is given as a common zero set of A_j for all j with $0 \le j \le q-1$.

Proof. We fix a point $x \in M$ and a neighborhood U of x. By a unitary change of basis $\mathcal{L}$ can be diagonalized. The pseudo-convexity implies that the diagonal elements are non-negative, and the number of zeros in the diagonal is the nullity of the Levi-form. To be precise, define a hermitian metric on $H^{1,0}_x(M)$ by declaring L_j's as in (1.1) are orthonormal. By the spectral theorem, there exists an orthonormal basis $Q_1, \ldots, Q_n$ that are eigen-vectors of T. Let d_j be the eigen-value corresponding to Q_j. Setting $Q_j = \sum_{k=1}^n Q_j^k L_k$ and $Q = [Q_j^k]$ we have

$$QT(x)Q^* = QT(x)Q^{-1} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix},\ d_j \ge 0,$$

so that

$$\mathcal{L}(Q_j, Q_k) = \begin{cases} d_j, & \text{if } j = k, \\ 0, & \text{if } j \ne k. \end{cases}$$

Suppose $x \in \mathcal{S}_1$. Then the Levi-form $\mathcal{L}(x)$ has a non-trivial null vector. Setting it by

$$V = \sum_{j=1}^n b^j Q_j,$$

we have

$$\begin{aligned} 0 &= \mathcal{L}(V, V) \\ &= \sum_{j=1}^n |b^j|^2 d_j. \end{aligned} \tag{2.2}$$

Since $d_j \geq 0$, (2.2) implies that at least one of d_j's is zero. Therefore,

$$\mathcal{A}_0(x) = d_1 \cdots d_n = 0.$$

Now suppose $x \in \mathcal{S}_q$. Then the Levi-form $\mathcal{L}(x)$ has at least q independent null vectors. Setting them by

$$V_\lambda = \sum_{j=1}^{n} b_\lambda^j Q_j,$$

we have

$$\begin{aligned} 0 &= \mathcal{L}(V_\lambda, V_\lambda) \\ &= \sum_{j=1}^{n} |b_\lambda^j|^2 d_j, \quad \lambda = 1, \dots, q. \end{aligned} \tag{2.3}$$

We restate (2.3) in matrices as

$$\underbrace{\begin{bmatrix} |b_1^1|^2 & \cdots & |b_1^n|^2 \\ \vdots & & \vdots \\ |b_q^1|^2 & \cdots & |b_q^n|^2 \end{bmatrix}}_{B} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.4}$$

Since V_λ's are independent

$$\beta := \begin{bmatrix} b_1^1 & \cdots & b_1^n \\ \vdots & & \vdots \\ b_q^1 & \cdots & b_q^n \end{bmatrix} \tag{2.5}$$

has rank q, so that β has q independent columns, which implies that q columns of B are non-zeros. Since $d_j \geq 0$, for all $j = 1, \dots, n$, (2.4) implies that at least q of d_j's are zeros. For $k = 0, 1, \dots, q-1$, $A_k(x)$ is the symmetric polynomial of degree $n-k$ in $d_1, \dots, d_n$, therefore, $\mathcal{A}_k(x) = 0$.

Conversely, if $A_0(x) = \cdots = A_{q-1}(x) = 0$, then there are at least q zeros in $\{d_1, \dots, d_n\}$, say

$$d_1 = \cdots = d_q = 0.$$

Then $Q_1, \dots, Q_q$ are null vectors of $\mathcal{L}(x)$. □

3. Integrable CR structures

The CR structure $(H(M), J)$ is said to be *integrable* if the module of smooth sections of $H^{1,0}(M)$, denoted by $\Gamma(H^{1,0}(M))$, is closed under the Lie bracket:

$$[L, L'] \in \Gamma(H^{1,0}(M)), \quad \forall L, L' \in \Gamma(H^{1,0}(M)). \tag{3.1}$$

(3.1) is equivalent to

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY], \quad \forall X, Y \in \Gamma(H(M)). \tag{3.2}$$

In this section M is assumed to be a CR manifold with an integrable CR structure. Then if a submanifold $\mathcal{S}$ of M is J-invariant, namely, if $JT(\mathcal{S}) \subset T(\mathcal{S})$, then $\mathcal{S}$ is a complex manifold with the complex structure J by the following.

Theorem 3.1 (Newlander-Nirenberg [10]). *Let $\mathcal{S}$ be a smooth real manifold of dimension $2m$ with a smooth almost complex structure J that is integrable in the sense that any vector fields X and Y defined locally on an open neighborhood $U \subset \mathcal{S}$ satisfy* (3.2)*. Then $\mathcal{S}$ is covered by coordinate patches with complex coordinates in which the coordinates in overlapping patches are related by holomorphic transformations, so that for a local coordinate system $(z_1, \dots, z_m)$, $z_j = x_j + \sqrt{-1}y_j$,*

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j},$$
$$J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \quad j = 1, \dots, m.$$

For the definition of the tangential Cauchy-Riemann operator $\bar{\partial}_b$ we refer the readers to [4]. For a function f, $\bar{\partial}_b f$ is a section of $H^{*0,1}(M) := (H^{0,1}(M))^*$ defined by

$$(\bar{\partial}_b f)(V) = df(V) \quad \text{for any } V \in \Gamma(H^{0,1}(M)).$$

Let $\pi_{1,0}$, $\pi_{0,1}$ and π_t be the projections of $\mathbb{C} \otimes T^*(M)$ onto each component of

$$\mathbb{C} \otimes T^*(M) = H^{*1,0}(M) \oplus H^{*0,1}(M) \oplus \langle \theta \rangle.$$

Then

$$\bar{\partial}_b f := \pi_{0,1}\, df,$$
$$\partial_b f := \pi_{1,0}\, df.$$

Theorem 3.2. *Suppose that M^{2n+1} is a smooth manifold with integrable CR structure $(H(M), J)$. For an integer k, $0 \le k \le n$, let $\rho_1, \dots, \rho_{2k+1}$ be a non-degenerate system of real-valued functions defined locally on an open subset $U \subset M$. Let $\mathcal{S}$ be the common zero set of ρ_j, $j = 1, \dots, 2k+1$. Then $\mathcal{S}$ is a complex manifold of complex dimension $n-k$ with J as its complex structure if and only if the linear span of $\{\bar{\partial}_b \rho_1, \dots, \bar{\partial}_b \rho_{2k+1}\}$ has constant rank k on $\mathcal{S}$.*

Proof. Suppose that $\mathcal{S}$ is a complex manifold of complex dimension $n-k$. Then

$$\mathbb{C} \otimes T(\mathcal{S}) = T^{1,0}(\mathcal{S}) \oplus T^{0,1}(\mathcal{S}), \tag{3.3}$$

where $T^{1,0}(\mathcal{S}) := H^{1,0}(M) \cap \mathbb{C} \otimes T(\mathcal{S})$ and so forth. Each of the direct summand of the right-hand side of (3.3) has rank $n-k$. For a local section V of $H^{0,1}(M)$ it is obvious that $V \in T^{0,1}(\mathcal{S})$ if and only if

$$d\rho_j(V) = 0, \quad j = 1, \dots, 2k+1. \tag{3.4}$$

Since $V \in H^{0,1}(M)$, (3.4) is equivalent to

$$\bar{\partial}_b \rho_j(V) = 0, \quad j = 1, \dots, 2k+1.$$

We note that there are $n-k$ independent vectors $V_1, \ldots, V_{n-k}$ in $T^{0,1}(\mathcal{S})$. Therefore, the linear span

$$\langle \bar\partial_b \rho_1, \ldots, \bar\partial_b \rho_{2k+1} \rangle (x) \subset H_x^{*0,1}(M), \quad \forall x \in \mathcal{S} \tag{3.5}$$

has rank $n-(n-k)=k$.

Conversely, suppose that (3.5) has rank k at $x \in \mathcal{S}$. Assuming the first k elements of $\{\bar\partial_b \rho_1, \ldots, \bar\partial_b \rho_{2k+1}\}$ are independent, we choose 1-forms $\omega^1, \ldots, \omega^{n-k} \in H^{*0,1}(M)$ on a neighborhood $U \subset M$ of x so that $H^{*0,1}(M)$ is spanned by

$$\{\bar\partial_b \rho_1, \ldots, \bar\partial_b \rho_k, \omega^1, \ldots, \omega^{n-k}\}.$$

Then

$$\underbrace{\partial_b \rho_1, \ldots, \partial_b \rho_k, \bar\omega^1, \ldots, \bar\omega^{n-k}}_{H^{*1,0}(M)}, \underbrace{\bar\partial_b \rho_1, \ldots, \bar\partial_b \rho_k, \omega^1, \ldots, \omega^{n-k}}_{H^{*0,1}(M)}, \theta \tag{3.6}$$

is a local basis of the whole cotangent bundle $\mathbb{C} \otimes T^*(M)$. Let

$$\underbrace{L_1, \ldots, L_k, \overline{V}_1, \ldots, \overline{V}_{n-k}}_{H^{1,0}(M)}, \underbrace{\overline{L}_1, \ldots, \overline{L}_k, V_1, \ldots, V_{n-k}}_{H^{0,1}(M)}, T \tag{3.7}$$

be the dual basis of (3.6) for $\mathbb{C} \otimes T(M)$. Now consider the bundle of maximal complex subspaces of $T(S)$ given by

$$H(S) := T(S) \cap JT(S).$$

Let $\mathcal{V}(\mathcal{S})$ be a sub-bundle of $T(\mathcal{S})$, locally defined on the neighborhood $U \cap \mathcal{S}$ of x, so that

$$T(S) = H(S) \oplus \mathcal{V}(\mathcal{S}).$$

Then the J-invariance of $\mathcal{S}$ comes from the dimension count of

$$\mathbb{C} \otimes T(\mathcal{S}) = H^{1,0}(\mathcal{S}) \oplus H^{0,1}(\mathcal{S}) \oplus (\mathbb{C} \otimes \mathcal{V}(\mathcal{S}))\,. \tag{3.8}$$

To count the dimension of $H^{0,1}(\mathcal{S})$, let $V := X + \sqrt{-1}JX \in H^{0,1}(M)$, X is a real vector, be any one of $V_1, \ldots, V_{n-k}$ in (3.7) and ρ be any one of $\rho_1, \ldots, \rho_{2k+1}$. We have

$$\begin{aligned} 0 = \bar\partial_b \rho(V) &= d\rho(V) \\ &= d\rho(X + \sqrt{-1}JX) \end{aligned}$$

which yields

$$d\rho(X) = 0 \quad \text{and} \quad d\rho(JX) = 0$$

so that X and JX are tangent to $\mathcal{S}$. Therefore, $V_1, \ldots, V_{n-k}$ are elements of $H^{0,1}(\mathcal{S})$. Hence $H^{0,1}(\mathcal{S})$ has complex dimension $\geq n-k$. But each fibre in the left-hand side of (3.8) has complex dimension $2(n-k)$. Therefore, the last component of the right-hand side of (3.8) has rank zero, which implies that $\mathcal{V}(\mathcal{S}) = 0$. Thus we have

$$T(\mathcal{S}) = H(\mathcal{S}),$$

which means that $\mathcal{S}$ is J-invariant and the conclusion follows from Theorem 3.1. □

If $(M, H(M), J)$ is pseudo-convex and $\mathcal{S}$ has real dimension $\geq 2q$, then the defining functions ρ_j's of Theorem 3.2 can be found from the coefficients and their radicals of the characteristic polynomial of the Levi-form. We adopt from [9] the following.

Definition 3.3. Let $\mathcal{J}$ be a subset of the ring $C^\infty(x_0)$ of germs at $x_0 \in M$ of smooth functions. The *real radical* of $\mathcal{J}$, denoted by $\sqrt[R]{\mathcal{J}}$, is the set of all $g \in C^\infty(x_0)$ such that there exist an integer m and an $f \in \mathcal{J}$ so that

$$|g|^m \leq |f|$$

on some neighborhood of x_0.

$\sqrt[R]{\mathcal{J}}$ is an ideal. We have:

Corollary 3.4. *Let M^{2n+1} be a smooth CR manifold with pseudo-convex integrable CR structure $(H(M), J)$. For an integer q, $0 \leq q \leq n$, let $\mathcal{S}_q$ be the set of points of M where the Levi-form has nullity $\geq q$. Suppose that the real dimension of $\mathcal{S}_q$ is greater than or equal to $2q$ and that there is a non-degenerate set of real-valued functions $\rho_1, \ldots, \rho_{2(n-q)+1}$ that generates $\sqrt[R]{\{A_0, \ldots, A_{q-1}\}}$ and satisfies*

$$\text{rank } \langle \bar{\partial}_b \rho_1, \ldots, \bar{\partial}_b \rho_{2(n-q)+1} \rangle = n - k$$

on the common zero set $\mathcal{S}$ of ρ_j's. Then $\mathcal{S}$ is a complex manifold of complex dimension q.

Proof. On $\mathcal{S}_q$, $A_j = 0$ for all $j = 0, 1, \ldots, q-1$. Since $\mathcal{S}$ is the common zero set of ρ_j's we have $\mathcal{S} \subset \mathcal{S}_q$. The conclusion follows from Theorem 3.2. □

Remark 3.5. In analytic (C^ω) category we check a finite set of non-degenerate functions $\vec{\rho} := (\rho_1, \rho_2, \ldots, \rho_{2(n-q)+1})$ including the generators of

$$\sqrt[R]{\{A_0, \ldots, A_{q-1}\}}$$

and determine the existence of a complex manifold by showing that

$$\bar{\partial}_b \rho_1 \wedge \cdots \wedge \bar{\partial}_b \rho_{n-q} \neq 0,$$
$$\bar{\partial}_b \rho_1 \wedge \cdots \wedge \bar{\partial}_b \rho_{n-q} \wedge \bar{\partial}_b \rho_\ell \equiv 0 \mod (\vec{\rho}), \quad \forall \ell.$$

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