Bull. Korean Math. Soc. **56** (2019), No. 1, pp. 151–167
https://doi.org/10.4134/BKMS.b180152
pISSN: 1015-8634 / eISSN: 2234-3016

FINDING RESULTS FOR CERTAIN RELATIVES OF THE APPELL POLYNOMIALS

Mahvish Ali and Subuhi Khan

Abstract. In this article, a hybrid family of polynomials related to the Appell polynomials is introduced. Certain properties including quasi-monomiality, differential equation and determinant definition of these polynomials are established. Further, applications of Carlitz's theorem to these polynomials and to certain other related polynomials are considered. Examples providing the corresponding results for some members belonging to this family are also considered.

1. Introduction and preliminaries

In 1956 Boas and Buck [7] studied a large class of generating functions of polynomial sets. Some of their work appeared also in their earlier mimeographed reports which are not generally available. A rough statement of one of the main results in Boas and Buck [7] is given below:

A polynomial set $\{p_n(x)\}_{n\geq 0}$ is said to be a Boas-Buck polynomial set, if it has the following generating function [7]:

$$A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \tag{1.1}$$

where $A(t)$, $\psi(t)$, $H(t)$ are power series such that

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad a_0 \neq 0; \tag{1.2}$$

$$\psi(t) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!}, \quad \gamma_n \neq 0, \quad \forall\ n, \tag{1.3}$$

with $\psi(t)$ not a polynomial and

$$H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}, \quad h_0 = 0 \quad \text{and} \quad h_1 \neq 0. \tag{1.4}$$

Received February 17, 2018; Revised July 11, 2018; Accepted August 16, 2018.
2010 *Mathematics Subject Classification.* 33C45, 33C99, 33E20.
Key words and phrases. Appell polynomials, Boas-Buck polynomials, operational techniques, monomiality principle, Carlitz's theorem.

©2019 Korean Mathematical Society

Remark 1.1. The Boas-Buck polynomials set include such general classes as Brenke polynomials $Y_n(x)$ [9] (for $H(t) = t$), Sheffer polynomials $s_n(x)$ [20] (for $\psi(t) = \exp(t)$), Appell polynomials $A_n(x)$ [4] (for $H(t) = t,\ \psi(t) = \exp(t)$) and some particular sets as certain constant multiples of the Jacobi, Laguerre and Hermite polynomials [2].

An important contribution on obtaining generating functions of the Boas and Buck type for orthogonal polynomials is given by Mourad Ismail [14]. Ceratin other contributions related to these polynomials are given in [5, 6, 11].

The Appell polynomials [4] constitute an important class of polynomials because of their remarkable applications in numerous fields. For recent applications of Appell polynomials in fields like probability theory and statistics, approximating 3D-mappings, quantum physics, see [3, 8, 17, 18, 22, 24].

In 1880, Appell [4] introduced a sequence of polynomials $A_n(x)$, called the Appell polynomials satisfying the relation

$$\frac{d}{dx}A_n(x) = nA_{n-1}(x),\ n = 1, 2, \ldots. \tag{1.5}$$

The Appell polynomials can also be defined by means of the following exponential generating function:

$$\mathcal{G}(t)\exp(xt) = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1.6}$$

where $\mathcal{G}(t)$ has the expansion

$$\mathcal{G}(t) = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!},\ \alpha_0 \neq 0. \tag{1.7}$$

Many properties of conventional and generalized polynomials have been shown to be derivable, in a straightforward way, within operational framework which is a consequence of the monomiality principle. The suggestion of the concept of poweroid by Steffensen [23] is behind the idea of monomiality. Further, Dattoli [13] reformulated and developed the principle of monomiality.

The strategy underlined this point of view is simple but efficient. According to the monomiality principle a polynomial set $r_n(x)$ ($n \in \mathbb{N}$, $x \in \mathbb{C}$) is quasi-monomial, if there exist two operators Φ^+ and Φ^-, called multiplicative and derivative operators respectively, which when acting on the polynomials $r_n(x)$ yield:

$$\Phi^+\{r_n(x)\} = r_{n+1}(x), \tag{1.8}$$

$$\Phi^-\{r_n(x)\} = n\ r_{n-1}(x). \tag{1.9}$$

The operators Φ^+ and Φ^- satisfy the commutation relation

$$[\Phi^-, \Phi^+] = \hat{1} \tag{1.10}$$

and thus display the Weyl group structure.

If Φ^+ and Φ^- have differential realization, then it easily shown that the differential equation satisfied by $r_n(x)$ is

$$\Phi^+\Phi^-\{r_n(x)\} = n\ r_n(x). \tag{1.11}$$

Assuming here and in the sequel $r_0(x) = 1$, then $r_n(x)$ can be explicitly constructed as:

$$r_n(x) = {\Phi^+}^n\{1\} \tag{1.12}$$

and consequently the generating function of $r_n(x)$ can be cast in the form

$$G(x,t) = \exp(t\Phi^+)\{1\} = \sum_{n=0}^{\infty} r_n(x)\frac{t^n}{n!}. \tag{1.13}$$

The Boas-Buck polynomials set defined by equation (1.1) is quasi-monomial under the action of the following multiplicative and derivative operators [5]:

$$\Phi_p^+ = \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))} + xD_xH'(H^{-1}(\sigma))\sigma^{-1}, \tag{1.14}$$

$$\Phi_p^- = H^{-1}(\sigma), \tag{1.15}$$

where $\sigma \in \Lambda^{(-1)}$ is given by

$$\sigma(1) = 0 \text{ and } \sigma(x^n) = \frac{\gamma_{n-1}}{\gamma_n}x^{n-1},\ n = 1, 2, \ldots \tag{1.16}$$

and $\Lambda^{(j)}$, $j \in \mathbb{Z}$ denotes the space of operators acting on analytic functions that augment or reduce the degree of every polynomial by exactly j for $j \geq 0$ or $j \leq 0$, respectively.

In this article, the Boas-Buck-Appell family is introduced and studied through different aspects. In Section 2, some important properties including quasi-monomiality and determinant definition of this family are established. In Section 3, Carlitz's theorem for mixed generating functions is extended to the Boas-Buck-Appell family of order λ and also to some of its members. In Section 4, examples of some members belonging to this family are given. A recursion relation characterizing the Boas-Buck-Appell polynomials is derived in the last section.

2. Boas-Buck-Appell polynomials

In this Section, the Boas-Buck-Appell polynomials are introduced by means of the generating function. Further, quasi-monomial properties and determinant definition of these polynomials are established.

On replacing x by the multiplicative operator Φ_p^+ of the Boas-Buck polynomials $p_n(x)$ in the l.h.s. of generating function (1.6) of the Appell polynomials $A_n(x)$, the following expression is obtained:

$$\mathcal{G}(t)\exp\left(\Phi_p^+ t\right) = \sum_{n=0}^{\infty} A_n\left(\Phi_p^+\right)\frac{t^n}{n!}, \tag{2.1}$$

which by virtue of equation (1.13) becomes

$$\mathcal{G}(t)\sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} A_n\left(\Phi_p^+\right)\frac{t^n}{n!}. \tag{2.2}$$

Using equation (1.1) in the l.h.s of equation (2.2) and denoting $A_n\left(\Phi_p^+\right)$ in the r.h.s. by the resultant Boas-Buck-Appell polynomials ${}_pA_n(x)$, that is

$$A_n\left(\Phi_p^+\right) = A_n\left(\frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))} + xD_xH'(H^{-1}(\sigma))\sigma^{-1}\right) = {}_pA_n(x), \tag{2.3}$$

we find the following generating function for the Boas-Buck-Appell polynomials:

$$\mathcal{G}(t)A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} {}_pA_n(x)\frac{t^n}{n!}. \tag{2.4}$$

In consideration of $H(t) = t$, the Boas-Buck polynomials $p_n(x)$ reduce to the Brenke polynomials $Y_n(x)$. Accordingly, taking $H(t) = t$ in (2.4), we get the following generating function for the Brenke-Appell polynomials ${}_YA_n(x)$:

$$\mathcal{G}(t)A(t)\psi(xt) = \sum_{n=0}^{\infty} {}_YA_n(x)\frac{t^n}{n!}. \tag{2.5}$$

Similarly, taking $\psi(t) = \exp(t)$ and $H(t) = t$; $\psi(t) = \exp(t)$ in (2.4), we get the following generating functions for the Sheffer-Appell polynomials ${}_sA_n(x)$ [16] and 2-Iterated Appell polynomials $A_n^{[2]}(x)$ [15], respectively:

$$\mathcal{G}(t)A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} {}_sA_n(x)\frac{t^n}{n!}, \tag{2.6}$$

$$\mathcal{G}(t)A(t)\exp(xt) = \sum_{n=0}^{\infty} A_n^{[2]}(x)\frac{t^n}{n!}. \tag{2.7}$$

Further, using expansion (1.7) of $\mathcal{G}(t)$ in the l.h.s. of equation (2.2), simplifying and then equating the coefficients of like powers of t in both sides of the resultant equation, we get the following series definition of the Boas-Buck-Appell polynomials ${}_pA_n(x)$:

$${}_pA_n(x) = \sum_{k=0}^{n}\binom{n}{k}\alpha_k\ p_{n-k}(x). \tag{2.8}$$

In order to establish the quasi-monomial properties of Boas-Buck-Appell polynomials ${}_pA_n(x)$, we prove the following results:

Theorem 2.1. *The Boas-Buck-Appell polynomials ${}_pA_n(x)$ are quasi-monomial with respect to the following multiplicative and derivative operators*:

$$\Phi_{pA}^+ = xH'(H^{-1}(\sigma))D_x\sigma^{-1} + \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))} + \frac{\mathcal{G}'(H^{-1}(\sigma))}{\mathcal{G}(H^{-1}(\sigma))}, \tag{2.9}$$

$$\Phi^{-}_{pA} = H^{-1}(\sigma). \tag{2.10}$$

Proof. Notice that equation (1.16) is equivalent to relation [5]

$$\sigma\psi(xt) = t\psi(xt), \tag{2.11}$$

so that, we can write

$$H^{-1}(\sigma)\mathcal{G}(t)A(t)\psi(xH(t)) = t\mathcal{G}(t)A(t)\psi(xH(t)). \tag{2.12}$$

Differentiating equation (2.4) partially with respect to t, the following expression is obtained:

$$\left(x\frac{H'(t)}{H(t)}D_x + \frac{A'(t)}{A(t)} + \frac{\mathcal{G}'(t)}{\mathcal{G}(t)}\right)\sum_{n=0}^{\infty} {}_pA_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_pA_{n+1}(x)\frac{t^n}{n!}, \tag{2.13}$$

which on using equation (2.12) and equating the coefficients of like powers of t in both sides of the resultant equation gives

$$\begin{aligned}&\left(xH'(H^{-1}(\sigma))D_x\sigma^{-1} + \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))} + \frac{\mathcal{G}'(H^{-1}(\sigma))}{\mathcal{G}(H^{-1}(\sigma))}\right){}_pA_n(x)\\ &= {}_pA_{n+1}(x).\end{aligned} \tag{2.14}$$

In view of equations (1.8) and (2.14), assertion (2.9) follows.

Again, since the polynomials sets generated, respectively, by $G(x,t)$ and $A(t)G(x,t)$, are associated with same lowering operator [5, Remark 3.2, p. 66]. Therefore, in view of equations (1.1) and (2.4), assertion (2.10) follows. □

Remark 2.1. Using equations (2.9) and (2.10) in monomiality equation (1.11), we deduce the following consequence of Theorem 2.1:

Corollary 2.1. *The Boas-Buck-Appell polynomials* ${}_pA_n(x)$ *satisfy the following differential equation*:

$$\left(xH'(H^{-1}(\sigma))D_x\sigma^{-1}H^{-1}(\sigma) + \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))}H^{-1}(\sigma) + \frac{\mathcal{G}'(H^{-1}(\sigma))}{\mathcal{G}(H^{-1}(\sigma))}H^{-1}(\sigma) - n\right){}_pA_n(x) = 0. \tag{2.15}$$

Remark 2.2. Taking $H(t) = t$ in equations (2.9), (2.10) and (2.15), the following consequences of Theorem 2.1 are deduced:

Corollary 2.2. *The Brenke-Appell polynomials* ${}_YA_n(x)$ *are quasi-monomial with respect to the following multiplicative and derivative operators*:

$$\Phi^{+}_{YA} = xD_x\sigma^{-1} + \frac{A'(\sigma)}{A(\sigma)} + \frac{\mathcal{G}'(\sigma)}{\mathcal{G}(\sigma)}, \tag{2.16}$$

$$\Phi^{-}_{YA} = \sigma \tag{2.17}$$

and satisfy the following differential equation:

$$\left(xD_x + \frac{A'(\sigma)}{A(\sigma)}\sigma + \frac{\mathcal{G}'(\sigma)}{\mathcal{G}(\sigma)}\sigma - n\right){}_YA_n(x) = 0. \tag{2.18}$$

Remark 2.3. Taking $\psi(t) = \exp(t)$ (for this case $\sigma = D_x$) in equations (2.9), (2.10) and (2.15), the analogous results for the Sheffer-Appell polynomials ${}_sA_n(x)$ are obtained [16]. Again, taking $H(t) = t$, $\psi(t) = \exp(t)$ (for this case $\sigma = D_x$) in equations (2.9), (2.10) and (2.15), the analogous results for the 2-iterated Appell polynomials $A_n^{[2]}(x)$ are obtained [15] .

In the last few years, the interest in Appell polynomials and their applications in different fields has significantly increased. Some authors were concerned with finding new characterizations of Appell polynomials through new approaches, see for example [1, 25].

The determinant form of the Appell polynomials is obtained by Costabile and Longo in [12]. By using a similar approach [12, p. 1531 (Theorem 6)] and taking help of equations (1.1) and (2.4), the following determinant definition for the Boas-Buck-Appell polynomials ${}_pA_n(x)$ is obtained:

Definition 2.1. The Boas-Buck-Appell polynomials ${}_pA_n(x)$ of degree n are defined by

$$ {}_pA_0(x) = \frac{1}{\beta_0}, \tag{2.19}$$

(2.20)

$$ {}_pA_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & p_1(x) & p_2(x) & \cdots & p_{n-1}(x) & p_n(x) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, \ldots, $$

where

$$\beta_0 = \frac{1}{\alpha_0} \neq 0,$$
$$\beta_n = -\frac{1}{\alpha_0}\Big(\sum_{k=1}^{n}\binom{n}{k}\alpha_k\ \beta_{n-k}\Big)\ (n = 1, 2, \ldots)$$

and $p_n(x)$ $(n = 1, 2, \ldots\ )$ are the Boas-Buck polynomials of degree n defined by equation (1.1).

Replacing $p_n(x)$ by $Y_n(x)(n = 1, 2, \ldots)$ in the first row in determinant on the r.h.s. of equation (2.20), the following determinant definition of the Brenke-Appell polynomials ${}_YA_n(x)$ is deduced:

Definition 2.2. The Brenke-Appell polynomials ${}_YA_n(x)$ are defined by

$$ {}_YA_0(x) = \frac{1}{\beta_0}, \tag{2.21}$$

(2.22)

$$
{}_Y A_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}}
\begin{vmatrix}
1 & Y_1(x) & Y_2(x) & \cdots & Y_{n-1}(x) & Y_n(x) \\
\beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\
0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\
0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\
. & . & . & \cdots & . & . \\
. & . & . & \cdots & . & . \\
0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1
\end{vmatrix}, \quad n = 1, 2, \ldots,
$$

where

$$
\beta_0 = \frac{1}{\alpha_0} \neq 0,
$$

$$
\beta_n = -\frac{1}{\alpha_0}\Big(\sum_{k=1}^{n} \binom{n}{k} \alpha_k \ \beta_{n-k}\Big) \ (n = 1, 2, \ldots)
$$

and $Y_n(x)$ $(n = 1, 2, \ldots)$ are the Brenke polynomials [9].

In the next Section, Carlitz Theorem is derived for the Boas-Buck-Appell polynomials ${}_pA_n(x)$.

3. Carlitz's Theorem for the Boas-Buck-Appell polynomials

The Appell polynomials appear in different applications in pure and applied mathematics. The typical examples of Appell polynomials besides the trivial example $\{x^n\}_{n=0}^{\infty}$ are the Bernoulli, Euler and Hermite polynomials [2]. In particular, the Hermite polynomials sequence defined by the generating function

$$
\exp\left(xt - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} \frac{He_n(x)t^n}{n!}, \ |t| < \infty; \ |x| < \infty, \tag{3.1}
$$

is a unique sequence of Appell polynomials that is also orthogonal with respect to a positive measure. We list certain members belonging to the Appell family (which have an order) in Table 1:

Corresponding to the special polynomials given in Table 1, we get the resultant Boas-Buck-Appell polynomials of order λ. We denote these polynomials by ${}_pA_n^{(\lambda)}(x)$. Expressing $\mathcal{G}(t)$ as $\mathfrak{B}(t))^{\lambda}$ in equation (2.4), the generating function for the Boas-Buck-Appell polynomials of order λ can be written as:

$$
(\mathfrak{B}(t))^{\lambda} A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} {}_pA_n^{(\lambda)}(x)\frac{t^n}{n!}. \tag{3.2}
$$

Carlitz derived generating functions for certain one-and two-parameter coefficients [10, p. 521 (Theorem 1)]. Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. Motivated by the importance of generating functions, here we

TABLE 1. Certain members belonging to the Appell family

S. No.	Name of the polynomials	$\mathcal{G}(t)$	Generating function
I.	Hermite polynomials of variance ν [21]	$e^{\frac{-\nu t^2}{2}}$ $(\nu \neq 0)$	$e^{-\frac{\nu t^2}{2}} e^{xt} = \sum\limits_{n=0}^{\infty} H_k^{(\nu)}(x) \frac{t^n}{n!}$
II.	Bernoulli polynomials of order a [21]	$\left(\frac{t}{e^t-1}\right)^a$ $(a \neq 0)$	$\left(\frac{t}{e^t-1}\right)^a e^{xt} = \sum\limits_{n=0}^{\infty} B_k^{(a)}(x) \frac{t^n}{n!}$
III.	Euler polynomials of order a [21]	$\left(\frac{2}{e^t+1}\right)^a$ $(a \neq 0)$	$\left(\frac{2}{e^t+1}\right)^a e^{xt} = \sum\limits_{n=0}^{\infty} E_k^{(a)}(x) \frac{t^n}{n!}$

extend the Carlitz theorem for the Boas-Buck-Appell polynomials of order λ by proving the following result:

Theorem 3.1. *Let $\mathfrak{B}(t)$, $A(t)$, $\psi(t)$, $H(t)$ be arbitrary functions which are analytic in the neighbourhood of the origin such that*

$$\mathfrak{B}(0) = b_0,\ A(0) = a_0,\ \psi(0) = \gamma_0,\ H(0) = h_0. \tag{3.3}$$

Then, for arbitrary μ independent of t, the following generating function for the Boas-Buck-Appell polynomials of order λ, ${}_pA_n^{(\lambda)}(x)$ holds true:

$$\sum_{n=0}^{\infty} {}_pA_n^{(\lambda+\mu n)}(x)\frac{u^n}{n!} = \frac{(\mathfrak{B}(z))^{\lambda+1}A(z)\psi(xH(z))}{\mathfrak{B}(z) - \mu z\mathfrak{B}'(z)}, \tag{3.4}$$

where $u = z(\mathfrak{B}(z))^{-\mu}$.

Proof. Applying Taylor's theorem in equation (3.2), we have

$$ {}_pA_n^{(\lambda)}(x) = D_t^n\{(\mathfrak{B}(t))^{\lambda}A(t)\psi(xH(t))\}\big|_{t=0}, \qquad D_t \equiv \frac{d}{dt}, \tag{3.5}$$

so that

$$ {}_pA_n^{(\lambda+\mu n)}(x) = D_t^n\{(\mathfrak{B}(t))^{\lambda+\mu n}A(t)\psi(xH(t))\}|_{t=0}. \tag{3.6}$$

Consider

$$ f(t) = (\mathfrak{B}(t))^{\lambda}A(t)\psi(xH(t)) \tag{3.7}$$

and

$$ \phi(t) = (\mathfrak{B}(t))^{\mu}. \tag{3.8}$$

Then equation (3.6) can be expressed as:

$$ {}_pA_n^{(\lambda+\mu n)}(x) = D_t^n\{f(t)\phi(t)^n\}\big|_{t=0}. \tag{3.9}$$

Next, consider the Lagrange's expansion [19, p. 146]:

$$\frac{f(z)}{1-u\phi'(z)} = \sum_{n=0}^{\infty} \frac{u^n}{n!}[D_t^n\{f(t)\phi(t)^n\}]\big|_{t=0}, \tag{3.10}$$

where the functions $f(t)$ and $\phi(t)$ are analytic about the origin and z is given by

$$z = u\phi(z),\ \ \phi(0) \neq 0. \tag{3.11}$$

Using equation (3.9) in the r.h.s. of equation (3.10), we have

$$\frac{f(z)}{1-u\phi'(z)} = \sum_{n=0}^{\infty} {}_pA_n^{(\lambda+\mu n)}(x)\frac{u^n}{n!}. \tag{3.12}$$

In view of equations (3.7), (3.8) and (3.11), assertion (3.4) follows. □

We note that for $\mu = 0$, generating function (3.4) reduces to generating function (3.2).

Remark 3.1. In view of Remark 1.1, the applications of Carlitz's theorem for the subclasses of the Boas-Buck-Appell polynomials of order λ are deduced as following consequences of Theorem 3.1:

Corollary 3.1. *Let $\mathfrak{B}(t)$, $A(t)$, $\psi(t)$ be arbitrary functions which are analytic in the neighbourhood of the origin such that*

$$\mathfrak{B}(0) = b_0,\ \ A(0) = a_0,\ \ \psi(0) = \gamma_0. \tag{3.13}$$

Then, for arbitrary μ independent of t, the following generating function for the Brenke-Appell polynomials of order λ, ${}_YA_n^{(\lambda)}(x)$ holds true:

$$\sum_{n=0}^{\infty} {}_YA_n^{(\lambda+\mu n)}(x)\frac{u^n}{n!} = \frac{(\mathfrak{B}(z))^{\lambda+1}A(z)\psi(xz)}{\mathfrak{B}(z)-\mu z\mathfrak{B}'(z)}, \tag{3.14}$$

where $u = z(\mathfrak{B}(z))^{-\mu}$.

Corollary 3.2. *Let $\mathfrak{B}(t)$, $A(t)$, $H(t)$ be arbitrary functions which are analytic in the neighbourhood of the origin such that*

$$\mathfrak{B}(0) = b_0,\ \ A(0) = a_0,\ \ H(0) = h_0. \tag{3.15}$$

Then, for arbitrary μ independent of t, the following generating function for the Sheffer-Appell polynomials of order λ, ${}_sA_n^{(\lambda)}(x)$ holds true:

$$\sum_{n=0}^{\infty} {}_sA_n^{(\lambda+\mu n)}(x)\frac{u^n}{n!} = \frac{(\mathfrak{B}(z))^{\lambda+1}A(z)\exp(xH(z))}{\mathfrak{B}(z)-\mu z\mathfrak{B}'(z)}, \tag{3.16}$$

where $u = z(\mathfrak{B}(z))^{-\mu}$.

Corollary 3.3. *Let $\mathfrak{B}(t)$, $A(t)$ be arbitrary functions which are analytic in the neighbourhood of the origin such that*

$$\mathfrak{B}(0) = b_0, \; A(0) = a_0. \tag{3.17}$$

Then, for arbitrary μ independent of t, following generating function for the 2-iterated Appell polynomials of order λ, $(A_n^{[2]})^{(\lambda)}(x)$ holds true:

$$\sum_{n=0}^{\infty} (A_n^{[2]})^{(\lambda+\mu n)}(x) \frac{u^n}{n!} = \frac{(\mathfrak{B}(z))^{\lambda+1} A(z) \exp(xz)}{\mathfrak{B}(z) - \mu z \mathfrak{B}'(z)}, \tag{3.18}$$

where $u = z(\mathfrak{B}(z))^{-\mu}$.

In the next section, examples of some members belonging to the Boas-Buck-Appell family are considered.

4. Examples

The results for some members belonging to the Boas-Buck-Appell family ${}_pA_n(x)$ are established by considering the following examples:

Example 4.1. Since, for $\mathcal{G}(t) = e^{\frac{-\nu t^2}{2}}$, the Appell polynomial $A_n(x)$ becomes the Hermite polynomials of variance ν (Table 1 (I)). Therefore, for the same choice of $\mathcal{G}(t)$, the Boas-Buck-Appell polynomials reduce to the Boas-Buck-Hermite polynomials of variance ν denoted by ${}_pH_n^{(\nu)}(x)$.

Thus, by taking the above expression of $\mathcal{G}(t)$ in equations (2.4), (2.9), (2.10), (2.15) and (3.4), we find the following results for the Boas-Buck-Hermite polynomials of variance ν:

Table 2. Results for ${}_pH_n^{(\nu)}(x)$

Generating function	$e^{\frac{-\nu t^2}{2}} A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} {}_pH_n^{(\nu)}(x)\frac{t^n}{n!}$
Multiplicative and derivative operators	$\Phi^+_{{}_pH^{(\nu)}} = xH'(H^{-1}(\sigma))D_x\sigma^{-1} + \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))} - \nu H^{-1}(\sigma)$, $\Phi^-_{{}_pH^{(\nu)}} = H^{-1}(\sigma)$
Differential equation	$\left(xH'(H^{-1}(\sigma))D_x\sigma^{-1}H^{-1}(\sigma) + \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))}H^{-1}(\sigma) - \nu(H^{-1}(\sigma))^2 - n\right){}_pH_n^{(\nu)}(x) = 0$
Carlitz type generating function	$\sum_{n=0}^{\infty} {}_pH_n^{(\nu+\mu n)}(x)\frac{u^n}{n!} = \frac{e^{\frac{-\nu z^2}{2}} A(z)\psi(xH(z))}{1+\mu z^2}$, $u = z\left(e^{\frac{-t^2}{2}}\right)^{-\mu}$

Example 4.2. Since, for $\mathcal{G}(t) = \left(\frac{t}{e^t-1}\right)^a$, the Appell polynomial $A_n(x)$ becomes the Bernoulli polynomials of order a (Table 1 (II)). Therefore, for the same choice of $\mathcal{G}(t)$, the Boas-Buck-Appell polynomials reduce to the Boas-Buck-Bernoulli polynomials of order a denoted by ${}_pB_n^{(a)}(x)$.

Thus, by taking the above expression of $\mathcal{G}(t)$ in equations (2.4), (2.9), (2.10), (2.15) and (3.4), we deduce the following results for the Boas-Buck-Bernoulli polynomials of order a:

TABLE 3. Results for ${}_pB_n^{(a)}(x)$

Generating function	$\left(\frac{t}{e^t-1}\right)^a A(t)\psi(xH(t)) = \sum\limits_{n=0}^{\infty} {}_pB_n^{(a)}(x)\frac{t^n}{n!}$
Multiplicative and derivative operators	$\Phi^+_{pB^{(a)}} = xH'(H^{-1}(\sigma))D_x\sigma^{-1} + \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))} + \frac{a\left(e^{H^{-1}(\sigma)}-1-H^{-1}(\sigma)e^{H^{-1}(\sigma)}\right)}{H^{-1}(\sigma)\left(e^{H^{-1}(\sigma)}-1\right)}$, $\Phi^-_{pB^{(a)}} = H^{-1}(\sigma)$
Differential equation	$\left(\left(xH'(H^{-1}(\sigma))D_x\sigma^{-1} + \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))}\right)H^{-1}(\sigma) + \frac{a\left(e^{H^{-1}(\sigma)}-1-H^{-1}(\sigma)e^{H^{-1}(\sigma)}\right)}{\left(e^{H^{-1}(\sigma)}-1\right)} - n\right)$ ${}_pB_n^{(a)}(x) = 0$
Carlitz type generating function	$\sum\limits_{n=0}^{\infty} {}_pB_n^{(a+\mu n)}(x)\frac{u^n}{n!} = \frac{\left(\frac{z}{e^z-1}\right)^a A(z)\psi(xH(z))}{1-\mu\left(\frac{e^z-1-ze^z}{e^z-1}\right)}$, $u = z\left(\frac{t}{e^t-1}\right)^{-\mu}$

It should be noted that for $a = 1$, the Bernoulli polynomials of order a reduce to the Bernoulli polynomials. Therefore, for $a = 1$, the Boas-Buck-Bernoulli polynomials of order a reduce to the Boas-Buck-Bernoulli polynomials ${}_pB_n(x)$.

It has been shown in [12], that for $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}$ $(i = 1, 2, \ldots, n)$, the determinant definition of the Appell polynomials $A_n(x)$, gives the determinant form of the Bernoulli polynomials $B_n(x)$. Therefore, taking $\beta_0 = 1$ and $\beta_i = \frac{1}{i+1}$ $(i = 1, 2, \ldots, n)$ in equations (2.19) and (2.20), we get the following determinant definition of the Boas-Buck-Bernoulli polynomials ${}_pB_n(x)$:

Definition 4.1. The Boas-Buck-Bernoulli polynomials ${}_pB_n(x)$ of degree n are defined by

$$ {}_pB_0(x) = 1, \tag{4.1}$$

$$ {}_pB_n(x) = (-1)^n \begin{vmatrix} 1 & p_1(x) & p_2(x) & \cdots & p_{n-1}(x) & p_n(x) \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{n-1} & \binom{n}{1}\frac{1}{n} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{n-2} & \binom{n}{2}\frac{1}{n-1} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}, \quad n = 1, 2, \ldots, \tag{4.2}$$

where $p_n(x)$ $(n = 1, 2, \ldots)$ are the Boas-Buck polynomials of degree n defined by equation (1.1).

Example 4.3. Since, for $\mathcal{G}(t) = \left(\frac{2}{e^t+1}\right)^a$, the Appell polynomial $A_n(x)$ becomes the Euler polynomials of order a (Table 1 (III)). Therefore, for the same choice of $\mathcal{G}(t)$, the Boas-Buck-Appell polynomials reduce to the Boas-Buck-Euler polynomials of order a denoted by ${}_pE_n^{(a)}(x)$.

Thus, by taking the above expression of $\mathcal{G}(t)$ in equations (2.4), (2.9), (2.10), (2.15) and (3.4), we find the following results for the Boas-Buck-Euler polynomials of order a:

TABLE 4. Results for ${}_pE_n^{(a)}(x)$

Generating function	$\left(\frac{2}{e^t+1}\right)^a A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} {}_pE_n^{(a)}(x)\frac{t^n}{n!}$
Multiplicative and derivative operators	$\Phi^+_{pE^{(a)}} = xH'(H^{-1}(\sigma))D_x\sigma^{-1} + \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))} + \frac{ae^{H^{-1}(\sigma)}}{e^{H^{-1}(\sigma)}+1}$, $\Phi^-_{pE^{(a)}} = H^{-1}(\sigma)$
Differential equation	$\left(xH'(H^{-1}(\sigma))D_x\sigma^{-1}H^{-1}(\sigma) + \frac{A'(H^{-1}(\sigma))}{A(H^{-1}(\sigma))}H^{-1}(\sigma) + \frac{ae^{H^{-1}(\sigma)}}{e^{H^{-1}(\sigma)}+1}H^{-1}(\sigma) - n\right) {}_pE_n^{(a)}(x) = 0$
Carlitz type generating function	$\sum_{n=0}^{\infty} {}_pE_n^{(a+\mu n)}(x)\frac{u^n}{n!} = \frac{\left(\frac{2}{e^z+1}\right)^a A(z)\psi(xH(z))}{1+\mu\left(\frac{ze^z}{e^z+1}\right)}$, $u = z\left(\frac{2}{e^t+1}\right)^{-\mu}$

It should be noted that for $a = 1$, the Euler polynomials of order a reduce to the Euler polynomials. Therefore, for $a = 1$, the Boas-Buck-Euler polynomials of order a reduce to the Boas-Buck-Euler polynomials ${}_pE_n(x)$.

It has been shown in [12, p. 1540 (60)-(61)], that for $\beta_0 = 1$ and $\beta_i = \frac{1}{2}$ $(i = 1, 2, \ldots, n)$ the determinant definition of the Appell polynomials $A_n(x)$, reduces to the determinant form of the Euler polynomials $E_n(x)$. Therefore, taking $\beta_0 = 1$ and $\beta_i = \frac{1}{2}$ $(i = 1, 2, \ldots, n)$ in equations (2.19) and (2.20), we get the following determinant definition of the Boas-Buck-Euler polynomials ${}_pE_n(x)$:

Definition 4.2. The Boas-Buck-Euler polynomials ${}_pE_n(x)$ of degree n are defined by

$$ {}_pE_0(x) = 1, \tag{4.3}$$

$$ {}_pE_n(x) = (-1)^n \begin{vmatrix} 1 & p_1(x) & p_2(x) & \cdots & p_{n-1}(x) & p_n(x) \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1} & \cdots & \frac{1}{2}\binom{n-1}{1} & \frac{1}{2}\binom{n}{1} \\ 0 & 0 & 1 & \cdots & \frac{1}{2}\binom{n-1}{2} & \frac{1}{2}\binom{n}{2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{n}{n-1} \end{vmatrix}, \quad n = 1, 2, \ldots, \tag{4.4}$$

where $p_n(x)$ $(n = 1, 2, \ldots)$ are the Boas-Buck polynomials of degree n defined by equation (1.1).

In the next section, characterization of the Boas-Buck-Appell polynomials is given by a recursion relation.

5. Concluding remarks

A rough statement of one of the main results in Boas and Buck [7] is that a necessary and sufficient condition for the polynomials $p_n(x)$ to have the generating function of the form (1.1) is that the sequences of numbers α_k and β_k

exist such that, for $n \geq 1$, the following recursion relation holds true:

$$xp'_n(x) - np_n(x) = -\sum_{k=0}^{n-1} \alpha_k \; p_{n-1-k}(x) - x\sum_{k=0}^{n-1} \beta_k \; p'_{n-1-k}(x). \tag{5.1}$$

The Boas' and Buck's work is presented with minor variations in notation by Rainville [20, p. 14 (Theorem 50)]. A natural question arises that whether, we can have the analogous result for the hybrid polynomials introduced as discrete convolution of the Boas-Buck polynomials? The answer to this question is given in the form of the following result:

Theorem 5.1. *For the Boas-Buck-Appell polynomials ${}_pA_n(x)$ defined by equation* (2.4)*, with equations* (1.2), (1.3), (1.4) *and* (1.7) *holding and $\gamma_n \neq 0$, there exist sequences of numbers σ_k, μ_k and ξ_k (independent of n) such that, for $n \geq 1$, the following recursion relation holds true*:

$$\begin{aligned} &x \; {}_pA'_n(x) - n \; {}_pA_n(x) \\ &\quad = -\sum_{k=0}^{n-1} \binom{n}{k+1} (\sigma_k + \mu_k) {}_pA_{n-1-k}(x) - x \sum_{k=0}^{n-1} \binom{n}{k+1} \xi_k \; {}_pA'_{n-1-k}(x), \end{aligned} \tag{5.2}$$

where

$$\frac{t\mathcal{G}'(t)}{\mathcal{G}(t)} = \sum_{k=0}^{\infty} \sigma_k \frac{t^{k+1}}{(k+1)!}, \tag{5.3}$$

$$\frac{tA'(t)}{A(t)} = \sum_{k=0}^{\infty} \mu_k \frac{t^{k+1}}{(k+1)!} \tag{5.4}$$

and

$$\frac{tH'(t)}{H(t)} = 1 + \sum_{k=0}^{\infty} \xi_k \frac{t^{k+1}}{(k+1)!}. \tag{5.5}$$

Proof. Consider

$$F = \mathcal{G}(t)A(t)\psi(xH(t)). \tag{5.6}$$

Then

$$\frac{\partial F}{\partial x} = H(t)\mathcal{G}(t)A(t)\psi' \tag{5.7}$$

and

$$\frac{\partial F}{\partial t} = \mathcal{G}(t)A'(t)\psi + A(t)\mathcal{G}'(t)\psi + xH'(t)\mathcal{G}(t)A(t)\psi'. \tag{5.8}$$

Elimination of ψ and ψ' from equations (5.6)-(5.8), gives

$$xt\frac{H'(t)}{H(t)}\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = \left(-t\frac{\mathcal{G}'(t)}{\mathcal{G}(t)} - t\frac{A'(t)}{A(t)}\right) F. \tag{5.9}$$

In view of equations (2.4) and (5.6), we have

$$F=\sum_{n=0}^{\infty} {}_{P}A_n(x)\frac{t^n}{n!}. \tag{5.10}$$

Use of equations (5.3)-(5.5) and (5.10) in equation (5.9) yields

$$\begin{aligned}\left[1+\sum_{k=0}^{\infty}\xi_k\frac{t^{k+1}}{(k+1)!}\right]\left[\sum_{n=0}^{\infty}x\ {}_{P}A'_n(x)\frac{t^n}{n!}\right]-\sum_{n=0}^{\infty}n\ {}_{P}A_n(x)\frac{t^n}{n!}\\ = -\left[\sum_{k=0}^{\infty}\sigma_k\frac{t^{k+1}}{(k+1)!}+\mu_k\frac{t^{k+1}}{(k+1)!}\right]\left[\sum_{n=0}^{\infty}{}_{P}A_n(x)\frac{t^n}{n!}\right],\end{aligned} \tag{5.11}$$

which on simplification gives

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[x\ {}_{P}A'_n(x)-n\ {}_{P}A_n(x)\right]\frac{t^n}{n!}\\ &= -\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{(\sigma_k+\mu_k)}{(k+1)!(n-k-1)!}\ {}_{P}A_{n-1-k}(x)t^n\\ &\quad -x\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{\xi_k}{(k+1)!(n-k-1)!}\ {}_{P}A'_{n-1-k}(x)t^n.\end{aligned} \tag{5.12}$$

Equating the coefficients of like powers of t in equation (5.12), we get assertion (5.2). □

Remark 5.1. Since for $H(t)=t$, the Boas-Buck-Appell polynomials ${}_{P}A_n(x)$ become the Brenke-Appell polynomials ${}_{Y}A_n(x)$, therefore, from equation (5.5), it follows that $\xi_k=0$ and thus, the following result is obtained as consequence of Theorem 5.1:

Corollary 5.1. *For the Brenke-Appell polynomials* ${}_{Y}A_n(x)$ *defined by equation* (2.5)*, with equations* (1.2), (1.3) *and* (1.7) *holding and* $\gamma_n\neq 0$*, there exist sequences of numbers* σ_k *and* μ_k (*independent on* n) *such that, for* $n\geq 1$*, the following recursion relation holds true*:

$$x\ {}_{Y}A'_n(x)-n\ {}_{Y}A_n(x)=-\sum_{k=0}^{n-1}\binom{n}{k+1}(\sigma_k+\mu_k)_{Y}A_{n-1-k}(x), \tag{5.13}$$

where

$$\frac{t\mathcal{G}'(t)}{\mathcal{G}(t)}=\sum_{k=0}^{\infty}\sigma_k\frac{t^{k+1}}{(k+1)!}, \tag{5.14}$$

$$\frac{tA'(t)}{A(t)}=\sum_{k=0}^{\infty}\mu_k\frac{t^{k+1}}{(k+1)!}. \tag{5.15}$$

Remark 5.2. Since for $\psi(t) = \exp(t)$, the Boas-Buck-Appell polynomials ${}_pA_n(x)$ become the Sheffer-Appell polynomials ${}_sA_n(x)$, therefore, from equation (1.3), it follows that $\gamma_n = 1 \neq 0$. It should be noted that the Sheffer-Appell polynomials ${}_sA_n(x)$ satisfy the equation identical to equation (5.2) with equations (5.3), (5.4) and (5.5) holding.

Remark 5.3. Since for $\psi(t) = \exp(t)$ and $H(t) = t$, the Boas-Buck-Appell polynomials ${}_pA_n(x)$ become the 2-iterated Appell polynomials $A_n^{[2]}(x)$, therefore, from equation (1.3), it follows that $\gamma_n = 1 \neq 0$ and from equation (5.5), it follows that $\xi_k = 0$. It should be noted that the 2-iterated Appell polynomials $A_n^{[2]}(x)$ satisfy the equation identical to equation (5.13) with equations (5.14) and (5.15) holding.

Finally, to give an application of the recursion relation (5.2), we consider the following example:

Example. For $A(t) = \frac{1}{1-t}$, $\psi(t) = \exp(xt)$ and $H(t) = \frac{-t}{(1-t)}$, the Boas-Buck polynomials become the Laguerre polynomials $L_n(x)$ [2] and for $\mathcal{G}(t) = \frac{1}{1-t}$, the Appell polynomials become the truncated exponential polynomials $e_n(x)$ [2]. Therefore, making these substitutions in equation (2.4), the following generating function for the Laguerre-truncated exponential polynomials ${}_Le_n(x)$ is obtained:

$$\frac{1}{(1-t)^2}\exp\left(\frac{-xt}{(1-t)}\right) = \sum_{n=0}^{\infty} {}_Le_n(x)\frac{t^n}{n!}. \tag{5.16}$$

From the expressions of the $\mathcal{G}(t)$, $A(t)$ and $H(t)$, we find

$$\frac{t\mathcal{G}'(t)}{\mathcal{G}(t)} = \sum_{k=0}^{\infty} t^{k+1},\ \frac{tA'(t)}{A(t)} = \sum_{k=0}^{\infty} t^{k+1},\ \frac{tH'(t)}{H(t)} = 1 + \sum_{k=0}^{\infty} t^{k+1}. \tag{5.17}$$

On comparing equation (5.17) with equations (5.3)-(5.5), the following values are obtained:

$$\sigma_k = \mu_k = \xi_k = (k+1)!. \tag{5.18}$$

Therefore, using equation (5.18) in equation (5.2), the following recurrence relation for the Laguerre-truncated exponential polynomials ${}_Le_n(x)$ is obtained:

$$\begin{aligned} &x\ {}_Le'_n(x) - n\ {}_Le_n(x) \\ &= -2\sum_{k=0}^{n-1} \frac{n!}{(n-k-1)!}\ {}_Le_{n-1-k}(x) - x\sum_{k=0}^{n-1} \frac{n!}{(n-k-1)!}\ {}_Le'_{n-1-k}(x). \end{aligned} \tag{5.19}$$

Examples of other members belonging to the Boas-Buck-Appell polynomials can also be considered in the same way.

The hybrid family of Boas-Buck polynomials introduced and studied in this article is important from the fact that the properties satisfied by this family are analogous to that of the parent family.

Acknowledgements. The authors are thankful to the Reviewer(s) for useful comments and suggestions towards the improvement of this paper.

References

[1] L. Aceto, H. R. Malonek, and G. Tomaz, *A unified matrix approach to the representation of Appell polynomials*, Integral Transforms Spec. Funct. **26** (2015), no. 6, 426–441.

[2] L. C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, Macmillan Co., New York, 1985.

[3] M. Anshelevich, *Appell polynomials and their relatives. III. Conditionally free theory*, Illinois J. Math. **53** (2009), no. 1, 39–66.

[4] P. Appell, *Sur une classe de polynômes*, Ann. Sci. École Norm. Sup. (2) **9** (1880), 119–144.

[5] Y. Ben Cheikh, *Some results on quasi-monomiality*, Appl. Math. Comput. **141** (2003), no. 1, 63–76.

[6] Y. Ben Cheikh and H. Chaggara, *Connection coefficients between Boas-Buck polynomial sets*, J. Math. Anal. Appl. **319** (2006), no. 2, 665–689.

[7] R. P. Boas and R. C. Buck, *Polynomials defined by generating relations*, Amer. Math. Monthly **63** (1956), 626–632.

[8] F. Brackx, H. De Schepper, R. Lávička, and V. Soucek, *Gel'fand-Tsetlin procedure for the construction of orthogonal bases in Hermitean Clifford analysis*, In: Simos TE, Psihoyios G, Tsitouras Ch, editors. Numerical analysis and applied mathematics-ICNAAM 2010, AIP Conference Proceedings (1281). Melville (NY): American Institute of Physics (AIP), (2010), 1508–1511.

[9] W. C. Brenke, *On generating functions of polynomial systems*, Amer. Math. Monthly **52** (1945), 297–301.

[10] L. Carlitz, *A class of generating functions*, SIAM J. Math. Anal. **8** (1977), no. 3, 518–532.

[11] H. Chaggara and I. Lamiri, *Linearization coefficients for Boas-Buck polynomial sets*, Appl. Math. Comput. **189** (2007), no. 2, 1533–1549.

[12] F. A. Costabile and E. Longo, *A determinantal approach to Appell polynomials*, J. Comput. Appl. Math. **234** (2010), no. 5, 1528–1542.

[13] G. Dattoli, *Hermite-Bessel and Laguerre-Bessel functions*: *a by-product of the monomiality principle*, in Advanced special functions and applications (Melfi, 1999), 147–164, Proc. Melfi Sch. Adv. Top. Math. Phys., 1, Aracne, Rome, 2000.

[14] M. E.-H. Ismail, *On obtaining generating functions of Boas and Buck type for orthogonal polynomials*, SIAM J. Math. Anal. **5** (1974), 202–208.

[15] S. Khan and N. Raza, *2-iterated Appell polynomials and related numbers*, Appl. Math. Comput. **219** (2013), no. 17, 9469–9483.

[16] S. Khan and M. Riyasat, *A determinantal approach to Sheffer-Appell polynomials via monomiality principle*, J. Math. Anal. Appl. **421** (2015), no. 1, 806–829.

[17] R. Lávička, *Canonical bases for $sl(2, C)$-modules of spherical monogenics in dimension 3*, Arch. Math. Tomus. **46** (2010), 339–349.

[18] H. R. Malonek and M. I. Falcão, *3D-Mappings using monogenic functions*, In: Simos TE, Psihoyios G, Tsitouras Ch, editors. Numerical analysis and applied mathematics-ICNAAM 2006. Weinheim: Wiley-VCH; (2006), 615–619.

[19] G. Pólya and G. Szegö, *Problems and Theorems in Analysis. Vol. I*, translated from the German by D. Aeppli, Springer-Verlag, New York, 1972.

[20] E. D. Rainville, *Special Functions*, reprint of 1960 first edition, Chelsea Publishing Co., Bronx, NY, 1971.

[21] S. Roman, *The Umbral Calculus*, Pure and Applied Mathematics, **111**, Academic Press, Inc., New York, 1984.

[22] P. Salminen, *Optimal stopping, Appell polynomials, and Wiener-Hopf factorization*, Stochastics **83** (2011), no. 4-6, 611–622.

[23] J. F. Steffensen, *The poweroid, an extension of the mathematical notion of power*, Acta Math. **73** (1941), 333–366.

[24] S. Weinberg, *The Quantum Theory of Fields. Vol. I*, corrected reprint of the 1995 original, Cambridge University Press, Cambridge, 1996.

[25] Y. Yang and H. Youn, *Appell polynomial sequences: a linear algebra approach*, JP J. Algebra Number Theory Appl. **13** (2009), no. 1, 65–98.

Mahvish Ali
Department of Mathematics
Aligarh Muslim University
Aligarh, India
Email address: mahvishali37@gmail.com

Subuhi Khan
Department of Mathematics
Aligarh Muslim University
Aligarh, India
Email address: subuhi2006@gmail.com