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NEW RESULTS ON THE PSEUDOREDUNDANCY

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Abstract. The concepts of pseudocodeword and pseudoweight play a fundamental role in the finite-length analysis of LDPC codes. The pseudoredundancy of a binary linear code is defined as the minimum number of rows in a parity-check matrix such that the corresponding minimum pseudoweight equals its minimum Hamming distance. By using the value assignment of Chen and Kløve we present new results on the pseudocodeword redundancy of binary linear codes. In particular, we give several upper bounds on the pseudoredundancies of certain codes with repeated and added coordinates and of certain shortened subcodes. We also investigate several kinds of k-dimensional binary codes and compute their exact pseudocodeword redundancy.

1. Introduction

The concept of a pseudocodeword plays a key role in the finite-length analysis of binary low-density parity-check (LDPC) codes under linear programming (LP) decoding (or, to some extent, under message-passing iterative decoding), see [3, 10]. The effect of pseudocodewords on the decoding behavior is measured by their pseudoweight [4, 10], which depends on the channel at hand. Accordingly, the pseudocodeword redundancy (or pseudoredundancy) of a binary linear code is of interest, which is defined as the minimum number of rows in a parity-check matrix such that the corresponding minimum pseudoweight is as large as its minimum Hamming distance. The pseudoredundancy for various channels has been studied, e.g., in [5], [9], and [11].

It is undoubtedly meaningful to determine either the pseudocodeword redundancy or to give bounds on the pseudocodeword redundancy of a binary linear code. However, it was shown in [11, Th. 3.2, Th. 3.5] that most codes have infinite AWGNC and BSC pseudoredundancy. In contrast to this result, we will determine the pseudoredundancies of some kinds of k-dimensional codes and give bounds for certain constructed codes. Our main tool to study the pseudoredundancy is the value assignment introduced by Chen and Kløve [1].

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Part of the paper has been presented at the 2014 IEEE International Symposium on Information Theory (ISIT 2014), Honolulu, HI, USA [8]. In the present version, we add Section 3, Theorem 9, and the detailed proof of some other theorems. We also give a new proof of Theorem 14 by using an alternative simple method.

The rest of the paper is organized as follows. In Section 2 we define pseudoweights for various channels and the notion of pseudoredundancy; we also present the concept of value assignment. Section 3 contains a discussion of codes based on repeating or adding coordinates and of shortened subcodes. In Section 4 we determine the pseudoredundancies of certain k-dimensional codes based on value assignment, generalising previous results significantly. Finally, we conclude in Section 5.

2. Preliminaries

For a binary linear code $\mathcal{C}$ of length n, when analyzing LP decoding for a binary-input output-symmetric channel, one may assume that the zero codeword $\mathbf{0}$ has been sent; then, the probability of correct LP decoding depends on the conic hull of the fundamental polytope, called the *fundamental cone* [3,10], which depends on the given parity-check matrix of $\mathcal{C}$.

Let H be an $m \times n$ parity-check matrix for $\mathcal{C}$, where the m rows may be linearly dependent. Let $\mathcal{I} = \{1, \ldots, n\}$ and $\mathcal{J} = \{1, \ldots, m\}$ be the set of column and row indices, respectively, and for each $j \in \mathcal{J}$ let $\mathcal{I}_j = \{i \in \mathcal{I} \mid H_{j,i} \neq 0\}$. Then, the fundamental cone $\mathcal{K}(H)$ with respect to the parity-check matrix H of $\mathcal{C}$ is given as the set of vectors $x \in \mathbb{R}^n$ that satisfy

$$\begin{aligned} \forall j \in \mathcal{J}\ \forall \ell \in \mathcal{I}_j :\ & x_\ell \leq \textstyle\sum_{i \in \mathcal{I}_j \setminus \{\ell\}} x_i, \\ \forall i \in \mathcal{I} :\ & x_i \geq 0. \end{aligned} \tag{2.1}$$

The vectors $x \in \mathcal{K}(H)$ are called *pseudocodewords* of $\mathcal{C}$ with respect to the parity-check matrix H.

The influence of a nonzero pseudocodeword on the decoding performance is measured by its *pseudoweight*, which depends on the underlying channel. For $x = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n$, let $\operatorname{supp}(x) = \{i \mid x_i \neq 0\}$. The BEC (binary erasure channel), AWGNC, BSC pseudoweights and max-fractional weight of a nonzero pseudocodeword $x \in \mathcal{K}(H)$ are defined as follows [4,10]:

$$w_{\mathrm{BEC}}(x) = |\operatorname{supp}(x)|,$$
$$w_{\mathrm{AWGNC}}(x) = \frac{(\sum_{i \in \mathcal{I}} x_i)^2}{\sum_{i \in \mathcal{I}} {x_i}^2},$$

letting x' be a vector in $\mathbb{R}^n$ with the same components as x but in nonincreasing order for $i - 1 < \xi \leq i$, where $1 \leq i \leq n$, letting $\phi(\xi) = x'_i$ and defining $\Phi(\xi) = \int_0^\xi \phi(\xi')d\xi'$,

$$w_{\mathrm{BSC}}(x) = 2\Phi^{-1}(\Phi(n)/2),$$

$$w_{\text{maxfrac}}(x) = \frac{\sum_{i \in \mathcal{I}} x_i}{\max_{i \in \mathcal{I}} x_i}.$$

For binary vectors $x \in \{0,1\}^n \setminus \{\mathbf{0}\}$ one has

$$w_{\text{BEC}}(x) = w_{\text{AWGNC}}(x) = w_{\text{BSC}}(x) = w_{\text{maxfrac}}(x) = w_H(x),$$

where $w_H(x)$ denotes the Hamming weight of x.

Define the *minimum pseudoweight* of a code $\mathcal{C}$ with respect to a parity-check matrix H as

$$w_{\min}(H) = \min_{x \in \mathcal{K}(H) \setminus \{\mathbf{0}\}} w(x),$$

where $w(x)$ may represent any one of the four pseudoweights (it is a fact that $w_{\min}(H)$ is indeed attained on $\mathcal{K}(H) \setminus \{\mathbf{0}\}$ [10]). The minimum pseudoweight $w_{\min}(H)$ can be seen as a first-order measure of decoding error-correcting performance of a code $\mathcal{C}$ given by the parity-check matrix H under LP decoding. We note that all four minimum pseudoweights are upper bounded by $d(\mathcal{C})$, the minimum distance of $\mathcal{C}$.

Definition 1. The *pseudocodeword redundancy*, or briefly the *pseudoredundancy*, $\rho(\mathcal{C})$, of a binary linear $[n,k,d]$ code $\mathcal{C}$ is defined as

$$\rho(\mathcal{C}) = \inf\{\#\text{rows}(H) \mid H \text{ is a parity-check matrix of } \mathcal{C}, w_{\min}(H) = d\},$$

where $\inf \varnothing$ is defined as ∞; here $w_{\min}(H)$ stands for one of the four pseudoweights, and we use accordingly the term BEC, AWGN, BSC, or max-fractional pseudoredundancy.

It is obvious that $\rho(\mathcal{C}) \geq n-k$ for any $[n,k,d]$ code $\mathcal{C}$. Furthermore, for any binary linear code $\mathcal{C}$ it holds [11, Th. 2.5] that

$$\begin{aligned} \rho_{\text{maxfrac}}(\mathcal{C}) \ &\geq\ \rho_{\text{AWGNC}}(\mathcal{C}) \ \geq\ \rho_{\text{BEC}}(\mathcal{C}), \\ \rho_{\text{maxfrac}}(\mathcal{C}) \ &\geq\ \rho_{\text{BSC}}(\mathcal{C}) \ \geq\ \rho_{\text{BEC}}(\mathcal{C}). \end{aligned} \tag{2.2}$$

The value assignment, which was first introduced in [1], is our main tool for investigating the pseudoredundancy. It is given as follows.

Definition 2. A *value assignment* is a map

$$m(\cdot) : PG(k-1,q) \longrightarrow \mathbb{N} = \{0,1,2,\dots\}$$

from the $(k-1)$-dimensional projective space $PG(k-1,q)$ over the finite field $\mathrm{GF}(q)$ to $\mathbb{N}$, the set of nonnegative integers. For a point $p \in PG(k-1,q)$, we call $m(p)$ the *value* of p.

In this paper, we view points of $PG(k-1,q)$ as column vectors, so that we may use them to construct generator matrices for linear codes. On the other hand, vectors over the real field (as we have seen earlier) or over finite fields (such as codewords) are all row vectors.

In particular, if G is a $k \times n$ generator matrix of a linear $[n,k]$ code over $\mathrm{GF}(q)$, then the columns of G can be viewed as points of $PG(k-1,q)$, and for any $p \in PG(k-1,q)$ we define $m(p)$ as the number of occurrences of p as columns of G (note that G may have repeating columns, for which the corresponding point p of $PG(k-1,q)$ has a value greater than 1, while if p does not appear in G, then $m(p)=0$). Thus viewing the columns of a generator matrix of a linear $[n,k]$ code as a multiset of points of the projective space $PG(k-1,q)$, this multiset defines a value assignment. Conversely, any value assignment $m(\cdot)$ (or equivalently, a sequence of nonnegative integers $z_1, z_2, \ldots, z_N$, where $z_i = m(p_i)$ for $p_i \in PG(k-1,q)$ and $N = (q^k-1)/(q-1)$ is the cardinality of $PG(k-1,q)$) uniquely determines a $k \times n$ matrix G (up to code equivalence), where n is the number of points p (each p counted $m(p)$ times) of the projective space $PG(k-1,q)$. The columns of G form a multiset of points of $PG(k-1,q)$, that is, the columns of G consist of the points p with positive values and each of them repeats $m(p)$ times. Therefore, the value assignment defines a generator matrix G and thus an $[n,k]$ code (up to code equivalence) if the matrix has rank k. Note that if $\mathcal{C}$ is the $[n,k]$ code determined by a value assignment $m(\cdot)$, then from the above discussion we deduce the important property $\sum_{p \in PG(k-1,q)} m(p) = n$.

Now, since equivalent codes lead to equivalent dual codes and thus equivalent codes have the same pseudoredundancy, it suffices to use the value assignment to construct different equivalent codes for studying the pseudoredundancy.

3. Codes based on repeating and adding coordinates and shortened subcodes

In this section, we will give several bounds on the pseudoredundancies of codes obtained by increasing the number of coordinates and of certain shortened subcodes by using the value assignment.

We remark that repeating coordinates is a useful method to construct a code. For example, any binary linear constant-weight code (that is, all the nonzero codewords have the same weight) is obtained from a simplex code by repeating each coordinate equally times [6], or equivalently, any binary constant-weight code consists of copies of the simplex code. Furthermore, a recent paper [7] shows that a large class of codes called relative constant-weight codes, which have applications to secret sharing schemes, can be obtained by repeating coordinates.

In [2] the authors studied the effect of repeating coordinates on the Tanner graph. In particular, [2, Lem. 3, Prop. 4] shows that any linear cycle-free code with rate ≤ 0.5 can be obtained from a linear cycle-free code with rate > 0.5 by repeating coordinates. Thus, repeating coordinates is also a useful operation, related to the Tanner graph itself.

Assume that G is a generator matrix of an $[n,k,d]$ code $\mathcal{C}$, and the value assignment $m(\cdot)$ is defined from G (as discussed in Section 2). Then any codeword $c \in \mathcal{C}$ may be written as $c = uG$ for some $u \in \mathrm{GF}(2)^k$; denote by $u^\perp$ the set of points in $PG(k-1,2)$ that are perpendicular to u (according to the usual inner product), and let

$$T_c = \{p \in u^\perp \mid m(p) \geq 1\} \quad \text{and} \quad \overline{T_c} = u^\perp \setminus T_c = \{p \in u^\perp \mid m(p) = 0\}.$$

Note that T_c corresponds to those column indices i of G where $c_i = 0$. Then, we have:

Theorem 3. *Let G, $\mathcal{C}$, and $m(\cdot)$ be as above and let c be any codeword of $\mathcal{C}$ with minimum weight d.*

(1) *Define an $[n', k]$ code $\mathcal{C}'$ generated by the matrix G' obtained from G by increasing the values of some of the points $p \in T_c$. Then,*

$$\rho(\mathcal{C}') \;\leq\; \rho(\mathcal{C}) + (n' - n)$$

for the max-fractional pseudoweight and for the BEC pseudoweight.

(2) *Define an $[n', k]$ code $\mathcal{C}'$ generated by the matrix G' obtained by adding the points of the set $\overline{T_c}$ to the columns of G. If each added point $p \in \overline{T_c}$ repeats at least $\lceil (1 - 1/k)d \rceil$ times in the columns of G', then*

$$\rho(\mathcal{C}') \;\leq\; \rho(\mathcal{C}) + (n' - n)$$

for the max-fractional pseudoweight and for the BEC pseudoweight.

Proof. The proof for the BEC pseudoweight is very similar to that for the max-fractional pseudoweight. We give a detailed proof for the max-fractional pseudoweight below.

To prove (1), up to code equivalence, we may assume that T_c is the set of the first t columns (points) of G, where $1 \leq t \leq n-1$. From the assumption, it follows that $\mathcal{C}'$ is an $[n', k, d]$ code that is generated by G', for which the value assignment $m'(p_i) \geq m(p_i)$ for any $1 \leq i \leq t$. Assume $m'(p_i) - m(p_i) = \theta_i$ for $1 \leq i \leq t$ and let H be the parity-check matrix with $\rho(\mathcal{C})$ rows of $\mathcal{C}$. Put the points p_i for $1 \leq i \leq t$ in order after the n-th column of G to obtain G', and each point p_i, $1 \leq i \leq t$, repeats θ_i times, respectively. Construct a matrix H' as

$$H' = \begin{pmatrix} H & & & \\ * & H_1 & & \\ * & & \ddots & \\ * & & & H_t \end{pmatrix},$$

where H_i is an identity matrix of order θ_i, and "$*$" corresponding to H_i has entries one at the i-th column of H for $1 \leq i \leq t$. It can be checked that H' is a matrix with $\rho(\mathcal{C}) + \sum_{i=1}^{t} \theta_i = \rho(\mathcal{C}) + (n'-n)$ rows and of rank $(n-k) + \sum_{i=1}^{t} \theta_i = (n-k) + (n'-n) = n'-k$. Furthermore, $H'G'^T = 0$, thus, H' is a parity-check matrix of $\mathcal{C}'$.

Assume now that $x \in \mathcal{K}(H')$. Then we may write $x = (y, z)$, where $y = (y_1, \ldots, y_n) \in \mathcal{K}(H)$ and $z = (z_1, \ldots, z_{n'-n}) \in \mathbb{R}^{n'-n}$. Note that z may be rewritten as $z = (y_1, \ldots, y_1, \ldots, y_t, \ldots, y_t)$ where each y_i, $1 \leq i \leq t$, repeats θ_i

times, respectively. Thus, for any $x \in \mathcal{K}(H')$, we have

$$\begin{aligned} w(x) &= \frac{\sum_{j=1}^{n} y_j + \sum_{i=1}^{n'-n} z_i}{\max_{j,i}\{y_j, z_i\}} \\ &= \frac{\sum_{j=1}^{n} y_j + \sum_{i=1}^{t} \theta_i y_i}{\max_j\{y_j\}} \\ &\geq \frac{\sum_{j=1}^{n} y_j}{\max_j\{y_j\}} \geq w_{\min}(H) = d. \end{aligned}$$

Thus, $w_{\min}(H') \geq d = d(\mathcal{C}')$, and so the result holds.

For (2), it follows from the assumption that $\mathcal{C}'$ is an $[n', k, d]$ code that is generated by G', where G' is obtained by adding t points p_i in order, $1 \leq i \leq t$, of the set $\overline{T_c}$ to the columns of G and assuming that each point p_i repeats $\theta_i \geq \lceil (1 - 1/k)d \rceil$ times, respectively. Since $\mathcal{C}$ is a k-dimensional code, there exist basis points $b_1, \ldots, b_k$ in the columns of G, and we may suppose without loss of generality that b_j is in the j-th position in G for $1 \leq j \leq k$. Write each point p_i as $p_i = \sum_{j=1}^{k} c_{ij} b_j$ and denote the support set by $A_i := \{j \mid c_{ij} = 1\} \subset \{1, \ldots, k\}$ for $1 \leq i \leq t$.

Let H be the parity-check matrix of $\mathcal{C}$ with $\rho(\mathcal{C})$ rows. Construct a matrix H' as

$$H' = \begin{pmatrix} H'' \\ h_1 \\ \vdots \\ h_t \end{pmatrix},$$

where

$$H'' = \begin{pmatrix} H & & & \\ & H_1 & & \\ & & \ddots & \\ & & & H_t \end{pmatrix};$$

here, each H_i, $1 \leq i \leq t$, is a $(\theta_i - 1) \times \theta_i$ submatrix whose entries are defined as follows

$$(H_i)_{st} = \begin{cases} 1 & \text{if } t \in \{s, s+1\}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

and each h_i, $1 \leq i \leq t$, is a binary vector with coordinate positions of h_i equal to one whenever the position is in A_i or the position corresponds to the first column of H_i.

Since $\sum_{i=1}^{t} \theta_i = n' - n$, the matrix H' is a $(\rho(\mathcal{C}) + (n' - n)) \times n'$ matrix of rank $n - k + (n' - n) = n' - k$. Furthermore, $H' G'^T = 0$, thus, H' is a parity-check matrix of $\mathcal{C}'$.

Let $x \in \mathcal{K}(H')$. Then, x may be written as $x = (y, z)$, where $y = (y_1, \ldots, y_n) \in \mathcal{K}(H)$, and $z \in \mathbb{R}^{n'-n}$. Note that z may be written as $z = (z_1, \ldots, z_1, \ldots, z_t, \ldots, z_t)$, and each z_i, $1 \leq i \leq t$, repeats θ_i times, respectively.

If $\max_{j,i}\{y_j, z_i\} = \max_j\{y_j\}$, where $1 \leq j \leq n$ and $1 \leq i \leq t$, then

$$\begin{aligned} w(x) &= \frac{\sum_{j=1}^n y_j + \sum_{i=1}^t \theta_i z_i}{\max_{j,i}\{y_j, z_i\}} \\ &= \frac{\sum_{j=1}^n y_j + \sum_{i=1}^t \theta_i z_i}{\max_j\{y_j\}} \\ &\geq \frac{\sum_{j=1}^n y_j}{\max_j\{y_j\}} \\ &\geq w_{\min}(H) = d = d(\mathcal{C}') \qquad (\text{by } y \in \mathcal{K}(H)). \end{aligned}$$

If $\max_{j,i}\{y_j, z_i\} = \max_i\{z_i\} = z_{i_0}$, then by the fundamental cone inequalities (2.1) we have $z_{i_0} \leq \sum_{j \in A_{i_0}} y_j \leq k \max_j\{y_j\}$, and since $\theta_{i_0} \geq (1 - 1/k)d$ we conclude

$$\begin{aligned} w(x) &= \frac{\sum_{j=1}^n y_j + \sum_{i=1}^t \theta_i z_i}{\max_{j,i}\{y_j, z_i\}} \\ &= \frac{\sum_{j=1}^n y_j + \sum_{i=1}^t \theta_i z_i}{z_{i_0}} \\ &\geq \frac{\sum_{j=1}^n y_j}{z_{i_0}} + \theta_{i_0} \\ &\geq \frac{\sum_{j=1}^n y_j}{k \max_j\{y_j\}} + \theta_{i_0} \\ &\geq (1/k)d + (1 - 1/k)d \qquad (\text{by } y \in \mathcal{K}(H)) \\ &= d = d(\mathcal{C}'). \end{aligned}$$

Thus, $w_{\min}(H') \geq d(\mathcal{C}')$ in any case, and so the result follows. □

Remark 4. It appears to be an open problem whether the above theorem also holds for the other two pseudoweights.

Let $\mathcal{C}$ be a binary linear code of length n and let $\mathcal{I}' \subset \mathcal{I} = \{1, 2, \ldots, n\}$ be a subset of $\mathcal{I}$. Define

$$\mathcal{C}_{\mathcal{I}'} = \{c \in \mathcal{C} \mid \mathrm{supp}(c) \subset \mathcal{I}'\},$$

which is the *shortened subcode* of $\mathcal{C}$ supported by $\mathcal{I}'$. Regarding the pseudoredundancy of the shortened subcode $\mathcal{C}_{\mathcal{I}'}$, the result is as follows.

Theorem 5. *Let $\mathcal{C}$ be an $[n, k, d]$ code and let $c \in \mathcal{C}$ be any codeword of minimum weight d. Then, for any shortened subcode $\mathcal{C}_{\mathcal{I}'}$ containing the codeword c, we have*

$$\rho(\mathcal{C}_{\mathcal{I}'}) \;\leq\; \rho(\mathcal{C}) + (n - |\mathcal{I}'|)$$

for all the four pseudoweights.

Proof. Let H be a parity-check matrix of $\mathcal{C}$ with $\rho(\mathcal{C})$ rows and let $H_{\mathcal{I}'}$ be the submatrix of H consisting of the columns corresponding to $\mathcal{I}'$. Define $\mathcal{C}'_{\mathcal{I}'}$ as the code obtained by puncturing those columns of $\mathcal{C}_{\mathcal{I}'}$ corresponding to $\mathcal{I} \setminus \mathcal{I}'$. Then it can be checked that $H_{\mathcal{I}'}$ is a parity-check matrix of $\mathcal{C}'_{\mathcal{I}'}$. Since $\mathcal{C}'_{\mathcal{I}'}$ is a linear code with minimum distance d according to the assumption, and since for any $x \in \mathcal{K}(H_{\mathcal{I}'})$ and $(x, 0) \in \mathbb{R}^n$ we have $(x, 0) \in \mathcal{K}(H)$, it follows that

$$w_{\min}(H_{\mathcal{I}'}) \geq w_{\min}(H) = d(\mathcal{C}) = d = d(\mathcal{C}'_{\mathcal{I}'}).$$

Thus, $\rho(\mathcal{C}'_{\mathcal{I}'}) \leq \rho(\mathcal{C})$. Then, using the proof of Lemma 4.1 in [11], we get $\rho(\mathcal{C}_{\mathcal{I}'}) \leq \rho(\mathcal{C}'_{\mathcal{I}'}) + (n - |\mathcal{I}'|) \leq \rho(\mathcal{C}) + (n - |\mathcal{I}'|)$. □

A code $\mathcal{C}$ is called *subcode-complete* if any subcode $\mathcal{D}$ of $\mathcal{C}$ can be written as $\mathcal{D} = \mathcal{C}_{\mathcal{I}'}$ for some $\mathcal{I}' \subset \mathcal{I}$. Define $\operatorname{supp}(\mathcal{D}) = \bigcup_{c \in \mathcal{D}} \operatorname{supp}(c)$. Since $\operatorname{supp}(\mathcal{D}) = \bigcap\{\mathcal{I}' \mid \mathcal{C}_{\mathcal{I}'} \supset \mathcal{D}\}$, it follows that a code $\mathcal{C}$ is subcode-complete if and only if $\mathcal{D} = \mathcal{C}_{\operatorname{supp}(\mathcal{D})}$ for any subcode $\mathcal{D}$ of $\mathcal{C}$. The following result gives a judging rule for a code to be subcode-complete by using the value assignment.

Theorem 6. *A code $\mathcal{C}$ with value assignment $m(\cdot)$ is subcode-complete if and only if $m(p) > 0$ for all $p \in PG(k-1, 2)$.*

Proof. Let $\mathcal{C}$ be subcode-complete. Assume that G is a generator matrix corresponding to $m(\cdot)$. If there exists a point p_0 such that $m(p_0) = 0$, then consider the $(k-1)$-dimensional subspace $(p_0)^{\perp}$ of $\mathrm{GF}(2)^k$, where

$$(p_0)^{\perp} = \{v \mid v \text{ is perpendicular to } p_0\}.$$

It follows that $\mathcal{D} = \{c \mid c = vG \text{ and } v \in (p_0)^{\perp}\}$ is a $(k-1)$-dimensional subcode of $\mathcal{C}$. Since $m(p_0) = 0$, we get $\operatorname{supp}(\mathcal{D}) = \operatorname{supp}(\mathcal{C})$. Thus, $\mathcal{D} \neq \mathcal{C}_{\operatorname{supp}(\mathcal{D})} = \mathcal{C}_{\operatorname{supp}(\mathcal{C})} = \mathcal{C}$, a contradiction to that $\mathcal{C}$ is subcode-complete.

Conversely, suppose that $m(p) > 0$ for each $p \in PG(k-1, 2)$ and consider any r-dimensional $(1 \leq r \leq k)$ subcode $\mathcal{D}$. Note that a generator matrix of $\mathcal{D}$ can be written as $U_{r\times k}G$ for some matrix $U_{r\times k}$. Denote

$$(U_{r\times k})^{\perp} = \{p \mid p \in PG(k-1, 2) \text{ and } p \text{ is perpendicular to each row of } U_{r\times k}\}.$$

Then, $(U_{r\times k})^{\perp}$ is a $(k-r-1)$-dimensional subspace of $PG(k-1, 2)$.

Since $m(p) > 0$ for each $p \in PG(k-1, 2)$, we get that the set

$$W = \{p \mid p \in (U_{r\times k})^{\perp} \text{ and } m(p) > 0\}$$

is equal to the $(k-r-1)$-dimensional projective subspace $(U_{r\times k})^{\perp}$. Observe that $\operatorname{supp}(\mathcal{D})$ corresponds to those columns of G (considered as points of $PG(k-1, 2)$) which are not contained in $W = (U_{r\times k})^{\perp}$. Thus,

$$\begin{aligned}\mathcal{C}_{\operatorname{supp}(\mathcal{D})} = \{c \mid c = vG &\text{ and } v \in \mathrm{GF}(2)^k \text{ is} \\ &\text{perpendicular to each point in } W = (U_{r\times k})^{\perp}\} \\ = \{c \mid c = vG &\text{ and } v \in \mathrm{GF}(2)^k \text{ is}\end{aligned}$$

$$\text{a linear combination of the rows of } U_{r\times k}\}$$
$$= \mathcal{D}.$$

Thus, $\mathcal{C}$ is subcode-complete. □

From Theorems 5 and 6, one gets the following result.

Corollary 7. *Let $\mathcal{C}$ be a subcode-complete $[n,k,d]$ code and let c be any codeword with minimum weight d. Then, for any subcode $\mathcal{D}$ containing c, there holds*

$$\rho(\mathcal{D}) \;\le\; \rho(\mathcal{C}) + (n - |\operatorname{supp}(\mathcal{D})|)$$

for all the four pseudoweights.

We may show that some special subcode-complete codes have finite pseudoredundancy and one example of such codes is a binary linear constant-weight code. Since any binary linear constant-weight code consists of copies of a binary simplex code, or equivalently, the value assignment of a linear constant-weight code takes the same value at each point $p \in PG(k-1,2)$, a linear constant-weight code is subcode-complete by Theorem 6. In [11] it is shown that a binary simplex code has finite pseudoredundancy as follows.

Lemma 8 ([11, Prop. 7.8])**.** *For $k \ge 2$, the $[2^k-1, k, 2^{k-1}]$ simplex code $\mathcal{C}$ satisfies*

$$\rho(\mathcal{C}) \;\le\; \frac{(2^k-1)(2^{k-1}-1)}{3}$$

for all the four pseudoweights.

In the proof of Lemma 8 (see [11]), a parity-check matrix H' of $\mathcal{C}$ is chosen such that the rows of H' consist of all the codewords of the Hamming code (the dual code of the simplex code $\mathcal{C}$) with Hamming weight equal to 3. In our framework, the value assignment of the simplex code $\mathcal{C}$ satisfies $m(p) = 1$ for any $p \in PG(k-1,2)$, that is, the columns of a generator matrix of $\mathcal{C}$ are exactly all the different points in $PG(k-1,2)$. By using such a framework, we may give an alternative explanation of the bound in $\rho(\mathcal{C})$ in Lemma 8. Since any row of H' can be viewed as a linear relation of three different columns of the generator matrix of $\mathcal{C}$, any row of H' can also be viewed as a *line* (spanned by two projective points) in $PG(k-1,2)$. Thus, the number of rows in H' equals the number of lines in $PG(k-1,2)$, which is $\frac{1}{3}(2^k-1)(2^{k-1}-1)$.

By using Lemma 8 and the structure of a linear constant-weight code, we obtain:

Theorem 9. *Any binary linear $[n,k,d]$ constant-weight code $\mathcal{C}$ satisfies*

$$\rho(\mathcal{C}) \;\le\; n + \frac{(2^k-1)(2^{k-1}-4)}{3}$$

for all the four pseudoweights.

Proof. Assume the value assignment of the given binary $[n, k, d]$ constant-weight code is $m(\cdot)$. Then, $m(\cdot)$ takes the same value at each point $p \in PG(k-1, 2)$, and then one may get that $n = (2^k - 1)m(p)$ and $d = 2^{k-1}m(p)$ for any point $p \in PG(k-1, 2)$.

Arrange a generator matrix G of the constant-weight code as follows: put each point $p \in PG(k-1, 2)$ once in some fixed order in the columns of G; and then, in the same order, repeat each of these points $m(p) - 1$ times in the columns of G. According to such a matrix G, a parity-check matrix H of $\mathcal{C}$ can be constructed as follows:

$$H = \begin{pmatrix} H' & \mathbf{0} \\ * & I \end{pmatrix},$$

where H' is the parity-check matrix of the simplex code given after Lemma 8, $\mathbf{0}$ stands for a $\frac{1}{3}(2^k - 1)(2^{k-1} - 1) \times (m(p) - 1)(2^k - 1)$ zero matrix, and I stands for a $(m(p) - 1)(2^k - 1) \times (m(p) - 1)(2^k - 1)$ identity matrix; finally, $*$ stands for a $(m(p) - 1)(2^k - 1) \times \frac{1}{3}(2^k - 1)(2^{k-1} - 1)$ matrix, which is written as

$$* = \begin{pmatrix} H_1 \\ \vdots \\ H_{2^k - 1} \end{pmatrix},$$

where each H_i, $1 \le i \le 2^k - 1$, is an $(m(p) - 1) \times \frac{1}{3}(2^k - 1)(2^{k-1} - 1)$ matrix, which has entries zero except for its i-th column, whose entries are all equal to one.

It can be checked that H is a matrix satisfying $HG^T = 0$ and

$$\begin{aligned} \operatorname{rank}(H) &= \operatorname{rank}(H') + (m(p) - 1)(2^k - 1) \\ &= (2^k - 1) - k + (m(p) - 1)(2^k - 1) \\ &= m(p)(2^k - 1) - k = n - k. \end{aligned}$$

Thus, H is a parity-check matrix of $\mathcal{C}$.

For this parity-check matrix H, let $x \in \mathcal{K}(H)$. Then, according to the fundamental cone inequalities (2.1), x may be written as $x = (y, z)$, where $y = (y_1, \ldots, y_{2^k - 1}) \in \mathcal{K}(H')$, and $z = (z_1, \ldots, z_{n-2^k+1}) \in \mathbb{R}^{n-2^k+1}$ is obtained from y by repeating $(m(p) - 1)$ times each coordinate of y. Thus, $w_{\text{maxfrac}}(x)$ can be computed as follows.

$$\begin{aligned} w_{\text{maxfrac}}(x) &= \frac{\sum_{i=1}^{2^k-1} y_i + \sum_{j=1}^{n-2^k+1} z_j}{\max_{i,j}\{y_i, z_j\}} \\ &= \frac{\sum_{i=1}^{2^k-1} y_i + (m(p) - 1)\sum_{i=1}^{2^k-1} y_i}{\max_i\{y_i\}} \\ &= m(p)\frac{\sum_{i=1}^{2^k-1} y_i}{\max_i\{y_i\}} \\ &= m(p)w(y) \end{aligned}$$

$$\geq m(p)2^{k-1} = d \quad \text{(by } y \in \mathcal{K}(H') \text{ and Lemma 8)}.$$

Thus, $w_{\min}(H) \geq d$ for the max-fractional pseudoweight. Since H has

$$\frac{1}{3}(2^k - 1)(2^{k-1} - 1) + (m(p) - 1)(2^k - 1) = n + \frac{1}{3}(2^k - 1)(2^{k-1} - 4)$$

rows, the result follows from (2.2). □

4. *k*-dimensional codes constructed by value assignment

In this section, we will proceed to determine the pseudocodeword redundancies of certain k-dimensional binary codes by making use of the value assignment.

Let $\mathcal{C}$ be an $[n, k]$ binary code determined by a value assignment $m(\cdot)$. Recall from Section 2 the basic fact that

$$\sum_{p \in PG(k-1,2)} m(p) = n.$$

We will use and extend the following results.

Lemma 10 ([11, Lem. 6.1]). *Let H be a parity-check matrix of $\mathcal{C}$ such that every row in H has weight* 2. *Then*:

(1) *There is an equivalence relation on the set $\mathcal{I}$ of column indices of H such that for a vector $x \in \mathbb{R}^n$ with nonnegative coordinates, we have $x \in \mathcal{K}(H)$ if and only if x has equal coordinates within each equivalence class.*

(2) *The minimum distance of $\mathcal{C}$ is equal to its minimum BEC, AWGNC, BSC, and max-fractional pseudoweights with respect to H, i.e., $d(\mathcal{C}) = w_{\min}(H)$.*

Lemma 11 ([11, Prop. 6.2]). *Let H be an $m \times n$ parity-check matrix of $\mathcal{C}$, and assume that the $m-1$ first rows in H have weight* 2. *Denote by $\widetilde{H}$ the $(m-1) \times n$ matrix consisting of these rows, and consider the equivalence relation of the second case of Lemma 10 with respect to $\widetilde{H}$, and assume that $\mathcal{I}_m$ (which is defined in Section 2) intersects each equivalence class in at most one element. Then, the minimum distance of $\mathcal{C}$ is equal to its minimum BEC, AWGNC, BSC, and max-fractional pseudoweights with respect to H, i.e., $d(\mathcal{C}) = w_{\min}(H)$.*

Using these lemmas, in [11, Cor. 6.4] it was shown that all 2-dimensional binary codes $\mathcal{C}$ with length n have pseudoredundancy $\rho(\mathcal{C}) = n-2$, and the proof was conducted according to the analysis of the *supports* of the two codewords generating the 2-dimensional code.

By the framework of the value assignment, we may consider the different cases of the supports of the two codewords generating the 2-dimensional code as *different points* in $PG(1, 2)$. Generalizing this idea, one may consider for each point occurring in the columns of a generator matrix of $\mathcal{C}$ the corresponding equivalence class from Lemma 10, and the size of this equivalence class is exactly the value of the corresponding value assignment at this point. In such a framework, $\mathcal{I}_m$ in the parity-check matrix in Lemma 11 is exactly the linear relation among different points in the columns of the generator matrix of $\mathcal{C}$.

Using the above stated techniques and Lemmas 10 and 11, we will in this section construct several kinds of $[n, k]$ codes whose pseudoredundancies are equal to $n - k$. The first result is:

Theorem 12.

(1) *For any k linearly independent points $p_i \in PG(k-1, 2)$, $1 \leq i \leq k$, if a value assignment $m(\cdot)$ satisfies*

$$m(p) = \begin{cases} z_i \geq 1 & \text{if } p = p_i, \ 1 \leq i \leq k, \\ 0 & \text{otherwise}, \end{cases}$$

and there exists some $1 \leq i_0 \leq k$ such that $m(p_{i_0}) = z_{i_0} \geq 2$, then the $[n, k]$ code $\mathcal{C}$ determined by $m(\cdot)$ satisfies

$$\rho(\mathcal{C}) = \sum_{i=1}^{k} z_i - k = n - k$$

for all the four pseudoweights.

(2) *For any $k+1$ points $p_i \in PG(k-1, 2)$, $1 \leq i \leq k+1$, such that the points p_i, $1 \leq i \leq k$, are linearly independent, if a value assignment satisfies*

$$m(p) = \begin{cases} z_i \geq 1 & \text{if } p = p_i, \ 1 \leq i \leq k+1, \\ 0 & \text{otherwise}, \end{cases}$$

then the $[n, k]$ code $\mathcal{C}$ determined by $m(\cdot)$ satisfies

$$\rho(\mathcal{C}) = \sum_{i=1}^{k+1} z_i - k = n - k$$

for all the four pseudoweights.

Proof. For (1), up to code equivalence, we may arrange a generator matrix G of $\mathcal{C}$ in such a way that the first $m(p_1)$ columns of G are the point p_1, the next $m(p_2)$ columns of G are the point p_2, and in such an order, one proceeds to put the point p_k in the last $m(p_k)$ columns of G. For this matrix G, we construct a matrix H in block diagonal form

$$H = \begin{pmatrix} H_1 & & \\ & \ddots & \\ & & H_k \end{pmatrix},$$

where H_i is an $(m(p_i) - 1) \times m(p_i)$ submatrix whose entries are defined as in (3.1). It can be checked that H is an $(n-k) \times n$ matrix of rank $n - k$ and $HG^T = 0$. Thus, H is a parity-check matrix of $\mathcal{C}$, and so $\rho(\mathcal{C}) = n - k$ by Lemma 10.

For (2), since the points p_i for $1 \le i \le k$ are a basis for $PG(k-1,2)$, one may write p_{k+1} as a linear combination of these basis points. Up to code equivalence, one may write $p_{k+1} = \sum_{j=1}^{s} p_j$ for $s \le k$. Arrange a generator matrix G of $\mathcal{C}$ similarly to the proof of (1), that is, the first $m(p_1)$ columns are the point p_1, the next $m(p_2)$ columns are the point p_2, and in such an order, the last $m(p_{k+1})$ columns are the point p_{k+1}. Then, we may construct a matrix H as

$$H = \begin{pmatrix} H' \\ h \end{pmatrix},$$

where the submatrix H' is the block diagonal one

$$H' = \begin{pmatrix} H_1 & & & \\ & \ddots & & \\ & & H_k & \\ & & & H_{k+1} \end{pmatrix},$$

and H_i for $1 \le i \le k+1$ is an $(m(p_i)-1) \times m(p_i)$ matrix defined as in (3.1), and h is a binary row vector whose coordinate positions corresponding to the first column of each H_i for $1 \le i \le t$ and to the first column of H_{k+1} are equal to one. It can be checked that H is an $(n-k) \times n$ matrix of rank $n-k$ and $HG^T = 0$, and so H is a parity-check matrix of $\mathcal{C}$. Thus, $\rho(\mathcal{C}) = n-k$ by Lemma 11. □

In order to determine the pseudocodeword redundancies of more kinds of k-dimensional codes, it is convenient to introduce the following notations. Let $p_1, \ldots, p_k$ be the points of a basis of $PG(k-1,2)$. Then, any $p \in PG(k-1,2)$ may be written as $p = \sum_{i=1}^{k} c_i p_i$, where $c_i \in \mathrm{GF}(2)$ for $1 \le i \le k$. Call the set of the basis points p_i whose coefficients are nonzero the *representing-set* of the point p with respect to the basis points $p_1, \ldots, p_k$. If the basis points are fixed, one may simply call this set representing-set of the point p.

In the following text of this section, for basis points $p_1, \ldots, p_k$ of $PG(k-1,2)$, let $S_1, \ldots, S_t$ stand for the representing-sets of $p_{k+1}, \ldots, p_{k+t}$, respectively.

Definition 13. The points $p_{k+1}, \ldots, p_{k+t}$ are called *representing-independent* if their representing-sets $S_1, \ldots, S_t$ are pairwise disjoint. They are called *representing-dependent* if for all $1 \le i \le t$ there exists $1 \le j \le t$, $j \ne i$ such that $S_i \cap S_j \ne \varnothing$.

For the points that are representing-independent, we have:

Theorem 14. *For any $k+t$ points $p_i \in PG(k-1,2)$, $1 \le i \le k+t$, such that the points $p_1, \ldots, p_k$ are basis points and the points $p_{k+1}, \ldots, p_{k+t}$ are representing-independent, if the value assignment $m(\cdot)$ satisfies*

$$m(p) = \begin{cases} z_i \ge 1 & \text{if } p = p_i,\ 1 \le i \le k+t, \\ 0 & \text{otherwise}, \end{cases}$$

then the $[n, k]$ *code* $\mathcal{C}$ *determined by* $m(\cdot)$ *satisfies* $\rho(\mathcal{C}) = \sum_{i=1}^{k+t} z_i - k = n - k$ *for all the four pseudoweights.*

Proof. We arrange a generator matrix G of $\mathcal{C}$ in such a way that the first $m(p_1)$ columns of G are the point p_1 and the last $m(p_{k+t})$ columns are the point p_{k+t}. Furthermore, up to code equivalence, we may assume that the representing-set of p_{k+j} is $S_j = \{p_{s_{j-1}+1}, p_{s_{j-1}+2}, \ldots, p_{s_j}\}$, $1 \le j \le t$, where $0 = s_0 < s_1 < s_2 < \cdots < s_t \le k$. Construct a matrix H as

$$H = \begin{pmatrix} H' \\ h_1 \\ \vdots \\ h_t \end{pmatrix}, \tag{4.1}$$

where

$$H' = \begin{pmatrix} H_1 & & & & & \\ & \ddots & & & & \\ & & H_k & & & \\ & & & H_{k+1} & & \\ & & & & \ddots & \\ & & & & & H_{k+t} \end{pmatrix}$$

is a block diagonal submatrix, and H_i for $1 \le i \le k+t$ is defined as in (3.1), and h_i for $1 \le i \le t$ is a binary row vector, and we demand that the coordinate position of h_j, $1 \le j \le t$, corresponding to the first column of H_i for $s_{j-1}+1 \le i \le s_j$ and to the first column of H_{k+j} be equal to one. Then, it can be checked that $HG^T = 0$ and $\operatorname{rank}(H) = n - k$, thus, H is a parity-check matrix of $\mathcal{C}$.

Consider the Tanner graph of the code $\mathcal{C}$ with respect to the parity-check matrix H. It is easy to see that this Tanner graph is a disjoint union of trees, i.e., it does not have any cycles. From [10, Lem. 28] it follows that the fundamental polytope equals the code polytope. Therefore, there do not exist any proper pseudocodewords. Hence, it holds that $\rho(\mathcal{C}) = n - k$ for all the four pseudoweights. $\square$

For the representing-dependent case, it is more complicated to determine the pseudocodeword redundancy, as the codes will have in general no cycle-free Tanner graph representation. However, we may get some results about some particular codes.

Assume that $p_1, \ldots, p_k$ are basis points of $PG(k-1, 2)$ and that $p_{k+1}, \ldots, p_{k+t}$ are representing-dependent. Denote by U_1 the basis points which belong to one and only one representing-set and denote by U_2 the basis points which belong to at least two representing-sets, so that $U_2 = (\bigcup_{i=1}^{t} S_i) \setminus U_1$. Define $U_3 = U_1 \cup \{p_{k+1}, p_{k+2}, \ldots, p_{k+t}\}$.

Theorem 15. *Let the notations be defined as above and assume one of the following conditions holds*:

(1) $S_i \cap U_1 \neq \varnothing$ *for each* $1 \leq i \leq t$, *and* $\min\{m(p) \mid p \in U_2\} \geq \max\{m(p) \mid p \in U_3\}$,

(2) $W_1 = \{j \mid S_j \cap U_1 \neq \varnothing\} \neq \varnothing$ *and* $W_2 = \{j \mid S_j \cap U_1 = \varnothing\} \neq \varnothing$ *and* $\min\{m(p) \mid p \in U_2\} \geq \max\{m(p) \mid p \in U_3\}$ *and* $\min\{m(p_{k+j}) \mid j \in W_1\} \leq \min\{m(p_{k+j}) \mid j \in W_2\}$,

(3) $|\bigcap_{i=1}^{t} S_i| \geq 2$ *and* $\max\{m(p) \mid p \in \bigcap_{i=1}^{t} S_i\} \leq \min\{m(p) \mid p \in ((\bigcup_{i=1}^{t} S_i) \setminus (\bigcap_{i=1}^{t} S_i)) \cup \{p_{k+1}, \ldots, p_{k+t}\}\}$.

Then, the $[n, k]$ *code* $\mathcal{C}$ *determined by*

$$m(p) = \begin{cases} z_i \geq 1 & \text{if } p = p_i,\ 1 \leq i \leq k+t, \\ 0 & \text{otherwise}, \end{cases}$$

satisfies $\rho(\mathcal{C}) = n - k$ *for all the four pseudoweights.*

Proof. The proof is similar for the three cases. We only give the proof for the first case. Arrange a generator matrix G of $\mathcal{C}$ as before, namely, put the points p_i for $1 \leq i \leq k+t$ in order in the columns of G, and each point p_i, $1 \leq i \leq k+t$, repeats $m(p_i)$ times.

Construct a matrix H as in (4.1), and the binary vector h_j in H, $1 \leq j \leq t$, is determined by the representing-set S_j of the point p_{k+j}. If $S_j = \{p_{i_1}, p_{i_2}, \ldots, p_{i_\theta}\}$, then h_j has a one in each coordinate position corresponding to the first column of H_{k+j} and to the first column of H_{i_ℓ}, $1 \leq \ell \leq \theta$. Then it can be checked that $HG^T = 0$ and $\operatorname{rank}(H) = \sum_{i=1}^{k+t} m(p_i) - k = n - k$. Thus, H is a parity-check matrix of $\mathcal{C}$.

For $1 \leq i \leq t$, let $T_i = S_i \cup \{p_{k+i}\}$, and let $V = \{p_1, \ldots, p_k\} \setminus (\bigcup_{i=1}^{t} S_i)$. Define

$$\begin{aligned} \gamma_i &= \min\{m(p_{j_1}) + m(p_{j_2}) \mid p_{j_1}, p_{j_2} \in T_i \setminus U_2,\ j_1 \neq j_2\}, \\ \gamma &= \min_{1 \leq i \leq t}\{\gamma_i\}, \\ \delta &= \min\{m(p) \mid p \in V\}. \end{aligned}$$

Different from the representing-independent case, the analysis of the codewords with minimum (Hamming) weight is tedious in the representing-dependent case. In general, according to the construction of the parity-check matrix H, one may divide the possible codewords with minimum weight into two classes: one class is the codewords with nonzero coordinate in the position corresponding to some point in U_2, and the other class is the ones with zero coordinate in the position corresponding to any point in U_2 (note that $U_2 = \varnothing$ in the representing-independent case). The confined condition $\min\{m(p) \mid p \in U_2\} \geq \max\{m(p) \mid p \in U_3\}$ plays a key role in determining the codewords with minimum weight.

More concretely, since $\min\{m(p) \mid p \in U_2\} \geq \max\{m(p) \mid p \in U_3\}$, it follows that $d(\mathcal{C}) = \min\{\gamma, \delta\}$ by analyzing the constructed parity-check matrix H, that is, the codewords with minimum weight should be ones with zero coordinates in the positions corresponding to any point in U_2.

On the other hand, for $x \in \mathcal{K}(H)$ and $p \in T_i$, $1 \le i \le t$, we have

$$x_p \le \sum_{p' \in T_i \setminus \{p\}} x_{p'} \qquad \text{for all } p \in T_i,$$

and thus by the assumption $\min\{m(p) \mid p \in U_2\} \ge \max\{m(p) \mid p \in U_3\}$, we get

$$\begin{aligned} m(p'')x_p &\le m(p'')\Big(\sum_{p' \in T_i \setminus \{p\}} x_{p'}\Big) \quad (\text{for some } p'' \in T_i \setminus U_2) \\ &\le \sum_{p' \in T_i \setminus \{p\}} m(p')x_{p'}. \end{aligned}$$

Thus,

$$(m(p'') + m(p))x_p \le \sum_{p' \in T_i} m(p')x_{p'},$$

that is,

$$m(p'') + m(p) \le \frac{\sum_{p' \in T_i} m(p')x_{p'}}{x_p}. \tag{4.2}$$

Since $\min\{m(p) \mid p \in U_2\} \ge \max\{m(p) \mid p \in U_3\}$, $\gamma_i \le m(p'') + m(p)$ always holds no matter $p \in (T_i \setminus U_2)$ or $p \in (T_i \cap U_2)$, $1 \le i \le t$. Thus, (4.2) can be rewritten as

$$\gamma_i \le m(p'') + m(p) \le \frac{\sum_{p' \in T_i} m(p')x_{p'}}{x_p}. \tag{4.3}$$

Therefore,

$$\begin{aligned} w_{\mathrm{maxfrac}}(x) &= \frac{\sum_{j=1}^n x_j}{\max_j\{x_j\}} \\ &= \frac{\sum_p m(p)x_p}{\max\{x_p\}}, \quad p \in \bigcup_{i=1}^t T_i \cup V \\ &\ge \begin{cases} \gamma_i & \text{if } \max\{x_p\} = x_{p_0} \text{ and } p_0 \in T_i,\ 1 \le i \le t \ \ (\text{by (4.3)}), \\ \delta & \text{if } \max\{x_p\} = x_{p_0} \text{ and } p_0 \in V. \end{cases} \end{aligned}$$

Thus, $w_{\min}(H) \ge d(\mathcal{C}) = \min\{\gamma, \delta\}$ for the max-fractional pseudoweight. From (2.2), we have $\rho(\mathcal{C}) \le n - k$ for all the four pseudoweights. Since $\rho(\mathcal{C}) \ge n - k$ by definition, we have $\rho(\mathcal{C}) = n - k$ for all the four pseudoweights. $\square$

Example 16. Let $k = 6$ and $t = 3$; consider the linear code $\mathcal{C}$ generated by the matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Denote the i-th column of the matrix G by p_i, $1 \le i \le 9$. Then, p_i for $1 \le i \le 6$ are basis points of $PG(5, 2)$, and the points p_7, p_8, and p_9 have representing-sets $S_1 = \{p_1, p_4, p_5\}$, $S_2 = \{p_2, p_5, p_6\}$, and $S_3 = \{p_3, p_4, p_6\}$, respectively. One sees that $U_1 = \{p_1, p_2, p_3\}$, $U_2 = (S_1 \cup S_2 \cup S_3) \setminus U_1 = \{p_4, p_5, p_6\}$, and $U_3 = U_1 \cup \{p_7, p_8, p_9\} = \{p_1, p_2, p_3, p_7, p_8, p_9\}$. Furthermore, $S_1 \cap S_2 \cap S_3 = \varnothing$ and $S_i \cap U_1 \neq \varnothing$ for each $1 \le i \le 3$ and $m(p_i) = 1$ for $1 \le i \le 9$. Thus, the points p_i, $1 \le i \le 9$, exactly satisfy the conditions of Case 1) in Theorem 15, and therefore, $\rho(\mathcal{C}) = n - k = 9 - 6 = 3$.

We remark that the code $\mathcal{C}$ is not *cycle-free*, i.e., the Tanner graph of any parity-check matrix of $\mathcal{C}$ has a cycle, as we will now demonstrate. According to the proof of Theorem 15, one may take a parity-check matrix H of $\mathcal{C}$ as

$$H = \begin{pmatrix} 1\,0\,0\,1\,1\,0\,1\,0\,0 \\ 0\,1\,0\,0\,1\,1\,0\,1\,0 \\ 0\,0\,1\,1\,0\,1\,0\,0\,1 \end{pmatrix}.$$

Obviously, the three rows of this H are linearly independent and H has a cycle located at the 3-th, 4-th, and 5-th coordinates.

Due to the fact that any parity-check matrix with 3 rows may be written as PH, where P is a 3×3 invertible matrix, it suffices to check that any matrix PH has a cycle for any invertible 3×3 matrix P. In fact, one may list all binary invertible 3×3 matrices P, and then check that PH has a cycle for each such P. A simpler argument is to make use of the form of the matrix H. Observe that H may be written as (I, M, I), where I is the 3×3 identity matrix, and

$$M = \begin{pmatrix} 1\,1\,0 \\ 0\,1\,1 \\ 1\,0\,1 \end{pmatrix}.$$

Thus, $PH = (P, PM, P)$. Since P is a binary 3×3 invertible matrix, the number of ones in P, denoted by $\mathcal{N}(P)$, should satisfy $\mathcal{N}(P) \ge 3$. If $\mathcal{N}(P) \ge 4$, then there exists a column in P such that the number of ones in the column is at least two. Since such a column will occur both in the first block P and in the last block P in the matrix $PH = (P, PM, P)$, the cycle can be found in these two same columns. The remaining case is $\mathcal{N}(P) = 3$, and in this case, the block PM in $PH = (P, PM, P)$ is just the permutations of the rows of M, thus, the block PM contains a cycle since M contains a cycle. These arguments show that $\mathcal{C}$ is not cycle-free.

Example 17. Let $k = 4$ and $t = 3$; consider the code $\mathcal{C}$ generated by the matrix

$$G = \begin{pmatrix} 1\,0\,0\,0\,1\,0\,0 \\ 0\,1\,0\,0\,1\,0\,1 \\ 0\,0\,1\,0\,0\,1\,1 \\ 0\,0\,0\,1\,1\,1\,0 \end{pmatrix}.$$

It can be checked that the first four columns of G, $p_1, \ldots, p_4$, are basis points in $PG(3,2)$, and the last three columns, p_5, p_6, and p_7, have representing-sets $S_1 = \{p_1, p_2, p_4\}$, $S_2 = \{p_3, p_4\}$, and $S_3 = \{p_2, p_3\}$, respectively. In addition, $U_1 = \{p_1\}$, $U_2 = \{p_2, p_3, p_4\}$, and $U_3 = \{p_1, p_5, p_6, p_7\}$. Furthermore, $S_1 \cap S_2 \cap S_3 = \varnothing$, $S_1 \cap U_1 = \{p_1\}$, $S_2 \cap U_1 = S_3 \cap U_1 = \varnothing$, and $m(p_i) = 1$ for $1 \le i \le 7$. Thus, the points p_i, $1 \le i \le 7$, exactly satisfy the conditions of Case 2) in Theorem 15, and therefore $\rho(\mathcal{C}) = n - k = 7 - 4 = 3$.

Similarly to Example 16, one may show that $\mathcal{C}$ is not cycle-free by taking a parity-check matrix H of $\mathcal{C}$ as

$$H = \begin{pmatrix} 1\,1\,0\,1\,1\,0\,0 \\ 0\,0\,1\,1\,0\,1\,0 \\ 0\,1\,1\,0\,0\,0\,1 \end{pmatrix}.$$

Example 18. Let $k = 4$ and $t = 3$; consider the code $\mathcal{C}$ generated by the matrix

$$G = \begin{pmatrix} 1\,0\,0\,0\,1\,1\,1 \\ 0\,1\,0\,0\,1\,1\,1 \\ 0\,0\,1\,0\,0\,1\,1 \\ 0\,0\,0\,1\,1\,1\,0 \end{pmatrix}.$$

Then, the first four columns of G, $p_1, \ldots, p_4$, are the basis points of $PG(3,2)$, and the last three columns of G, p_5, p_6, and p_7, have representing-sets $S_1 = \{p_1, p_2, p_4\}$, $S_2 = \{p_1, p_2, p_3, p_4\}$, and $S_3 = \{p_1, p_2, p_3\}$, respectively. In addition, $U_1 = \varnothing$, $U_2 = \{p_1, p_2, p_3, p_4\}$, $U_3 = \{p_5, p_6, p_7\}$, $S_1 \cap S_2 \cap S_3 = \{p_1, p_2\}$, and $m(p_i) = 1$ for $1 \le i \le 7$. Thus, the points p_i for $1 \le i \le 7$ exactly satisfy the conditions of Case 3) in Theorem 15, so that $\rho(\mathcal{C}) = n - k = 7 - 4 = 3$.

Similarly to the above two examples, one may show that $\mathcal{C}$ is not cycle-free by taking a parity-check matrix H of $\mathcal{C}$ as

$$H = \begin{pmatrix} 1\,1\,0\,1\,1\,0\,0 \\ 1\,1\,1\,1\,0\,1\,0 \\ 1\,1\,1\,0\,0\,0\,1 \end{pmatrix}.$$

Remark 19. Along the line of Theorem 15, we may use the value assignment to get other kinds of codes whose pseudoredundancies can be determined, however, the conditions will be too tedious to get more information. So, we omit them.

Summing up Theorems 14 and 15 and using similar arguments as in these two theorems, we get in general the following (the detailed proof is omitted):

Theorem 20. *Assume that the points p_{k+i}, $1 \le i \le t$, can be divided into ℓ subsets such that*:

(1) *the representing-sets of points from different classes do not intersect*;
(2) *each of these ℓ classes is either an representing-independent one or an representing-dependent one satisfying the conditions of Theorem 15.*

Then, the $[n, k]$ code $\mathcal{C}$ determined by

$$m(p) = \begin{cases} z_i \geq 1 & \text{if } p = p_i,\ 1 \leq i \leq k + t, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies $\rho(\mathcal{C}) = n - k$ for all the four pseudoweights.

5. Conclusion

Making use of the value assignment, we derived upper bounds on the pseudoredundancies for certain binary codes with repeated and added coordinates and for certain shortened subcodes. Also, we constructed several kinds of k-dimensional binary linear codes by using the value assignment; the pseudoredundancies for all of the four pseudoweights of these binary linear codes are fully determined.

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