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EXTREMAL CONFIGURATIONS OF THREE OR FOUR SYMMETRIES ON A RIEMANN SURFACE

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Abstract. We consider Riemann surfaces with three or four symmetries, assuming that they have a maximal total number of ovals and find all the possible topological types of the symmetries realizing such a configuration.

1. Introduction

A Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \geq 2$, where Γ is a Fuchsian surface group, will be called *symmetric* if it admits an antiholomorphic involution $\tau \in G = \mathrm{Aut}^{\pm}(X)$, called a *symmetry* of X. Now the set of points fixed by τ consists of at most $g+1$ disjoint simple closed curves called *ovals*. If the set $X \setminus \mathrm{Fix}(\tau)$ is disconnected, then we call τ to be *separating* and we call it *non-separating* in the other case. Moreover, we define a *topological type* of τ to be a symbol $\pm k$, where $k \geq 0$ denotes the number of ovals of τ, and the sign depends on the separability of τ: $+$ for separating, $-$ for non-separating symmetry.

It is known that three symmetries on a Riemann surface of genus g have at most $2g+4$ ovals in total and if the bound is attained, then g is odd and the three symmetries commute. Similarly, for four symmetries the bound is $2g+8$ and the group generated by the symmetries is $\mathrm{D}_n \times \mathrm{Z}_2^2$ (see Natanzon [7], Bujalance-Costa [1]). Furthermore, for even values of g the bound for three symmetries is $2g+3$ and it is sharp for arbitrary even g also only with commuting symmetries. For four symmetries the bound is $2g+2$ in such a case and the group is $\mathrm{D}_n \times \mathrm{Z}_2$ (Gromadzki-Izquierdo [5]). Our aim in this paper is to find all the possible topological types for three or four symmetries that realize the bound on the maximal total number of ovals. Not surprisingly, in the commuting case there are numerous possible configurations, while in the other case the maximal total number of ovals is realized in a single possible way.

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2. Preliminaries

The main tool in our paper is the theory of non-Euclidean crystallographic groups (*NEC groups* in short), which are just the discrete and cocompact subgroups of the group $\mathcal{G}$ of all the isometries of the hyperbolic plane $\mathcal{H}$. The algebraic structure of such a group Λ is determined by the so-called *signature*:

$$(1)\qquad s(\Lambda) = (h; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k}), (-)^l\}),$$

where the brackets $(n_{i1}, \dots, n_{is_i})$ are called *the period cycles*, the integers n_{ij} are the *link periods*, m_i are the *proper periods* and finally h is the *orbit genus* of Λ. We shall also denote $s = s_1 + \cdots + s_k$. The algebraic presentation for the group Λ with signature (1) is as follows, where generators used are called *canonical*:

$$x_1, \dots, x_r,\ e_i,\ c_{ij},\ 1 \le i \le k+l,\ 0 \le j \le s_i$$

and $a_1, b_1, \dots, a_h, b_h$ if the sign is $+$ or $d_1, \dots, d_h$ otherwise. Moreover, we have relators: $x_i^{m_i}, i = 1, \dots, r,\ c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i,\ i = 1, \dots, k+l,$ $j = 0, \dots, s_i$ and

$$x_1 \cdots x_r e_1 \cdots e_{k+l} a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1} \text{ or } x_1 \cdots x_r e_1 \cdots e_{k+l} d_1^2 \cdots d_h^2,$$

according to whether the sign is $+$ or $-$. Every element of finite order in Λ is conjugate either to a canonical reflection or to a power of some canonical elliptic element x_i or else to a power of the product of two consecutive canonical reflections. An abstract group with such a presentation can be realized as an NEC group Λ if and only if the value

$$2\pi\left(\varepsilon h + k + l - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

where $\varepsilon = 2$ or 1 according to the sign being $+$ or $-$, is positive. The value above is just the hyperbolic area $\mu(\Lambda)$ of any fundamental region for the group Λ and the Hurwitz-Riemann formula holds:

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda),$$

where Λ' is a subgroup of finite index in an NEC group Λ.

Particularly important in this theme are the torsion free Fuchsian groups. Recall that such a group Γ is called a *surface group* and it has signature $(g; -)$. In such a case $\mathcal{H}/\Gamma$ is a compact Riemann surface of genus $g \ge 2$ and conversely, any compact Riemann surface of genus $g \ge 2$ can be represented as such an orbit space for some Fuchsian surface group Γ of genus g. Furthermore, given a Riemann surface so represented, a finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ containing Γ as a normal subgroup. Let $C(G, g)$ denote the centralizer of an element g in G. The following result from [3] is crucial for the paper.

Theorem 2.1. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with a group G of all automorphisms of X, let $G = \Lambda/\Gamma$ for some NEC group Λ and let $\theta : \Lambda \to G$ be the canonical epimorphism. Then the number of ovals of a symmetry τ of X equals*

$$\sum [C(G, \theta(c_i)) \,:\, \theta(C(\Lambda, c_i))],$$

the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to τ.

Now to determine the number of ovals we also need to know the order of the centralizer of a reflection in an NEC group. This can be done with the following result of Singerman from [8].

Theorem 2.2. *Let $c_0, c_1, \ldots, c_s, e$ be the system of canonical reflections corresponding to a period cycle $(n_1, \ldots, n_s)$ of an NEC group Λ with signature* (1). *If all n_i are even, then the centralizer $C(\Lambda, c_i)$ equals*

$$\begin{array}{ll} \langle c_i\rangle \times \big(\langle (c_{i-1}c_i)^{n_i/2}\rangle * \langle (c_ic_{i+1})^{n_{i+1}/2}\rangle\big) = \mathrm{Z}_2 \times (\mathrm{Z}_2 * \mathrm{Z}_2) & \textit{for } i \neq 0, \\ \langle c_0\rangle \times \big(\langle (c_0c_1)^{n_1/2}\rangle * \langle e^{-1}(c_{s-1}c_s)^{n_s/2}e\rangle\big) = \mathrm{Z}_2 \times (\mathrm{Z}_2 * \mathrm{Z}_2) & \textit{for } i = 0, \\ \langle c_0\rangle \times \langle e\rangle = \mathrm{Z}_2 \times \mathrm{Z} & \textit{for } s = 0. \end{array}$$

With the two above results we are in position to compute the number of ovals of a symmetry. To complete our task with the topological type, we shall need the following result, which can be found in [2], and will allow us to determine the separability character of the symmetries in question. Let Λ' be a normal subgroup of an NEC group Λ. A canonical generator of Λ is *proper* (with respect to Λ') if it does not belong to Λ'. The elements of Λ expressible as a composition of proper generators of Λ are the *words* of Λ (with respect to Λ'). We have:

Lemma 2.3 (c.f. Theorem 2.1.3)**.** *Suppose that $[\Lambda : \Lambda']$ is even and Λ has sign $+$. Then Λ' has sign $+$ if and only if no orientation reversing word belongs to Λ'. If $[\Lambda : \Lambda']$ is even and Λ has the sign $-$, then Λ' has the sign $-$ if and only if either a glide reflection of the canonical generators of Λ or an orientation reversing word belongs to Λ'.*

The above result can be used to determine separability for these symmetries, which are central in the automorphism group. For the non-central symmetries one can use the Schreier coset graph method described for example in [6].

3. Possible topological types of the symmetries

The starting points for this section are the results concerning the maximal total number of ovals of three or four symmetries and the group structure for surfaces realizing that bound:

Theorem 3.1 (Natanzon [7])**.** *The maximal total number of ovals of symmetries of a Riemann surface of genus g is $2g+4$ for three symmetries and $2g+8$ for four symmetries.*

Theorem 3.2 (Bujalance, Costa [1]). *If three symmetries on a Riemann surface of genus g have $2g+4$ ovals in total, then they commute. However, if four symmetries on a Riemann surface of genus g have $2g+8$ ovals, then they generate the group $\mathrm{D}_n \times \mathrm{Z}_2^2$.*

Theorem 3.3 (Gromadzki, Izquierdo [5]). *Three non-conjugate symmetries of a Riemann surface of even genus g have at most $2g+3$ ovals in total and this bound is attained for arbitrary even g and only for commuting symmetries. In the case of four symmetries the bound is $2g+2$ and it is attained for arbitrary even g with the group of automorphisms being of the form $\mathrm{D}_n \times \mathrm{Z}_2$.*

Let us begin with the case of three symmetries. As our symmetries commute, we actually have four anticonformal involutions x, y, z, xyz where $G = \langle x, y, z \rangle = \mathrm{Z}_2^3$. As we shall see, one of these, say xyz will be fixed-point free, while all the others will be separating. First we shall find the only possible signature of an NEC group Λ, see also [5].

To begin with, let us assume that $X = \mathcal{H}/\Gamma$ is a Riemann surface of genus $g \geq 2$ admitting three symmetries x, y, z with the maximal total number of ovals. Now $G = \Lambda/\Gamma$ for some NEC group Λ with signature

$$(h; \pm; [2, \overset{m}{\ldots}, 2]; \{(2, \overset{s_1}{\ldots}, 2), \overset{k}{\ldots}, (2, \overset{s_k}{\ldots}, 2), (-)^l\}), \tag{2}$$

where $s = s_1 + \cdots + s_k$. Observe that all the proper and link periods must be equal 2 as the group G only has nontrivial elements of order 2. Now let $\theta : \Lambda \to G$ be the canonical epimorphism and let t denote the total number of ovals of symmetries x, y, z. We know that $t = 2g+3$ for g even and $t = 2g+4$ for g odd. Now by Theorems 2.1 and 2.2 a reflection c in a non-empty period cycle contributes to $\theta(c)$ with 2 ovals if its neighbors have the same image under θ and with 1 oval otherwise. Similarly, a reflection c in an empty period cycle contributes with 4 ovals if $\theta(e) = 1$ for the corresponding generator e and with 2 ovals otherwise. Summing up, $2g+3 \leq t \leq 2s+4l$ and by the Hurwitz-Riemann formula

$$\frac{g-1}{4} = \varepsilon h + k + l - 2 + \frac{m}{2} + \frac{s}{4} \geq \varepsilon h + k + \frac{l}{2} + \frac{m}{2} - 2 + \frac{t}{8}$$

which in turn gives $2g+3 \leq t \leq 2g+14-8\varepsilon h-8k-4l-4m$ and $8\varepsilon h + 8k + 4l + 4m \leq 11$, which is only possible if $k=1, h=m=l=0$. Indeed, $k \geq 1$ forces all the other parameters to be equal 0. Now if $k=0$, then there are only empty period cycles and so the total number of ovals is even, which is impossible for the case g even. If g is odd and $k=0$, then we have $t \leq 4l$ and

$$\frac{g-1}{4} = l - 2 + \frac{m}{2} \geq \frac{t}{4} - 2 + \frac{m}{2}$$

and in turn $3 - 2m \geq g$ as $t = 2g+4$. This is only possible for $g=3$ and $m=0$. But then $\frac{g-1}{4} = \frac{1}{2}$ and so the normalized hyperbolic area of Λ is not an integer, hence $m \neq 0$, a contradiction. Therefore we are left with a signature for Λ being of the form

$$(0; +; [-]; \{(2, \overset{g+3}{\ldots}, 2)\}). \tag{3}$$

Here we are in position to discuss the separability character of the symmetries x, y, z. By Lemma 2.3 we see, that there are no orientation reversing words for any of the symmetries. Indeed, the sign of Λ is $+$, there are no proper periods and no other canonical generators mapped nontrivially to orientation preserving elements of G. Hence the only non-separating symmetry is xyz, while the others are separating. Let us also note that this means that for g even, the numbers of ovals are odd and for g odd, the numbers of ovals are even.

Now, to give the possible topological types, we have to see how many of the reflections have neighbors with distinct images under θ. This can be done in several ways (see for example [5]). As we already know the maximal number of ovals, we can use this fact to simplify the proof. Observe, that the only non-empty period cycle allows at most $2(g+3)$ ovals. Now for the odd genus, we have $2g+4$ ovals, hence two of the reflections must contribute with 1 oval. Moreover, these two contribute to the same symmetry, as the numbers of ovals are even. For the case of g being even, we have $2g+3$ ovals, hence there are three reflections which contribute with only 1 oval to the respective symmetry. Also, these must contribute to three different symmetries as the numbers of ovals are all odd. Now we are ready to specify the epimorphism θ and find the topological types of the symmetries.

Let first g be odd. As exactly two of the reflections contribute with 1 oval and they contribute to the same symmetry - say x, our epimorphism θ must map the canonical reflections respectively to

$$\underbrace{x, y, \ldots, x, y}_{2\alpha}, \underbrace{x, z, \ldots, x, z}_{2\beta},$$

where $2\alpha + 2\beta = g + 3$. Symmetries y and z always appear with the same neighbors. Observe here that x must have $2\alpha + 2\beta - 2 = g + 1$ ovals, y has 2α ovals and z has 2β ovals.

Now if g is even, then we have 3 commuting symmetries with $2g+3$ ovals in total and the epimorphism is of the form:

$$x, \underbrace{y, x, \ldots, y, x}_{2\alpha}, y, \underbrace{z, y, \ldots, z, y}_{2\beta}, z, \underbrace{x, z, \ldots, x, z}_{2\gamma},$$

where $2\alpha + 2\beta + 2\gamma = g$. Here x has $2\alpha + 2\gamma + 1$ ovals, y has $2\alpha + 2\beta + 1$ ovals, z has $2\beta + 2\gamma + 1$ ovals. Summing up, we found all the triples of symmetry types that realize the bound on the total number of ovals, hence we have proved the following theorem.

Theorem 3.4. *If a Riemann surface of genus g admits three non-conjugate symmetries with a maximal total number of ovals, then the symmetries commute and their topological types are*:

1. *for odd* g: $+(g+1), +2\alpha, +2\beta$ *with* $2\alpha+2\beta = g+3$, *where* $\alpha, \beta > 0$ *are integers*;
2. *for even* g: $+(2\alpha+2\beta+1), +(2\alpha+2\gamma+1), +(2\beta+2\gamma+1)$ *with* $2\alpha+2\beta+2\gamma = g$, *where* $\alpha, \beta, \gamma \geq 0$ *are integers.*

Conversely, for all such sets of integers α, β, γ, g *as above we constructed a Riemann surface having three commuting symmetries with specified types and the maximal configuration of ovals.*

Now we shall consider the case of four symmetries. Let first g be even. In such a case, by the results of Gromadzki and Izquierdo [5] we know that the maximal total number of ovals is $2g+2$ and when attained, the symmetries generate the group $G = \mathrm{D}_n \times \mathrm{Z}_2$. Let us assume first that the four symmetries have $2g+2$ ovals in total and two of them do not commute, that is $n > 2$. As usual, we have an NEC group Λ with signature (1) and an epimorphism $\theta : \Lambda \to G$. Observe, that by Theorems 2.1 and 2.2 a reflection corresponding to the non-empty period cycle can give at most n ovals if its image is a central symmetry or at most 2 ovals otherwise. Similarly, a reflection corresponding to the empty period cycle gives at most $2n$ ovals - for a central image - or 4 ovals otherwise. Now we shall find the only possible NEC signature for Λ and the only possible epimorphism θ. For, observe first that as g is even, then by the Hurwitz-Riemann formula there is an odd number of link periods equal to n in the signature of Λ. Hence $k \neq 0$. Now if $h+r+l > 0$, then

$$\frac{g-1}{2n} \geq -1 + \frac{1}{2} + \frac{2l+s-1}{4} + \frac{1}{2} - \frac{1}{2n}$$

and in turn $2l+s \leq \frac{2g}{n}+1$. Now the total number of ovals t holds $t \leq (2l+s-2)n+4 \leq 2g-n+4 < 2g+2$, therefore $h = r = 0$ and there is only one, non-empty, period cycle in the signature of Λ. Now we may treat our reflections and its images, being the symmetries, as situated on a circle and by Lemma 3.3 in [4], at least three of the reflections have images with distinct neighbors under θ. Observe that at least two of the ones with distinct neighbors must be central. Indeed, it is impossible that all the central symmetries appearing in the cycle have the same neighbors. In such a case either all the symmetries of the cycle would be central or at least 4 of the symmetries in the cycle are non-central with at least three having distinct neighbors and hence $t \leq (s-4)n+2+3\cdot 1 \leq 2g-n+5 < 2g+2$ as $s \leq \frac{2g}{n}+3$ by the Hurwitz-Riemann formula.

Now if only one of the central symmetries has distinct neighbors, then for the epimorphism to exist these neighbors must be non-central (otherwise as above we have only central symmetries in the cycle and we get a contradiction). But then again the non-central symmetries appear at least 4 times, at least two times with distinct neighbors, and so $t \leq (s-5)n + \frac{n}{2} + 6 \leq 2g - 2n + \frac{n}{2} + 6 < 2g+2$.

Therefore indeed at least two of the central symmetries have distinct neighbors. Now if there would be at least three non-central symmetries, then $t \leq (s-5)n + 2\cdot\frac{n}{2} + 5 \leq 2g - 2n + n + 5 < 2g+2$.

Hence each of the non-central symmetries appears only once, every time with distinct neighbors as they must be consecutive - recall that there is a link period equal n. Now if at least three of the central symmetries have distinct neighbors, then $t \leq (s-5)n + 3n/2 + 2 \leq 2g - 2n + 3n/2 + 2 < 2g + 2$.

Summing up, for $G = \mathrm{D}_n \times \mathrm{Z}_2 = \langle a, b \,|\, a^2, b^2, (ab)^n \rangle \times \langle x | x^2 \rangle$, the epimorphism θ maps, without loss of generality, the consecutive canonical reflections to

$$a,\ b,\ x,\ x(ab)^{n/2},\ x,\ x(ab)^{n/2}, \ldots, x \text{ or } x(ab)^{n/2}.$$

In both cases it is easy to see that a and b have 1 oval each, while x and $x(ab)^{n/2}$ have g ovals each. Clearly $x, x(ab)^{n/2}$ are non-separating. Also, by considering the Schreier coset graph, like in [6], it is easy to see that a, b are separating.

Let now g be even and the symmetries x, y, z, xyz commute, we have $G = \mathrm{Z}_2^3 = \langle x, y, z \rangle$ here. Assume that these symmetries have the maximal number of $2g+2$ ovals in total. Now again we shall find the only NEC signature for Λ and the epimorphism θ. Observe first, that by Theorems 2.1 and 2.2, a symmetry corresponding to an non-empty period cycle gives 2 ovals if it has the same neighbors and 1 oval if it has distinct neighbors. A symmetry corresponding to the empty period cycle gives 4 or 2 ovals accordingly. Therefore the total number of ovals holds $t \leq 4l + 2s$ for Λ having the signature (1), and observe again that $k > 0$, as there is an odd number of link periods in the signature. Now if $l > 0, k \geq 2$, then

$$\frac{g-1}{4} \geq -1 + 1 + \frac{1}{2} + \frac{2l+s}{4}$$

and in turn $t \leq 2g - 6$, a contradiction. Also, if $l \geq 2, k = 1$, then similarly $t \leq 2g - 2$, a contradiction again. Now if $l = k = 1$, then by Lemma 3.3 in [4], at least two of the symmetries in the non-empty period cycle have distinct neighbors and so $t \leq 4 + 2s - 2$ and by the Hurwitz-Riemann formula

$$\frac{g-1}{4} \geq -1 + \frac{1}{2} + \frac{2+s}{4}$$

and so $t \leq 2g$, a contradiction. Hence again Λ has signature (3). As there are exactly $2g+2$ ovals, it follows that exactly four of the symmetries in the cycle have neighbors with distinct labels. To find the only possible type of epimorphism, we shall analyze the spots corresponding to the symmetries with distinct neighbors in our cycle. Observe first that it is impossible that all the spots with distinct neighbors belong to one symmetry, say x. Indeed, between two such spots we always have an odd number of symmetries, which leads to an even length of the period cycle, a contradiction as $s = g + 3$. Similarly, if each of the spots belongs to a different symmetry, then the numbers of symmetries are always even leading to the same result. Now if three of these spots belong to one symmetry, say x then, as we treat the symmetries as situated on a circle, we have a situation where between two appearances of x with distinct neighbors there is exactly one spot with distinct neighbors belonging to another symmetry. This clearly is impossible. Therefore we are left with a situation where exactly two of the spots with distinct neighbors contribute to one symmetry, say x. It cannot be that these two spots are separated by another one belonging to a symmetry different than x, so without loss of generality we are left with the epimorphism of the type:

$$\underbrace{x,\ y,\ x,\ldots,y,}_{2\alpha}\underbrace{x,\ z,\ x,\ldots,z,}_{2\beta}\underbrace{xyz,\ z,\ldots,z,}_{2\gamma}\underbrace{xyz,\ x,\ xyz,\ldots,x,}_{2\delta}xyz.$$

We easily compute the numbers of ovals by Theorems 2.1 and 2.2 and obtain that x has $2\alpha+2\beta+2\delta-2$ ovals, y has 2α ovals, z has $2\beta+2\gamma-1$ ovals and xyz has $2\gamma+2\delta+1$ ovals. By Lemma 2.3, all the symmetries are non-separating as for each one we can easily find the orientation reversing word in the corresponding NEC subgroup of Λ.

Let us now assume that g is odd. Recall that the maximal total number of ovals for four symmetries is $2g+8$ and it is attained with the group generated by the symmetries being $G = \mathrm{D}_n \times \mathrm{Z}_2^2$. Again, we shall find the only epimorphism that realizes the bound on the total number of ovals. As usual, we begin with an NEC group Λ with signature (1) and an epimorphism $\theta : \Lambda \to G$. Now by Theorems 2.1 and 2.2, a central symmetry corresponding to the non-empty period cycle contributes at most $2n$ ovals if it has the same neighbors, n ovals if it has distinct neighbors; in addition, a central symmetry contributes at most $4n$ ovals if it corresponds to an empty period cycle. For a non-central symmetry the respective values are 4, 2 and 8 ovals. Therefore, as the non-central symmetries appear at least twice, the total number of ovals holds $t \le n(4l+2s)-4$. Now if $k=0$ in the signature of Λ, then by the Hurwitz-Riemann formula $\frac{g-1}{4n} \ge -2+l$ and so $t \le 4n(l-2)+2\cdot 8 \le g+15$ and $g+15 \ge 2g+8$ for $7 \ge g$ only. On the other hand, there must be at least four period cycles, hence $g \ge 8n+1 \ge 17$, which gives a contradiction.

If $k>0$, $l \ge 2$, then $\frac{g-1}{4n} \ge \frac{2l+s}{4}$ and so $t \le 2g-2-4$, a contradiction. If $k>0$, $l=1$, then there are two possible cases. If the empty period cycle contributes to a central symmetry, then there are at least two non-central symmetries in the non-empty period cycles and so $t < (s-2)2n+8+4n = 2g+8$, a contradiction. Similarly, if the empty period cycle contributes to a non-central symmetry, then $t \le (s-1)2n+2+4 \le 2g-2-2n+6$, a contradiction. If $l=0, k\ge 2$ or $k=1, h+r>0$, then by the Hurwitz-Riemann formula $\frac{g-1}{4n} \ge -\frac{1}{2}+\frac{s}{4}$ and in turn $t \le (s-2)2n+2\cdot 4 \le 2g+6$, a contradiction.

Hence the signature of an NEC group Λ has genus 0, has no proper periods and has only one, non-empty, period cycle of length s. Let us assume first that the symmetries commute and denote them by x, y, z, w. As $t = 2g+8$ and $s = \frac{g-1}{2}+4$ by the Hurwitz-Riemann formula, it follows easily that exactly three of the symmetries in the cycle have distinct neighbors. It is not hard to see that, without loss of generality, the only possible epimorphism is:

$$\underbrace{x,\ y,\ldots,y}_{2\alpha},\ \underbrace{x,\ z,\ x,\ldots,z}_{2\beta},\ \underbrace{x,\ w,\ x,\ldots,x,\ w}_{2\gamma}.$$

By Theorems 2.1 and 2.2 it follows easily that x has $4\alpha+4\beta+4\gamma-6=g+1$ ovals, y has 4α, z has 4β and w has 4γ ovals where $s=2\alpha+2\beta+2\gamma=\frac{g-1}{2}+4$. By Lemma 2.3, all the symmetries are separating.

Let us now move to the case of non-commuting symmetries. For, let a, b be the generating symmetries of D_n and x, y be the generators of Z_2's in the presentation of G as the direct product. Observe first, that as we have only one period cycle and at least two non-central symmetries, then at least two times central symmetries appear with distinct neighbors. The explanation for this fact is similar as the one in case of g being even. Now if non-central symmetries appear at least 4 times in the cycle, then $t \leq (s-6)2n+2\cdot n+14 \leq 2g-2-2n+14<2g+8$ as $n>2$ and $s \leq \frac{g-1}{n}+4$, a contradiction.

Assume now that non-central symmetries appear three times in the cycle. If at least four of the symmetries have distinct neighbors, then $t \leq (s-5)2n+2n+8 \leq 2g-2+8$, a contradiction. Hence exactly three of the symmetries have distinct neighbors and two of them are central. Observe that no two non-central symmetries are consecutive, as only one of them may have distinct neighbors and we would obtain more than 3 consecutive non-central symmetries, a contradiction. Now if no two non-central symmetries are consecutive, then at least three of their central neighbors must have distinct neighbors, a contradiction.

Therefore we are left with the case where there are only two non-central symmetries in the cycle, we may assume these to be a, b. If at least four of the central symmetries have distinct neighbors, then $t \leq (s-6)2n+4n+8 \leq 2g+6$, a contradiction. If the non-central symmetries are consecutive, then by the Hurwitz-Riemann formula $\frac{g-1}{4n}=-1+\frac{s-1}{4}+\frac{1}{2}-\frac{1}{2n}$ and so $s=\frac{g-1}{n}+\frac{2}{n}+3$ and $t \leq (s-4)2n+2n+4=2g-2+4+6n-8n+2n+4=2g+6$, a contradiction. Therefore non-central symmetries are not consecutive. As no more than three central symmetries have distinct neighbors, then the non-central symmetries share common neighbor, say x. Observe, that this appearance of x contributes only $8n/4n=2$ ovals to the symmetry x. Now if any of the non-central symmetries has distinct neighbors, then $t \leq (s-5)2n+2n+2+6=2g+6$, a contradiction. Finally, we are left with the epimorphism of the form

$$a,\ x,\ b,\ \underbrace{x,\ y,\ x, \ldots, y}_{2\alpha},\ x.$$

We can easily count that a and b have 4 ovals each, y has $2\alpha n$ ovals and x has $2\alpha n+2$ ovals, where by the Hurwitz-Riemann formula $2\alpha=\frac{g-1}{n}$. By Lemma 2.3 and the results of [6] all the symmetries are separating.

Theorem 3.5. *If a Riemann surface X_g admits four non-conjugate symmetries with a maximal total number of ovals, then their topological types are*:

1. *if g is even and the symmetries do not commute*: $+1,\ +1,\ -g,\ -g$;

2. *if g is even and the symmetries commute*: $-(2\alpha+2\beta+2\delta-2)$, -2α, $-(2\beta+2\gamma-1)$, $-(2\gamma+2\delta+1)$ *with* $2\alpha+2\beta+2\gamma+2\delta=g+2$, *where* $\alpha,\beta,\gamma,\delta>0$ *are integers*;
3. *if g is odd and the symmetries do not commute*: $+4$, $+4$, $+(g-1)$, $+(g+1)$;
4. *if g is odd and the symmetries commute*: $+(g+1)$, $+4\alpha$, $+4\beta$, $+4\gamma$ *with* $2\alpha+2\beta+2\gamma=\frac{g-1}{2}+4$, *where* α, β, γ, $\delta>0$ *are integers.*

Conversely, for all such sets of integers $\alpha,\beta,\gamma,\delta,g$ as above, we constructed a Riemann surface having four symmetries with specified types and the maximal configuration of ovals.

References

[1] E. Bujalance and A. F. Costa, *On the group generated by three and four anticonformal involutions of Riemann surfaces with maximal number of fixed curves*, in Mathematical contributions in honor of Professor Enrique Outerelo Domínguez (Spanish), 73–76, Homen. Univ. Complut, Editorial Complutense, Madrid, 2004.

[2] E. Bujalance, J. J. Etayo, J. M. Gamboa, and G. Gromadzki, *Automorphism groups of compact bordered Klein surfaces*, Lecture Notes in Mathematics, **1439**, Springer-Verlag, Berlin, 1990.

[3] G. Gromadzki, *On a Harnack-Natanzon theorem for the family of real forms of Riemann surfaces*, J. Pure Appl. Algebra **121** (1997), no. 3, 253–269.

[4] ———, *On ovals on Riemann surfaces*, Rev. Mat. Iberoamericana **16** (2000), no. 3, 515–527.

[5] G. Gromadzki and M. Izquierdo, *On ovals of Riemann surfaces of even genera*, Geom. Dedicata **78** (1999), no. 1, 81–88.

[6] A. H. M. Hoare and D. Singerman, *The orientability of subgroups of plane groups*, in Groups—St. Andrews 1981 (St. Andrews, 1981), 221–227, London Math. Soc. Lecture Note Ser., 71, Cambridge Univ. Press, Cambridge, 1982.

[7] S. M. Natanzon, *Finite groups of homeomorphisms of surfaces and real forms of complex algebraic curves*, Trans. Moscow Math. Soc. **1989**, 1–51; translated from Trudy Moskov. Mat. Obshch. **51** (1988), 3–53, 258.

[8] D. Singerman, *On the structure of non-Euclidean crystallographic groups*, Proc. Cambridge Philos. Soc. **76** (1974), 233–240.

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