Bull. Korean Math. Soc. **56** (2019), No. 1, pp. 45–56
https://doi.org/10.4134/BKMS.b180076
pISSN: 1015-8634 / eISSN: 2234-3016

MEROMORPHIC FUNCTIONS SHARING 1CM+1IM CONCERNING PERIODICITIES AND SHIFTS

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Abstract. The aim of this paper is to investigate the problems of meromorphic functions sharing values concerning periodicities and shifts. In this paper we prove the following result: Let $f(z)$ and $g(z)$ be two nonconstant entire functions, let $c \in \mathbb{C}\backslash\{0\}$, and let a_1, a_2 be two distinct finite complex numbers. Suppose that $\mu(f) \neq 1$, $\rho_2(f) < 1$, and $f(z) = f(z+c)$ for all $z \in \mathbb{C}$. If $f(z)$ and $g(z)$ share a_1 CM, a_2 IM, then $f(z) \equiv g(z)$. Moreover, examples are given to show that all the conditions are necessary.

1. Introduction

We use $\mathbb{C}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to denote the whole complex plane and the extended complex plane, respectively. Throughout this paper, a meromorphic (resp. entire) function always means a meromorphic (resp. analytic) function in $\mathbb{C}$. It is assumed that the reader is familiar with the basic concepts of Nevanlinna theory and in particular with its standard terms and symbols (see, for example, [14, 18]).

Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions. Denoting by $E(a,f)$ (resp. $\overline{E}(a,f)$) the set of those points $z \in \overline{\mathbb{C}}$ where $f(z) = a$ counting multiplicities (resp. ignoring multiplicities), we say that $f(z)$ and $g(z)$ share a CM (resp. IM) if $E(a,f) = E(a,g)$ (resp. $\overline{E}(a,f) = \overline{E}(a,g)$).

The following definitions are also needed in this paper.

Definition 1.1. Let $f(z)$ be nonconstant meromorphic. Then the order $\rho(f)$, hyper-order $\rho_2(f)$, lower order $\mu(f)$ and low hyper-order $\mu_2(f)$ of $f(z)$ are

Received January 25, 2018; Accepted May 30, 2018.
2010 *Mathematics Subject Classification.* 30D35, 30D30.
Key words and phrases. meromorphic function, shared value, periodicity, shift, uniqueness.
Project supported by the National Natural Science Foundation of China (Grant No. 11301076), the Natural Science Foundation of Fujian Province, China (Grant No. 2018J01658) and Key Laboratory of Applied Mathematics of Fujian Province University (Putian University) (NO. SX201801).

defined in turn as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r},$$
$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \mu_2(f) = \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Definition 1.2. Let $f(z)$ be nonconstant meromorphic. If $\rho(f) < +\infty$, then we denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = O(\log r) \quad (r \to \infty).$$

If $\rho(f) = +\infty$, then we denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = O(\log(rT(r,f))) \quad (r \to \infty, r \notin E),$$

where E is a set of finite linear measure not necessarily the same at every occurrence.

The study of the uniqueness theory of meromorphic functions began with famous Nevanlinna's five value theorem, which claims that if two non-constant meromorphic functions $f(z)$ and $g(z)$ share five distinct complex numbers IM, then $f(z) \equiv g(z)$. Also Nevanlinna's four value theorem points out that if two meromorphic functions $f(z)$ and $g(z)$ share four distinct complex numbers CM, then $f(z)$ and $g(z)$ are much related by a fractional linear transformation (see, for example, [14, 18]). The condition 4CM in the four value theorem has been weakened to 2CM+2IM due to Gundersen [9]. It is well-known that 4CM cannot be further relaxed to 4IM [8], while 1CM+3IM remains open [10]. In the case of less than four shared values (even 3CM), the quantified relations between two meromorphic functions $f(z)$ and $g(z)$ are difficult to establish in general [18]. But it is still interesting to put forward the following questions.

Question 1.1. What can be said if one nonconstant meromorphic function and another nonconstant periodic meromorphic function share less than or equal to three values?

Question 1.2. What can be said if two nonconstant periodic meromorphic functions with the same nonzero period share less than or equal to three values?

As for Question 1.1, in 1989 Brosch [1, 18] proved the following result in his PhD thesis.

Theorem 1.1 (See [1] or [18, Theorem 5.15])**.** *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, let $c \in \mathbb{C}\backslash\{0\}$, and let a_1, a_2, a_3 be three distinct complex numbers. If $f(z)$ and $g(z)$ share a_1, a_2, a_3 CM, and if $f(z)$ is a periodic function with period c, then $g(z)$ is also a periodic function with period c.*

In 1992, Zheng [18, 19] improved a result given by Brosch and obtained the following theorem, which dealt with Question 1.1 and Question 1.2.

Theorem 1.2 (See [19, Theorem] or [18, Theorem 5.18]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions sharing 0, 1, ∞ CM, and let $c \in \mathbb{C}\backslash\{0\}$. If $f(z)$ is a periodic function with period c, then $g(z)$ is also a periodic function with period c. Furthermore, if $\mu_2(f) < 1$, then $f(z) \equiv g(z)$ or $f(z)$ and $g(z)$ assume the following form $f(z) = \frac{e^{a_1 z + b_1} - 1}{e^{a_2 z + b_2} - 1}$ and $g(z) = \frac{e^{-a_1 z - b_1} - 1}{e^{-a_2 z - b_2} - 1}$, where $a_1 = \frac{2m\pi i}{c}$, $a_2 = \frac{2k\pi i}{c}$, b_1, b_2 are constants, and m, k are some integers.*

Based on the definition of periodic functions and above theorems, it seems natural to study shared value problems between a meromorphic function $f(z)$ and its shift $f(z+c)$ or between one meromorphic function $f(z)$ and another function's shift $g(z+c)$, where $c \in \mathbb{C}\backslash\{0\}$. The background for these considerations lies in the recent great interest of studying difference analogues of Nevanlinna theory for meromorphic functions of finite order, see, e.g., the papers [11, 12] by Halburd and Korhonen and, independently, [6, 7] by Chiang and Feng. Currently fundamental theorems of these difference analogues of Nevanlinna theory were extended by Halburd, Korhonen, and Tohge [13] to meromorphic functions of hyper-order strictly less than one. Recently, a number of papers (see, for example, [2,15,16]) focus on the problem of value sharing for shifts of meromorphic functions.

In 2012, Chen and Xu [5] replaced the assumption 3CM in Theorem 1.2 by 2CM+1IM with some additional assumptions and obtained the following theorem.

Theorem 1.3 (See [5, Theorem 2]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, let $c \in \mathbb{C}\backslash\{0\}$, and let a_1, a_2, a_3 be three distinct complex numbers. Suppose that $1 < \mu(f) \leq \rho(f) < \infty$, $\limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-a_1}\right)}{T(r,f)} < 1$, and $f(z) = f(z+c)$ for all $z \in \mathbb{C}$. If $f(z)$ and $g(z)$ share a_1, a_2 CM, and a_3 IM, then $f(z) \equiv g(z)$.*

In 2017, Chen [3] proposed another result closely related to Theorem 1.3.

Theorem 1.4 (See [3, Corollary 1.7]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let $c \in \mathbb{C}\backslash\{0\}$. Suppose that $\mu(f) \neq 1$, $\rho_2(f) < 1$, $f(z) = f(z+c)$ and $g(z) = g(z+c)$ for all $z \in \mathbb{C}$. If $f(z)$ and $g(z)$ share 0, ∞ CM, and 1 IM, then $f(z) \equiv g(z)$.*

More recently, Chen [4] further considered the case of 1CM+2IM in Theorems 1.3-1.4 by deriving the following theorem in his PhD thesis.

Theorem 1.5 (See [4, Theorem 3.4.1]). *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, let $c \in \mathbb{C}\backslash\{0\}$, and let a_1, a_2 be two nonzero distinct finite complex numbers. Suppose that $\mu(f) \neq 1$, $\rho_2(f) < 1$, and $f(z) = f(z+c)$ for all $z \in \mathbb{C}$. If $f(z)$ and $g(z)$ share 0 CM and a_1, a_2 IM, then $f(z) \equiv g(z)$.*

It is natural to ask whether the conclusion of Theorem 1.5 is still valid if 1CM+2IM is replaced by 1CM+1IM. In this paper, we give an affirmative answer to this question, where the following theorem is established.

Theorem 1.6. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, let $c \in \mathbb{C}\backslash\{0\}$, and let a_1, a_2 be two distinct finite complex numbers. Suppose that $\mu(f) \neq 1$, $\rho_2(f) < 1$, and $f(z) = f(z+c)$ for all $z \in \mathbb{C}$. If $f(z)$ and $g(z)$ share a_1 CM, a_2 IM, then $f(z) \equiv g(z)$.*

All the conditions in Theorem 1.6 are necessary, as is seen below.

Remark 1.1. The following example shows that the condition "$\mu(f) \neq 1$" in Theorem 1.6 is necessary.

Example 1.1. Let $f(z) = \frac{e^z - e^{2z}+1}{e^z}$ and $g(z) = \frac{e^z-e^{2z}+1}{e^{-z}}$. Clearly, $f(z)$ is a periodic function with period $2\pi i$; $f(z)$ and $g(z)$ are both entire functions satisfying $\mu(f) = 1$ and $\rho_2(f) < 1$. It is easy to verify that $f(z)$ and $g(z)$ share 0 CM, 1 IM. But $f(z) \not\equiv g(z)$.

Remark 1.2. The following example shows that the condition "$\rho_2(f) < 1$" in Theorem 1.6 is necessary.

Example 1.2. According to the result obtained by Ozawa (see [17, Theorem 1]): for an arbitrary real number $\gamma \in [1, \infty)$, there exists a periodic entire function $\Pi(z)$ with period $c \neq 0$ such that $\rho(\Pi) = \gamma \in [1,\infty)$. Set $f(z) = \frac{e^{\Pi(z)}-e^{2\Pi(z)}+1}{e^{\Pi(z)}}$ and $g(z) = \frac{e^{\Pi(z)}-e^{2\Pi(z)}+1}{e^{-\Pi(z)}}$. Clearly, $f(z)$ is a periodic function with period $c \neq 0$; $f(z)$ and $g(z)$ are both entire functions satisfying $\mu(f) \neq 1$ and $\rho_2(f) \geq 1$. It is easy to verify that $f(z)$ and $g(z)$ share 0 CM, 1 IM. But $f(z) \not\equiv g(z)$.

Remark 1.3. The following example shows that the condition that "$f(z) = f(z+c)$ for all $z \in \mathbb{C}$" in Theorem 1.6 is necessary.

Example 1.3. Let $f(z) = \frac{e^{z^l}-e^{2z^l}+1}{e^{z^l}}$ and $g(z) = \frac{e^{z^l}-e^{2z^l}+1}{e^{-z^l}}$, where $l \geq 2$ is a positive integer. Clearly, there does not exist any finite value $c \neq 0$ such that $f(z) = f(z+c)$ for all $z \in \mathbb{C}$; $f(z)$ and $g(z)$ are both entire functions satisfying $\mu(f) = l$ and $\rho_2(f) < 1$. It is easy to verify that $f(z)$ and $g(z)$ share 0 CM, 1 IM. But $f(z) \not\equiv g(z)$.

Remark 1.4. The following example shows that the condition "1CM+1IM" in Theorem 1.6 cannot be replaced by "1CM".

Example 1.4. Let $\Pi(z)$ be the same as in Example 1.2 with $\mu(\Pi) \neq 1$. Set $f(z) = \Pi(z)$ and $g(z) = \frac{\Pi(z)}{e^z}$. Clearly, $f(z)$ is a periodic function with period $c \neq 0$; $f(z)$ and $g(z)$ are both entire functions satisfying $\mu(f) \neq 1$ and $\rho_2(f) < 1$. It is easy to verify that $f(z)$ and $g(z)$ only share 0 CM. But $f(z) \not\equiv g(z)$.

Remark 1.5. The following examples show that the condition that "$f(z)$ and $g(z)$ are both entire functions" in Theorem 1.6 is necessary.

Example 1.5. Let $\Pi(z)$ be the same as in Example 1.4. Set $f(z) = \Pi(z)$ and $g(z) = \frac{2\Pi(z)}{\Pi^2(z)+1}$. Clearly, $f(z)$ is a periodic entire function with period $c \neq 0$

satisfying $\mu(f) \neq 1$ and $\rho_2(f) < 1$. Moreover, by Picard's theorem $g(z)$ has infinitely many poles. It is easy to verify that $f(z)$ and $g(z)$ share 0 CM, 1 IM. But $f(z) \not\equiv g(z)$.

Example 1.6. Let $\Pi(z)$ be the same as in Example 1.4. Set $f(z) = \frac{\Pi^2(z)}{\Pi^2(z)-4}$ and $g(z) = \frac{\Pi^2(z)}{\Pi^2(z)-1}$. Clearly, $f(z)$ is a periodic function with period $c \neq 0$ satisfying $\mu(f) \neq 1$ and $\rho_2(f) < 1$. Moreover, by Picard's theorem $f(z)$ and $g(z)$ have infinitely many poles. It is easy to verify that $f(z)$ and $g(z)$ share 0 CM, 1 IM. But $f(z) \not\equiv g(z)$.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, and the proof of the main theorem can be found in Section 3.

2. Some lemmas

Lemma 2.1 (See [18, Theorem 1.19 and Corollary of Theorem 1.19])**.** *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions. If*

$$T(r,f) = O(T(r,g)) \quad (r \to \infty,\ r \notin E,\ mesE < \infty),$$

then $\mu(f) \leq \mu(g)$, $\rho(f) \leq \rho(g)$, $\mu_2(f) \leq \mu_2(g)$, $\rho_2(f) \leq \rho_2(g)$.

Lemma 2.2 (See [18, Lemma 5.1])**.** *Let $f(z)$ be a nonconstant periodic meromorphic function. Then $\rho(f) \geq 1$, $\mu(f) \geq 1$.*

Lemma 2.3 (See [18, Theorem 1.42])**.** *Let $f(z)$ be a nonconstant meromorphic function. If 0 and ∞ are two Picard exceptional values of $f(z)$, then $f(z) = e^{h(z)}$, where $h(z)$ is a nonconstant entire function.*

Lemma 2.4 (see [4, Theorem 2.3.1])**.** *Let $f(z)$ be a nonconstant meromorphic function such that $N(r,f) = S(r,f)$ and $\rho_2(f) < 1$. Let $c \in \mathbb{C}\backslash\{0\}$, and let a_1, a_2 be two distinct finite complex numbers. If $f(z)$ and $f(z+c)$ satisfy $E(a_1, f(z)) \subset E(a_1, f(z+c))$ and $\overline{E}(a_2, f(z)) \subset \overline{E}(a_2, f(z+c))$, then $f(z) \equiv f(z+c)$.*

Lemma 2.5 (See [18, Theorem 1.45])**.** *Let $h(z)$ be a nonconstant entire function and $f(z) = e^{h(z)}$. Then $\rho_2(f) = \rho(h)$.*

By using the same argument as in Theorem 1.14 of [18], we can easily obtain the following result.

Lemma 2.6. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. Then*

$$\rho_2(f \cdot g) \leq \max\{\rho_2(f), \rho_2(g)\},$$
$$\rho_2(f + g) \leq \max\{\rho_2(f), \rho_2(g)\}.$$

Lemma 2.7 (See [18, Theorem 1.21])**.** *Let $f(z)$ be a nonconstant meromorphic function. Then $\rho(f) = \rho(f')$ and $\mu(f) = \mu(f')$.*

Lemma 2.8 (See [18, Theorem 1.14]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. Then*

$$\rho\left(f\cdot g\right)\leq \max\left\{\rho\left(f\right),\rho\left(g\right)\right\},$$
$$\rho\left(f+g\right)\leq \max\left\{\rho\left(f\right),\rho\left(g\right)\right\}.$$

3. Proof of Theorem 1.6

Suppose on the contrary that $f(z)\not\equiv g(z)$. Since $f(z)$ and $g(z)$ share a_1 CM, a_2 IM, and $f(z)$ and $g(z)$ are two nonconstant entire functions, by the second fundamental theorem we have

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-a_2}\right)+\overline{N}\left(r,f\right)+S(r,f)\\ &= \overline{N}\left(r,\frac{1}{g-a_1}\right)+\overline{N}\left(r,\frac{1}{g-a_2}\right)+S(r,f)\\ &\leq T\left(r,\frac{1}{g-a_1}\right)+T\left(r,\frac{1}{g-a_2}\right)+S(r,f)\\ &= 2T\left(r,g\right)+S(r,f)\quad (r\to\infty,\ r\notin E,\ mesE<\infty). \end{aligned}\tag{3.1}$$

Similarly,

$$T(r,g)\leq 2T\left(r,f\right)+S(r,g)\quad (r\to\infty,\ r\notin E,\ mesE<\infty).\tag{3.2}$$

From the assumption, (3.1)-(3.2), and Lemmas 2.1-2.2 we get

$$\rho(g)=\rho(f)\geq 1,\ \mu(g)=\mu(f)\geq 1,\ \rho_2(g)=\rho_2(f)<1,\ S(r,f)=S(r,g).\tag{3.3}$$

For convenience, we set $S(r):=S(r,f)=S(r,g)$. By Lemma 2.3, we know that there exists an entire function $h_1(z)$ such that

$$V_1=\frac{f(z)-a_1}{g(z)-a_1}=e^{h_1(z)}.\tag{3.4}$$

Now it follows from (3.4) that

$$\frac{f(z+c)-a_1}{g(z+c)-a_1}=e^{h_1(z+c)},\tag{3.5}$$

where $c\in\mathbb{C}\backslash\{0\}$. Since $f(z)$ and $g(z)$ share a_1 CM, a_2 IM, and $f(z)=f(z+c)$ for all $z\in\mathbb{C}$, we have

$$E(a_1,g(z))=E(a_1,f(z))=E(a_1,f(z+c))=E(a_1,g(z+c)),$$
$$\overline{E}\left(a_2,g(z)\right)=\overline{E}\left(a_2,f(z)\right)=\overline{E}\left(a_2,f(z+c)\right)=\overline{E}\left(a_2,g(z+c)\right).$$

Noting that $\rho_2(g)<1$ and $N(r,g)=S(r,g)$ by (3.3) and the assumption that $g(z)$ is entire, respectively, this together with Lemma 2.4 yields

$$g(z)\equiv g(z+c).\tag{3.6}$$

Then we deduce by (3.4)-(3.6) that

$$e^{h_1(z)-h_1(z+c)}\equiv 1.\tag{3.7}$$

This implies that $h_1(z)-h_1(z+c)$ must be a constant. Let $h_1(z)-h_1(z+c)\equiv\eta$ for some constant $\eta\in\mathbb{C}$. Then by (3.7) we get

$$e^{\eta}\equiv 1, \tag{3.8}$$

and $h_1'(z)-h_1'(z+c)\equiv 0$. Consequently $h_1'(z)$ is a periodic function with period $c\neq 0$. Now from (3.3)-(3.4), and Lemmas 2.5-2.6, we obtain $\rho(h_1)<1$. Noting $\rho(h_1')=\rho(h_1)<1$ by Lemma 2.7, we thus deduce by Lemma 2.2 that $h_1'(z)$ must be a constant. Hence we can assume that

$$h_1(z)=\alpha z+\beta, \tag{3.9}$$

where $\alpha,\beta\in\mathbb{C}$ are two constants. Substituting (3.9) into (3.4) gives

$$V_1=\frac{f(z)-a_1}{g(z)-a_1}=e^{h_1(z)}=e^{\alpha z+\beta}. \tag{3.10}$$

Next we introduce another auxiliary function

$$V_2=(f-g)\left(\frac{f'}{(f-a_1)(f-a_2)}-\frac{g'}{(g-a_1)(g-a_2)}\right). \tag{3.11}$$

Firstly, we need to prove two properties of V_2.

Property 1. For $\xi_0\in\mathbb{C}$, $f(\xi_0)=g(\xi_0)=a_1\Rightarrow V_2(\xi_0)=0$.

Proof. Because $f(z)$ and $g(z)$ share a_1 CM, let ξ_0 is a zero of $f(z)-a_1$ of multiplicity p and so a zero of $g(z)-a_1$ also of multiplicity p. Then we have, near ξ_0,

$$f(z)-a_1=s_p(z-\xi_0)^p+s_{p+1}(z-\xi_0)^{p+1}+s_{p+2}(z-\xi_0)^{p+2}+\cdots, \tag{3.12}$$

$$g(z)-a_1=t_p(z-\xi_0)^p+t_{p+1}(z-\xi_0)^{p+1}+t_{p+2}(z-\xi_0)^{p+2}+\cdots, \tag{3.13}$$

where $s_\imath$ $(\imath=p,p+1,\ldots)$ and $t_\imath$ $(\imath=p,p+1,\ldots)$ are finite complex numbers with $s_p\neq 0$ and $t_p\neq 0$. Thus by (3.12)-(3.13) we get

$$f'(z)=ps_p(z-\xi_0)^{p-1}+(p+1)s_{p+1}(z-\xi_0)^p+(p+2)s_{p+2}(z-\xi_0)^{p+1}+\cdots, \tag{3.14}$$

$$g'(z)=pt_p(z-\xi_0)^{p-1}+(p+1)t_{p+1}(z-\xi_0)^p+(p+2)t_{p+2}(z-\xi_0)^{p+1}+\cdots. \tag{3.15}$$

Combining (3.12) with (3.14) we have

(3.16)
$$\begin{aligned}
&\frac{f'}{(f-a_1)(f-a_2)}\\
&=\frac{1}{a_1-a_2}\cdot\frac{ps_p(z-\xi_0)^{p-1}+(p+1)s_{p+1}(z-\xi_0)^p+(p+2)s_{p+2}(z-\xi_0)^{p+1}+\cdots}{s_p(z-\xi_0)^p+s_{p+1}(z-\xi_0)^{p+1}+s_{p+2}(z-\xi_0)^{p+2}+\cdots}\\
&=\frac{p}{(a_1-a_2)(z-\xi_0)}+\frac{s_{p+1}}{s_p(a_1-a_2)}+O(z-\xi_0).
\end{aligned}$$

On the other hand, by (3.13) and (3.15) we get

(3.17)

$$\begin{aligned}&\frac{g'}{(g-a_1)\,(g-a_2)}\\&=\frac{1}{a_1-a_2}\cdot\frac{pt_p(z-\xi_0)^{p-1}+(p+1)t_{p+1}(z-\xi_0)^p+(p+2)t_{p+2}(z-\xi_0)^{p+1}+\cdots}{t_p(z-\xi_0)^p+t_{p+1}(z-\xi_0)^{p+1}+t_{p+2}(z-\xi_0)^{p+2}+\cdots}\\&=\frac{p}{(a_1-a_2)(z-\xi_0)}+\frac{t_{p+1}}{t_p(a_1-a_2)}+O(z-\xi_0).\end{aligned}$$

Since $f(z)$ and $g(z)$ share a_1 CM, combining (3.11), (3.16), and (3.17), we see that $V_2(\xi_0)=0$. □

Property 2. For $\xi_1\in\mathbb{C}$, $f(\xi_1)=g(\xi_1)=a_2\Rightarrow V_2(\xi_1)\neq\infty$.

Proof. Because $f(z)$ and $g(z)$ share a_2 IM, let ξ_1 is a zero of $f(z)-a_2$ of multiplicity i and so a zero of $g(z)-a_2$ also of multiplicity j, which is possibly different from i. Then we have, near ξ_1,

$$f(z)-a_2=q_i(z-\xi_1)^i+q_{i+1}(z-\xi_1)^{i+1}+q_{i+2}(z-\xi_1)^{i+2}+\cdots,\tag{3.18}$$

$$g(z)-a_2=w_j(z-\xi_1)^j+w_{j+1}(z-\xi_1)^{j+1}+w_{j+2}(z-\xi_1)^{j+2}+\cdots,\tag{3.19}$$

where q_ι $(\iota=i,i+1,\ldots)$ and w_κ $(\kappa=j,j+1,\ldots)$ are finite complex numbers with $q_i\neq 0$ and $w_j\neq 0$. Hence from (3.18)-(3.19) we get

$$f'(z)=iq_i(z-\xi_1)^{i-1}+(i+1)q_{i+1}(z-\xi_1)^i+(i+2)q_{i+2}(z-\xi_1)^{i+1}+\cdots,\tag{3.20}$$

$$\begin{aligned}g'(z)=&\,jw_j(z-\xi_1)^{j-1}+(j+1)w_{j+1}(z-\xi_1)^j\\&+(j+2)w_{j+2}(z-\xi_1)^{j+1}+\cdots.\end{aligned}\tag{3.21}$$

Thus by (3.18) and (3.20) we have

$$\begin{aligned}&\frac{f'}{(f-a_1)\,(f-a_2)}\\&=\frac{1}{a_2-a_1}\cdot\frac{iq_i(z-\xi_1)^{i-1}+(i+1)q_{i+1}(z-\xi_1)^i+(i+2)q_{i+2}(z-\xi_1)^{i+1}+\cdots}{q_i(z-\xi_1)^i+q_{i+1}(z-\xi_1)^{i+1}+q_{i+2}(z-\xi_1)^{i+2}+\cdots}\\&=\frac{i}{(a_2-a_1)(z-\xi_1)}+\frac{q_{i+1}}{q_i(a_2-a_1)}+O(z-\xi_1).\end{aligned}\tag{3.22}$$

On the other hand, by (3.19) and (3.21) we get

(3.23)

$$\begin{aligned}&\frac{g'}{(g-a_1)\,(g-a_2)}\\&=\frac{1}{a_2-a_1}\cdot\frac{jw_j(z-\xi_1)^{j-1}+(j+1)w_{j+1}(z-\xi_1)^j+(j+2)w_{j+2}(z-\xi_1)^{j+1}+\cdots}{w_j(z-\xi_1)^j+w_{j+1}(z-\xi_1)^{j+1}+w_{j+2}(z-\xi_1)^{j+2}+\cdots}\\&=\frac{j}{(a_2-a_1)(z-\xi_1)}+\frac{w_{j+1}}{w_j(a_2-a_1)}+O(z-\xi_1).\end{aligned}$$

Since $f(z)$ and $g(z)$ share a_2 IM, it follows from (3.11), (3.22), and (3.23) that $V_2(\xi_1) \not\equiv \infty$. □

Secondly, we discuss the following two cases.

Case 1. Suppose that $V_2 \not\equiv 0$.

According to Property 1 and Property 2, we can deduce by (3.11) that all possible poles of V_2 only occur at the poles of $f(z)$ and $g(z)$, which are only finitely many poles, so

$$N(r, V_2) = O(\log r). \tag{3.24}$$

It follows from (3.4), (3.11), and the first fundamental theorem that

$$\begin{aligned}
m(r, V_2) &= m\left(r, (f-g)\left(\frac{f'}{(f-a_1)(f-a_2)} - \frac{g'}{(g-a_1)(g-a_2)}\right)\right)\\
&\le m\left(r, \frac{(f-g)f'}{(f-a_1)(f-a_2)}\right) + m\left(r, \frac{(f-g)g'}{(g-a_1)(g-a_2)}\right) + O(1)\\
&\le m\left(r, \frac{f-g}{f-a_1}\right) + m\left(r, \frac{f'}{f-a_2}\right) + m\left(r, \frac{f-g}{g-a_1}\right) + m\left(r, \frac{g'}{g-a_2}\right)\\
&\quad + O(1)\\
&= m\left(r, 1-\frac{g-a_1}{f-a_1}\right) + m\left(r, \frac{f-a_1}{g-a_1} - 1\right) + S(r)\\
&\le m\left(r, \frac{g-a_1}{f-a_1}\right) + m\left(r, \frac{f-a_1}{g-a_1}\right) + S(r)\\
&= m\left(r, \frac{1}{V_1}\right) + m(r, V_1) + S(r)\\
&\le T\left(r, \frac{1}{V_1}\right) + T(r, V_1) + S(r)\\
&= 2T(r, V_1) + S(r),
\end{aligned}$$

which together with (3.24) yields

$$T(r, V_2) \le 2T(r, V_1) + S(r). \tag{3.25}$$

By Property 1 and the first fundamental theorem we have

$$\overline{N}\left(r, \frac{1}{f-a_1}\right) \le N\left(r, \frac{1}{V_2}\right) \le T(r, V_2) + O(1). \tag{3.26}$$

Since $f(z)$ and $g(z)$ share a_2 IM, it follows from (3.4) and (3.10) that

$$\overline{N}\left(r, \frac{1}{f-a_2}\right) \le N\left(r, \frac{1}{V_1-1}\right) \le T(r, V_1) + O(1). \tag{3.27}$$

Thus, we deduce by (3.25), (3.26), (3.27) and the second fundamental theorem that

$$T(r,f) \le \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + \overline{N}(r,f) + S(r)$$

$$\leq T(r, V_2) + T(r, V_1) + S(r)$$
$$\leq 3T(r, V_1) + S(r) \quad (r \to \infty,\ r \notin E,\ mesE < \infty),$$

which implies from Lemma 2.1 that

$$\rho(f) \leq \rho(V_1). \tag{3.28}$$

On the other hand, by (3.2), (3.10), and the first fundamental theorem we obtain

$$T(r, V_1) = T\left(r, \frac{f - a_1}{g - a_1}\right)$$
$$\leq T(r, f) + T(r, g) + O(1)$$
$$\leq 3T(r, f) + S(r),$$

which yields from Lemma 2.1 that

$$\rho(V_1) \leq \rho(f). \tag{3.29}$$

Combining (3.28), (3.29), and (3.10) we have

$$\rho(f) = \rho(V_1) = \rho\left(e^{\alpha z + \beta}\right) \leq 1. \tag{3.30}$$

Noting that $1 \leq \mu(f) \leq \rho(f)$ by (3.3), we deduce by (3.30) that $\mu(f) = 1$, which contradicts the condition $\mu(f) \neq 1$.

Case 2. Suppose that $V_2 \equiv 0$.

Then by the original assumption $f(z) \not\equiv g(z)$ and (3.11) we get

$$\frac{f'}{(f - a_1)(f - a_2)} - \frac{g'}{(g - a_1)(g - a_2)} \equiv 0. \tag{3.31}$$

Now assume that ξ_1 is a zero of $f(z) - a_2$ of multiplicity i and so a zero of $g(z) - a_2$ also of multiplicity j, which is possibly different from i because $f(z)$ and $g(z)$ share a_2 IM. Combining (3.18)-(3.23) in proving Property 2, and (3.31), it follows that $i = j$, which means that $f(z)$ and $g(z)$ must share a_2 CM. Using the same argument as in the proof of (3.10), we have

$$\hat{V_1} = \frac{f(z) - a_2}{g(z) - a_2} = e^{h_2(z)}, \tag{3.32}$$

where $h_2(z) = \hat{\alpha} z + \hat{\beta}, \hat{\alpha}, \hat{\beta} \in \mathbb{C}$ are constants. Combining (3.10) with (3.32), we get

$$f(z) = \frac{(a_1 - a_2)\, e^{h_2(z)} - a_1 e^{h_2(z) - h_1(z)} + a_2}{1 - e^{h_2(z) - h_1(z)}}. \tag{3.33}$$

Then it follows from (3.33) and Lemma 2.8 that

$$\rho(f) \leq \max\left\{\rho(e^{h_2}), \rho(e^{h_2 - h_1})\right\} \leq \max\left\{\rho(e^{h_1}), \rho(e^{h_2})\right\} \leq 1.$$

Thus using the same argument as in Case 1, we have $\mu(f) = 1$, a contradiction. This contradiction shows that $f(z) \equiv g(z)$. Theorem 1.6 is proved.

Acknowledgments. The authors would like to thank the referees and the editor for their thorough comments and helpful suggestions.

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