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TREES WITH EQUAL STRONG ROMAN DOMINATION NUMBER AND ROMAN DOMINATION NUMBER

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Abstract. A graph theoretical model called Roman domination in graphs originates from the historical background that any undefended place (with no legions) of the Roman Empire must be protected by a stronger neighbor place (having two legions). It is applicable to military and commercial decision-making problems. A Roman dominating function for a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ has at least a neighbor w in G for which $f(w) = 2$. The Roman domination number of a graph is the minimum weight $\sum_{v \in V} f(v)$ of a Roman dominating function. In order to deal a problem of a Roman domination-type defensive strategy under multiple simultaneous attacks, Álvarez-Ruiz et al. [1] initiated the study of a new parameter related to Roman dominating function, which is called strong Roman domination. Álvarez-Ruiz et al. posed the following problem: Characterize the graphs G with equal strong Roman domination number and Roman domination number. In this paper, we construct a family of trees. We prove that for a tree, its strong Roman dominance number and Roman dominance number are equal if and only if the tree belongs to this family of trees.

1. Introduction

For notation and graph-theoretical terminology not defined here we follow [1]. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d_G(v)$, $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. If the graph G is clear from context, we simply write $d(v)$, $N(v)$ and $N[v]$, respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *diameter* $diam(G)$ of a connected graph G is the maximum distance between two vertices of G. Roman domination number was defined and discussed by Stewart [4] in 1999. It was developed by ReVelle and Rosing [3] in 2000 and

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Cockayne et al. [2] in 2004. A *Roman dominating function* of a graph G is defined as a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an Roman dominating function is defined as the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is equal to the minimum weight of a Roman dominating function of G. In fact, Roman domination is of both historical and mathematical interest. Emperor Constantine had the requirement that an army or legion could be sent from its home to defend a neighbouring location only if there was a second army which would stay and protect the home. Thus, there were two types of armies: stationary and travelling. Each vertex with no army must have a neighbouring vertex with a travelling army. Stationary armies then dominate their own vertices, and a vertex with two armies is dominated by its stationary army, and its open neighbourhood is dominated by the travelling army. This is applicable to military and commercial decision-making problems.

In order to deal with a problem of a Roman domination-type defensive strategy under multiple simultaneous attacks, Álvarez-Ruiz et al. [1] initiated the study of a new parameter related to Roman dominating function, which is called a strong Roman domination.

Let $f : V(G) \to \{0, 1, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ be a function that labels the vertices of G. Let $B_0 = \{v \in V : f(v) = 0\}$. Then f is a *strong Roman dominating function* for G, if every $v \in B_0$ has a neighbor w, such that $f(w) \geq 1 + \lceil \frac{1}{2} |N(w) \cap B_0| \rceil$. The weight of a strong Roman dominating function is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a strong Roman dominating function of G is called the *strong Roman domination number* of G and is denoted by $\gamma_{StR}(G)$. A strong Roman dominating function of G with weight $\gamma_{StR}(G)$ is called a γ_{StR}-function of G. For any $S \subseteq V$, denotes $f(S) = \sum_{v \in S} f(v)$. The graph induced by $S \subseteq V$ is denoted by $G[S]$. A *path* on n vertices is denoted by P_n. A vertex of degree one is called a *leaf*. A vertex is called a *support vertex* if it is adjacent to a leaf. We let $L(T)$ and $S(T)$ denote the set of leaves and support vertices of a tree T, respectively. Let T be a tree. If $\gamma_R(T) = \gamma_{StR}(T)$, then T is called a (γ_R, γ_{StR})-tree.

Álvarez-Ruiz et al. [1] showed the relationship between strong Roman domination and Roman domination as follows.

Observation 1 ([1]). *For any graph G, $\gamma_R(G) \leq \gamma_{StR}(G)$.*

Observation 2 ([1]). *For any connected graph G with $\Delta(G) \leq 2$, $\gamma_{StR}(G) = \gamma_R(G)$.*

According to this, they posed the following problem.

Problem 1 ([1]). *Characterize the graphs G with equal strong Roman domination and Roman domination numbers.*

As a consequence of the definition of strong Roman domination number and Observation 1, we have the following two observations.

Observation 3. *Let G be a connected graph. Then $\gamma_R(G) = \gamma_{StR}(G)$ if and only if every γ_{StR}-function of G is a γ_R-function of G.*

Observation 4. *Let G be a connected graph. Then $\gamma_R(G) = \gamma_{StR}(G)$ if and only if there exists a γ_R-function f of G such that f is a γ_{StR}-function G.*

The paper is organized as follows. In Section 2, we study the properties of trees in which strong Roman domination number and Roman domination number are the same. In Section 3, we construct a family $\mathcal{F}$ of trees consisting of $\{P_1, P_2, P_3\} \cup \{T : T$ is a tree obtained from P_1, P_2, P_3 by a finite sequence of operations τ_i for $i \in \{1, 2, \ldots, 9\}\}$. By this family, we characterize all trees for which strong Roman domination and Roman domination numbers are the same as follows:

Main Theorem. *A tree T is a (γ_R, γ_{StR})-tree if and only if T belongs to the family $\mathcal{F}$.*

2. Properties of (γ_R, γ_{StR})-trees

In this section, we give a series of lemmas about (γ_R, γ_{StR})-trees for operations $\tau_1 - \tau_9$ that will be used to prove the main theorem.

Lemma 1. *Let T be a (γ_R, γ_{StR})-tree. Then every support vertex is adjacent to at most two leaves.*

Proof. Assume that vertex u is a support vertex and $|N(u) \cap L(T)| \geq 3$. Let $N(u) \cap L(T) = \{v_i : i = 1, 2, \ldots, l\}$ for $l \geq 3$. Let f be a γ_{StR}-function of T. If $f(v_i) \geq 1$ for $1 \leq i \leq l$, then $f(N(u) \cap L(T)) \geq 3$. If $f(v_i) = 0$ for $1 \leq i \leq l$, then $f(u) \geq 1 + \lceil \frac{l}{2} \rceil \geq 1 + \lceil \frac{3}{2} \rceil = 3$. Without loss of generality, we may assume that $f(v_1) = 0$ and $f(v_2) \geq 1$. Then $f(u) \geq 2$. Hence, in all cases $f(u) + f(N(u) \cap L(T)) \geq 3$. By Observation 3, f is also a γ_R-function of T. Define f' on $V(T)$ by $f'(x) = f(x)$ for $x \in V(T) - (\{u\} \cup (N(u) \cap L(T)))$, $f'(u) = 2$ and $f'(x) = 0$ for $x \in N(u) \cap L(T)$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. $\square$

If a support vertex u is adjacent to two leaves and $d(u) = 3$, then u is called an *end strong support vertex.* If a support vertex u is adjacent to exactly one leaf and $d(u) = 2$, then u is called an *end weak support vertex.*

Lemma 2. *Let T be a (γ_R, γ_{StR})-tree. Suppose that u is an end strong support vertex, $N(u) \cap L(T) = \{v, v'\}$ and $N(u) \setminus \{v, v'\} = \{w\}$. Then for any γ_{StR}-function f of T, $f(u) = 2$ and $f(w) = 2$.*

Proof. Let f be a γ_{StR}-function of T. Suppose that $f(w) \leq 1$. Then, $f(N[u]) \geq 3$. Define f' on $V(T)$ by $f'(x) = f(x)$ for $x \in V(T) - N[u]$, $f'(u) = 2$ and $f'(x) = 0$ for $x \in N(u)$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. Hence, we can assume that $f(w) \geq 2$. Since $\gamma_R(T) = \gamma_{StR}(T)$, f is a γ_R-function of T. So $f(w) = 2$. Suppose that $f(u) = 1$. Then, $f(v) = f(v') = 1$. Define f' on

$V(T)$ by $f'(x) = f(x)$ for $x \in V(T) - \{u, v, v'\}$, $f'(u) = 2$, $f'(v) = 0$ and $f'(v') = 0$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. For the other case, let $f(u) = 0$. Now $f(v) \geq 1$ and $f(v') \geq 1$. If $f(v) \geq 2$ or $f(v') \geq 2$, then define f' on $V(T)$ by $f'(x) = f(x)$ for $x \in V(T) - \{u, v, v'\}$, $f'(u) = 2$, $f'(v) = 0$ and $f'(v') = 0$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. Hence, $f(v) = f(v') = 1$. Since f is a γ_{StR}-function T, w is adjacent to at most two vertices in B_0. If $|N(w) \cap B_0| = 1$, then define f' on $V(T)$ by $f'(x) = f(x)$ for $x \in V(T) - \{u, v, v', w\}$, $f'(u) = 2$ and $f'(x) = 0$ for $x \in N(u)$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. If $|N(w) \cap B_0| = 2$, then assume that $u' \in (N(w) \cap B_0) \setminus \{u\}$. Define f' on $V(T)$ by $f'(x) = f(x)$ for $x \in V(T) - \{u, u', v, v', w\}$, $f'(u) = 2$, $f'(x) = 0$ for $x \in N(u)$ and $f'(u') = 1$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. Hence, $f(u) = 2$. □

Lemma 3. *Let T be a (γ_R, γ_{StR})-tree. Suppose that u is an end weak support vertex, $N(u) \cap L(T) = \{v\}$ and $N(u) - \{v\} = \{w\}$. For any γ_{StR}-function f of T, the following hold.*

(1) *$f(w) \neq 1$.*

(2) *If $f(w) = 2$, then $f(u) = 0$ and $f(v) = 1$.*

(3) *If $f(w) = 0$, then there exists a γ_{StR}-function f' of T such that $f'(w) = 0$, $f'(u) = 2$ and $f'(v) = 0$.*

Proof. Let f be a γ_{StR}-function of T.

(1) Suppose that $f(w) = 1$. It is obvious that $f(u) + f(v) \geq 2$. Define f' on $V(T)$ by $f'(x) = f(x)$ for $x \in V(T) - \{u, v, w\}$, $f'(u) = 2$, $f'(w) = 0$ and $f'(v) = 0$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. Hence, $f(w) \neq 1$.

(2) Suppose that $f(w) = 2$. If $f(u) + f(v) \geq 2$, then define f' on $V(T)$ by $f'(x) = f(x)$ for $x \in V(T) - \{u, v\}$, $f'(u) = 0$ and $f'(v) = 1$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. Hence, $f(u) + f(v) \leq 1$. Since f is a γ_{StR}-function T, $f(u) + f(v) = 1$. So, $f(u) = 0$ and $f(v) = 1$.

(3) If $f(w) = 0$, then it is obvious that $(f(u), f(v)) \in \{(1, 1), (2, 0), (0, 2)\}$. Define f' on $V(T)$ by $f'(x) = f(x)$ for $x \in V(T) - \{u, v\}$, $f'(u) = 2$ and $f'(v) = 0$. Obviously f' is a γ_{StR}-function of T such that $f'(w) = 0$, $f'(u) = 2$ and $f'(v) = 0$. □

Lemma 4. *Let T be a tree. Assume that $P_4 : vuwx$ is an induced subgraph of T with $d(v) = 1$, $d(u) = 2$ and $d(w) = 2$. Let $T' = T - \{w, u, v\}$. Then T is a (γ_R, γ_{StR})-tree if and only if T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(x) \leq 1$.*

Proof. It is obvious that $\gamma_R(T) = \gamma_R(T') + 2$. Suppose that T is a (γ_R, γ_{StR})-tree. Then $\gamma_R(T) = \gamma_{StR}(T)$. By Lemma 3, there exists a γ_{StR}-function f

of T such that $(f(w), f(u), f(v)) \in \{(2,0,1),(0,2,0)\}$. If $(f(w), f(u), f(v)) = (2,0,1)$, then $f(x) = 0$. Otherwise, if $f(x) \geq 1$, then define f' on $V(T)$ by $f'(y) = f(y)$ for $y \in V(T) - \{u, v, w\}$, $f'(w) = 0$, $f'(u) = 2$ and $f'(v) = 0$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. Hence, define f' on $V(T')$ by $f'(y) = f(y)$ for $y \in V(T') - \{x\}$ and $f'(x) = 1$. Obviously f' is a strong Roman dominating function of T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - 2$. Suppose that $(f(w), f(u), f(v)) = (0,2,0)$. If $f(x) = 2$, then there exists exactly one vertex $u' \in N(x) - \{w\}$ such that $f(u') = 0$. Otherwise, define f' on $V(T)$ by $f'(y) = f(y)$ for $y \in V(T) - \{x\}$ and $f'(x) = 1$. Obviously f' is a Roman dominating function of T with weight less than $\gamma_{StR}(T)$, which is a contradiction. Hence, define f' on $V(T')$ by $f'(y) = f(y)$ for $y \in V(T') - \{x, u'\}$, $f'(u') = 1$ and $f'(x) = 1$. Obviously f' is a strong Roman dominating function of T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - 2$. If $f(x) \leq 1$, then define f' on $V(T')$ by $f'(y) = f(y)$ for $y \in V(T')$. Obviously f' is a strong Roman dominating function for T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - 2$. Hence, in all cases, $\gamma_{StR}(T') \leq \gamma_{StR}(T) - 2$. It follows that $\gamma_R(T) = \gamma_R(T') + 2 \leq \gamma_{StR}(T') + 2 \leq \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + 2$. Hence, T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(x) \leq 1$.

Conversely, let f' be a γ_{StR}-function of T' such that $f'(x) \leq 1$. Define f on $V(T)$ by $f(y) = f'(y)$ for $y \in V(T')$, $f(w) = 0$, $f(u) = 2$ and $f(v) = 0$. Obviously f is a strong Roman dominating function of T. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + 2$. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + 2 = \gamma_R(T') + 2 = \gamma_R(T)$. By Observation 1, $\gamma_R(T) = \gamma_{StR}(T)$ and T is a (γ_R, γ_{StR})-tree. □

Lemma 5. *Let T be a tree. Suppose that w is adjacent to two end strong support vertices u_1 and u_2. Say $N(u_i) - \{w\} = \{v_i, t_i\}$ for $i = 1, 2$. Let $T' = T - \{u_1, v_1, t_1\}$. Then T is a (γ_R, γ_{StR})-tree if and only if T' is a (γ_R, γ_{StR})-tree.*

Proof. It is obvious that $\gamma_R(T) = \gamma_R(T') + 2$. Suppose that T is a (γ_R, γ_{StR})-tree. Then $\gamma_R(T) = \gamma_{StR}(T)$. By Lemma 2, for any γ_{StR}-function f of T, $f(w) = f(u_1) = f(u_2) = 2$. Define f' on $V(T')$ by $f'(x) = f(x)$ for $x \in V(T')$. Obviously f' is a strong Roman dominating function of T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - 2$. It follows that $\gamma_R(T) = \gamma_R(T') + 2 \leq \gamma_{StR}(T') + 2 \leq \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and T' is a (γ_R, γ_{StR})-tree.

Conversely, let f' be a γ_{StR}-function of T'. By Lemma 2, $f'(w) = 2$. Define f on $V(T)$ by $f(x) = f'(x)$ for $x \in V(T')$, $f(u_1) = 2$, $f(v_1) = 0$ and $f(t_1) = 0$. Obviously f is a strong Roman dominating function of T. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + 2$. Hence $\gamma_{StR}(T) \leq \gamma_{StR}(T') + 2 = \gamma_R(T') + 2 = \gamma_R(T)$. By Observation 1, $\gamma_R(T) = \gamma_{StR}(T)$ and T is a (γ_R, γ_{StR})-tree. □

Lemma 6. *Let T be a tree. Suppose that $d(w) = 3, 4$ and $\{u_1, u_2, u_3\} \subseteq N(w)$, where u_1 is an end strong support vertex, u_i is a leaf or an end weak support vertex for $i = 2, 3$. Let $T' = T - ((N[u_1] \cup N(u_2) \cup N(u_3)) - \{w\})$. Then T is a (γ_R, γ_{StR})-tree if and only if T' is a (γ_R, γ_{StR})-tree.*

Proof. Let $l = |\{u_i : u_i \text{ is an end weak support vertex for } i \in \{2,3\}\}|$, where $l \in \{0,1,2\}$. It is obvious that $\gamma_R(T) = \gamma_R(T') + l + 2$. Suppose that T is a (γ_R, γ_{StR})-tree. Then $\gamma_R(T) = \gamma_{StR}(T)$. By Lemma 2, for any γ_{StR}-function f of T, $f(w) = f(u_1) = 2$. Define f' on $V(T')$ by $f'(x) = f(x)$ for $x \in V(T')$. Obviously f' is a strong Roman dominating function of T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - l - 2$. It follows that $\gamma_R(T) = \gamma_R(T') + l + 2 \leq \gamma_{StR}(T') + l + 2 \leq \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and T' is a (γ_R, γ_{StR})-tree.

Conversely, let f' be a γ_{StR}-function of T'. By Lemma 2, $f'(w) = 2$. Define f on $V(T)$ by $f(x) = f'(x)$ for $x \in V(T')$, $f(u_1) = 2$, $f(x) = 0$ for $x \in N(u_1) \cap L(T)$ and $f(x) = 1$ for $x \in N(\{u_2, u_3\}) \cap L(T)$. Obviously f is a strong Roman dominating function of T. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 2$. Hence $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 2 = \gamma_R(T') + l + 2 = \gamma_R(T)$. So $\gamma_R(T) = \gamma_{StR}(T)$ and T is a (γ_R, γ_{StR})-tree. □

Lemma 7. *Let T be a tree. Suppose that $d(w) = 3$ and $\{u_1, u_2\} \subseteq N(w)$, where u_1 is an end strong support vertex, u_2 is an end weak support vertex. Let $T' = T - (N[u_1] - \{w\})$. Then T is a (γ_R, γ_{StR})-tree if and only if T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(w) = 2$.*

Proof. It is obvious that $\gamma_R(T) = \gamma_R(T') + 2$. Suppose that T is a (γ_R, γ_{StR})-tree. By Lemma 2, for any γ_{StR}-function f of T, $f(w) = f(u_1) = 2$. Define f' on $V(T')$ by $f'(x) = f(x)$ for $x \in V(T')$. Obviously f' is a strong Roman dominating function for T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - 2$. It follows that $\gamma_R(T) = \gamma_R(T') + 2 \leq \gamma_{StR}(T') + 2 \leq \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + 2$. Hence T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(w) = 2$.

Conversely, let f' be a γ_{StR}-function of T' with $f'(w) = 2$. Define f on $V(T)$ by $f(x) = f'(x)$ for $x \in V(T')$, $f(u_1) = 2$ and $f(x) = 0$ for $x \in N(u_1) \cap L(T)$. Obviously f is a strong Roman dominating function of T. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + 2$. Hence $\gamma_{StR}(T) \leq \gamma_{StR}(T') + 2 = \gamma_R(T') + 2 = \gamma_R(T)$. So $\gamma_R(T) = \gamma_{StR}(T)$ and T is a (γ_R, γ_{StR})-tree. □

Lemma 8. *Let T be a tree. Suppose that $N(w) = \{u_1, u_2, x\}$, where u_1 is an end strong support vertex and u_2 is a leaf. Let $T' = T - (N[u_1] \cup \{u_2\})$. Then T is a (γ_R, γ_{StR})-tree if and only if T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(x) = 1$.*

Proof. It is obvious that $\gamma_R(T) = \gamma_R(T') + 3$. Suppose that T is a (γ_R, γ_{StR})-tree. By Lemma 2, for any γ_{StR}-function f of T, $f(w) = f(u_1) = 2$. It is obvious that $f(x) = 0$. Define f' on $V(T')$ by $f'(y) = f(y)$ for $y \in V(T') - \{x\}$ and $f'(x) = 1$. Obviously f' is a strong Roman dominating function of T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - 3$. It follows that $\gamma_R(T) = \gamma_R(T') + 3 \leq \gamma_{StR}(T') + 3 \leq \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + 3$. Hence, T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(x) = 1$.

Conversely, let f' be a γ_{StR}-function of T' with $f'(x) = 1$. Since T' is a (γ_R, γ_{StR})-tree, it follows that $f'(y) \leq 1$ for $y \in N(x) - \{w\}$. Define f on

$V(T)$ by $f(y) = f'(y)$ for $y \in V(T') - \{x\}$, $f(x) = 0$, $f(w) = 2$, $f(u_1) = 2$ and $f(y) = 0$ for $y \in N(\{w, u_1\}) \cap L(T)$. Obviously f is a strong Roman dominating function of T. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + 3$. Hence $\gamma_{StR}(T) \leq \gamma_{StR}(T') + 3 = \gamma_R(T') + 3 = \gamma_R(T)$. So $\gamma_R(T) = \gamma_{StR}(T)$ and T is a (γ_R, γ_{StR})-tree. □

Lemma 9. *Let T be a tree. Suppose that $d(w) = 3$ and $\{u_1, u_2\} \subseteq N(w)$, where u_1 is an end weak support vertex, u_2 is a leaf or an end weak support vertex. Let $T' = T - ((N(u_1) \cup N(u_2)) \cap L(T))$. Then T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w) = 2$ if and only if T' is a (γ_R, γ_{StR})-tree.*

Proof. Define $l = 1$ if u_2 is an end weak support vertex, otherwise $l = 0$. It is obvious that $\gamma_R(T) = \gamma_R(T') + l + 1$. Let f be a γ_{StR}-function of T such that $f(w) = 2$. Define f' on $V(T')$ by $f'(x) = f(x)$ for $x \in V(T')$. Obviously f' is a strong Roman dominating function of T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - l - 1$. It follows that $\gamma_R(T) = \gamma_R(T') + l + 1 \leq \gamma_{StR}(T') + l + 1 \leq \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and T' is a (γ_R, γ_{StR})-tree.

Conversely, let f' be a γ_{StR}-function of T'. By Lemma 2, $f'(w) = 2$. Define f on $V(T)$ by $f(x) = f'(x)$ for $x \in V(T')$ and $f(x) = 1$ for $x \in N(\{u_1, u_2\}) \cap L(T)$. Obviously f is a strong Roman dominating function of T. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 1$. Hence $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 1 = \gamma_R(T') + l + 1 = \gamma_R(T)$. So $\gamma_R(T) = \gamma_{StR}(T)$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + l + 1$. Hence, T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w) = 2$. □

Lemma 10. *Let T be a tree. Suppose that $\{w_1, w_2\} \subseteq N(x)$ and $N(w_1) = \{x, u_1, u_2\}$, where w_2 and u_1 are end weak support vertices and u_2 is a leaf or an end weak support vertex. Let $T' = T - (N[u_1] \cup N[u_2])$. Then T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w_1) = 0$ if and only if T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(x) = 0$.*

Proof. Define $l = 1$ if u_2 is an end weak support vertex, otherwise $l = 0$. It is obvious that $\gamma_R(T) = \gamma_R(T') + l + 3$. Let f be a γ_{StR}-function of T such that $f(w_1) = 0$. By Lemma 3, $f(x) = 0$. Define f' on $V(T')$ by $f'(y) = f(y)$ for $y \in V(T')$. Obviously f' is a strong Roman dominating function for T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - l - 3$. It follows that $\gamma_R(T) = \gamma_R(T') + l + 3 \leq \gamma_{StR}(T') + l + 3 \leq \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + l + 3$. Hence, T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(x) = 0$.

Conversely, let f' be a γ_{StR}-function of T' with $f'(x) = 0$. Define f on $V(T)$ by $f(z) = f'(z)$ for $z \in V(T')$, $f(u_1) = 2$ and $f(z) = 0$ for $z \in N(\{u_1, u_2\})$. If u_2 is an end weak support vertex, $f(u_2) = 2$, otherwise, $f(u_2) = 1$. Obviously f is a strong Roman dominating function of T. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 3$. Hence $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 3 = \gamma_R(T') + l + 3 = \gamma_R(T)$. So $\gamma_R(T) = \gamma_{StR}(T)$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + l + 3$. Hence, T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w_1) = 0$. □

Lemma 11. *Let T be a tree. Suppose that $d(x) = 3$ and $N(x) = \{y, w_1, w_2\}$ and $N(w_1) = \{x, u_1, u_2\}$, where u_1 is an end weak support vertex, w_2 is a leaf and u_2 is a leaf or an end weak support vertex. Let $T' = T - ((N[u_1] \cup N[u_2]) \cup \{x, w_2\})$. Then T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w_1) = 0$ if and only if T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(y) = 2$ and y is a leaf of T'.*

Proof. Define $l = 1$ if u_2 is an end weak support vertex, otherwise $l = 0$. It is obvious that $\gamma_R(T) = \gamma_R(T') + l + 4$. Let f be a γ_{StR}-function of T such that $f(w_1) = 0$. Then by Lemma 3, $f(u_1) = 2$. Since $\gamma_R(T) = \gamma_{StR}(T)$, f is a γ_R-function of T. Hence, $f(v) \le 2$ for any $v \in V(T)$. If $f(x) = 2$, then $f(w_2) = 0$. Since f is a γ_{StR}-function of T, it follows that $f(y) \ge 1$. Define f_1 on $V(T)$ by $f_1(z) = f(z)$ for $z \in V(T')$, $f_1(w_1) = 2$, $f_1(z) = 0$ for $z \in N(w_1)$, and $f_1(z) = 1$ for $z \in N(N(w_1)) - \{y\}$. It is obvious that f_1 is a Roman dominating function of T. So $\gamma_R(T) \le f_1(V(T)) = f(V(T)) - 1 = \gamma_{StR}(T) - 1 = \gamma_R(T) - 1$, which is a contradiction. Hence, $f(x) \le 1$. If $f(x) = 1$, then $f(w_2) = 1$. Then define f_1 on $V(T)$ as above. It is obvious that f_1 is a Roman dominating function of T. So $\gamma_R(T) \le f_1(V(T)) = f(V(T)) - 1 = \gamma_R(T) - 1$, which is a contradiction. Hence, $f(x) = 0$. Then $f(w_2) \ge 1$. If $f(w_2) = 2$, then define f_1 on $V(T)$ as above. It is obvious that f_1 is a Roman dominating function of T. So $\gamma_R(T) \le f_1(V(T)) = f(V(T)) - 1 = \gamma_R(T) - 1$, which is a contradiction. Hence, $f(w_2) = 1$. By the definition of Roman domination, $f(y) = 2$. Suppose that $d_T(y) \ge 3$. Say $N_T(y) - \{x\} = \{z_1, z_2, \dots, z_l\}$. Then $l \ge 2$. Since f is a γ_{StR}-function of T, it follows that there exists at most one vertex $z_i \in N(y) - \{x\}$ with $f(z_i) = 0$. Without loss of generality, we can assume that $f(z_i) \ge 1$ for $2 \le i \le l$. If there exists z_i such that $f(z_i) = 1$, where $i \in \{2, \dots, l\}$, then define f_1 on $V(T)$ by $f_1(z) = f(z)$ for $z \in V(T) - \{z_i\}$ and $f_1(z_i) = 0$. It is obvious that f_1 is a Roman dominating function of T whose weight is less than $\gamma_R(T)$, which is a contradiction. Hence, we can assume that $f(z_i) = 2$ for $2 \le i \le l$.

Define f_1 on $V(T)$ by $f_1(z) = f(z)$ for $z \in V(T') - \{y, z_1\}$, $f_1(w_1) = 2$, $f_1(z) = 0$ for $z \in N(w_1)$, and $f_1(z) = 1$ for $z \in N(N(w_1)) - \{y\}$, $f_1(y) = 0$. If $f(z_1) = 0$, $f_1(z_1) = 1$, otherwise, $f_1(z_1) = f(z_1)$. It is obvious that f_1 is a Roman dominating function of T. So $\gamma_R(T) \le f_1(V(T)) < f(V(T)) = \gamma_{StR}(T) = \gamma_R(T)$, which is a contradiction. Hence, $d_T(y) = 2$. So, $f(u_1) = 2$, $f(x) = 0$, $f(y) = 2$, $f(w_2) = 1$ and $d_T(y) = 2$.

Define f' on $V(T')$ by $f'(z) = f(z)$ for $z \in V(T')$. Obviously f' is a strong Roman dominating function for T'. So $\gamma_{StR}(T') \le \gamma_{StR}(T) - l - 4$. It follows that $\gamma_R(T) = \gamma_R(T') + l + 4 \le \gamma_{StR}(T') + l + 4 \le \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + l + 4$. Hence, T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(y) = 2$ and $d_{T'}(y) = 1$.

Conversely, let f' be a γ_{StR}-function of T' with $f'(y) = 2$. Define f on $V(T)$ by $f(z) = f'(z)$ for $z \in V(T')$, $f(u_1) = 2$, $f(w_2) = 1$, $f(x) = 0$ and $f(z) = 0$ for $z \in N(\{u_1, u_2\})$. If u_2 is an end weak support vertex, $f(u_2) = 2$, otherwise, $f(u_2) = 1$. Obviously f is a strong Roman dominating function for T. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 4$. Hence $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 4 = \gamma_R(T') + l + 4 = \gamma_R(T)$. So $\gamma_R(T) = \gamma_{StR}(T)$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + l + 4$. Hence, T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w_1) = 0$. □

Lemma 12. *Let T be a tree. Suppose that $N(x) = \{y, w\}$ and $N(w) = \{x, u_1, u_2\}$, where u_1 is an end weak support vertex and u_2 is a leaf or an end weak support vertex. Let $T' = T - (N[u_1] \cup N[u_2] \cup \{x\})$. Then T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w) = 0$ if and only if T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(y) = 2$ and y is a leaf of T'.*

Proof. Define $l = 1$ if u_2 is an end weak support vertex, otherwise $l = 0$. It is obvious that $\gamma_R(T) = \gamma_R(T') + l + 3$. Let f be a γ_{StR}-function of T such that $f(w) = 0$. Then $f(u_1) = 2$, $f(x) = 0$ and $f(y) = 2$. Furthermore, $d_T(y) = 2$ by a similar reason in the proof of Lemma 11. Define f' on $V(T')$ by $f'(z) = f(z)$ for $z \in V(T')$. Obviously f' is a strong Roman dominating function of T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - l - 3$. It follows that $\gamma_R(T) = \gamma_R(T') + l + 3 \leq \gamma_{StR}(T') + l + 3 \leq \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + l + 3$. Hence, T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(y) = 2$ and $d_{T'}(y) = 1$.

Conversely, let f' be a γ_{StR}-function of T' with $f'(y) = 2$. Define f on $V(T)$ by $f(z) = f'(z)$ for $z \in V(T')$, $f(u_1) = 2$, $f(x) = 0$ and $f(z) = 0$ for $z \in N(\{u_1, u_2\})$. If u_2 is an end weak support vertex, $f(u_2) = 2$, otherwise, $f(u_2) = 1$. Obviously f is a strong Roman dominating function of T. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 3$. So $\gamma_{StR}(T) \leq \gamma_{StR}(T') + l + 3 = \gamma_R(T') + l + 3 = \gamma_R(T)$. Hence $\gamma_R(T) = \gamma_{StR}(T)$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + l + 3$. So, T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w) = 0$. □

Lemma 13. *Let T be a tree. Suppose that $\{w_1, w_2\} \subseteq N(x)$ and $N(w_i) - \{x\} = \{u_i, t_i\}$ for $i = 1, 2$, where u_1, t_1, u_2 are end weak support vertices and t_2 is a leaf or an end weak support vertex. Let $T' = T - (N[u_2] \cup N[t_2])$. Then T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w_1) = 0$ if and only if T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(w_1) = 0$.*

Proof. Define $l = 1$ if t_2 is an end weak support vertex, otherwise $l = 0$. It is obvious that $\gamma_R(T) = \gamma_R(T') + l + 3$. Let f be a γ_{StR}-function of T such that $f(w_1) = 0$. Then $f(x) = 0$, $f(w_2) = 0$ and $f(u_2) = 2$. If t_2 is an end weak support vertex, $f(t_2) = 2$, otherwise, $f(t_2) = 1$. Define f' on $V(T')$ by $f'(z) = f(z)$ for $z \in V(T')$. Obviously f' is a strong Roman dominating function for T'. So $\gamma_{StR}(T') \leq \gamma_{StR}(T) - l - 3$. It follows that $\gamma_R(T) = \gamma_R(T') + l + 3 \leq \gamma_{StR}(T') + l + 3 \leq \gamma_{StR}(T)$. So $\gamma_R(T') = \gamma_{StR}(T')$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + l + 3$. Hence, T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(w_1) = 0$.

Conversely, let f' be a γ_{StR}-function of T' with $f'(w_1) = 0$. Define f on $V(T)$ by $f(z) = f'(z)$ for $z \in V(T')$, $f(u_2) = 2$ and $f(z) = 0$ for $z \in N(\{u_2, t_2\})$. If t_2 is an end weak support vertex, $f(t_2) = 2$, otherwise, $f(t_2) = 1$. Obviously f is a strong Roman dominating function of T. So $\gamma_{StR}(T) \le \gamma_{StR}(T') + l + 3$. So $\gamma_{StR}(T) \le \gamma_{StR}(T') + l + 3 = \gamma_R(T') + l + 3 = \gamma_R(T)$. Hence $\gamma_R(T) = \gamma_{StR}(T)$ and $\gamma_{StR}(T) = \gamma_{StR}(T') + l + 3$. So T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w_1) = 0$. $\square$

By a similar proof as Lemma 13, we give the following result. The proof is omitted.

Lemma 14. *Let T be a tree. Suppose that $\{w_1, w_2\} \subseteq N(x)$ and $N(w_i) - \{x\} = \{u_i, t_i\}$ for $i = 1, 2$, where u_1, u_2 are end weak support vertices and t_1, t_2 are leaves. Let $T' = T - (N[u_2] \cup \{t_2\})$. Then T is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f of T such that $f(w_1) = 0$ if and only if T' is a (γ_R, γ_{StR})-tree and there exists a γ_{StR}-function f' of T' such that $f'(w_1) = 0$.*

3. A characterization of (γ_R, γ_{StR})-trees

In the following, we construct a family $\mathcal{F}$ of trees with equal strong Roman domination and Roman domination numbers. For this purpose, we introduce some additional notation. Let $P_k : v_1, v_2, \ldots, v_k$ be a path with order k. Let a double star $S(1, 2)$ be obtained from P_4 and a vertex v_5 by joining an edge v_3v_5. Let T be a tree. Let f be a γ_{StR}-function of T. Let's denote f by $f = (f(v_1), f(v_2), \ldots, f(v_n))$. For any tree $T \in \mathcal{F}$, we denote $\digamma(T)$ by a collection of γ_{StR}-functions of T. Define $\mathcal{B}_i^T = \{v :$ there exists $f \in \digamma(T)$ such that $f(v) = i\}$ for $i = 0, 1, 2$. It is obvious that if $\digamma(T)$ is given, then $\mathcal{B}_i^T$ is determined. Hence we only give $\digamma(T)$ for any tree T.

Firstly, if $T = P_i$ for $i = 1, 2, 3$, then $\digamma(T) = \{f : f$ is a γ_{StR}-function of T such that $f(v_i) \le 2$ for $i = 1, 2, \ldots, n\}$. For example, $\digamma(P_1) = \{(1)\}$, $\digamma(P_2) = \{(1, 1), (0, 2), (2, 0)\}$, $\digamma(P_3) = \{(0, 2, 0)\}$.

Let $T' \in \mathcal{F}$ be a tree with $\digamma(T')$. We construct a new tree T with $\digamma(T)$ by the following operations on the tree T' as follows:

• Operation τ_1 (Lemma 5). Suppose that u is an end strong support vertex of a tree T'. Say $N(u) - L(T') = \{w\}$. A tree T is obtained from the tree T' by adding a P_3 and an edge between w and v_2. For any $f \in \digamma(T)$, f can be obtained from a $f' \in \digamma(T')$ by defining $f(x) = f'(x)$ for $x \in V(T')$ and $f(P_3) = (0, 2, 0)$.

• Operation τ_2 (Lemma 6). Suppose that w is a support vertex with degree two or an end strong support vertex. Say $N(w) \cap L(T') = \{u_1, u_2\}$. A tree T is obtained from the tree T' by adding a P_3 and an edge between w and v_2. For each u_i, do nothing or add a new vertex t_i and give an edge between u_i and t_i. For any $f \in \digamma(T)$, f can be obtained from a $f' \in \digamma(T')$ by defining $f(x) = f'(x)$ for $x \in V(T')$, $f(P_3) = (0, 2, 0)$ and $f(t_i) = 1$ if u_i is adjacent to a leaf t_i for $i = 1, 2$.

• Operation τ_3 (Lemma 7). Suppose that $w \in \mathcal{B}_2^{T'}$, $N(w) = \{x, u\}$ and u is an end weak support vertex. A tree T is obtained from the tree T' by adding a P_3 and an edge between w and v_2. For any $f \in F(T)$, f can be obtained from a $f' \in F(T')$ with $f'(w) = 2$ by defining $f(x) = f'(x)$ for $x \in V(T')$ and $f(P_3) = (0, 2, 0)$.

• Operation τ_4 (Lemma 8). Suppose that $x \in \mathcal{B}_1^{T'}$. A tree T is obtained from the tree T' by adding a double star $S(1, 2)$ and an edge between x and v_2. For any $f \in F(T)$, f can be obtained from a $f' \in F(T')$ with $f'(x) = 1$ by defining $f(z) = f'(z)$ for $z \in V(T') - \{x\}$, $f(x) = 0$ and $f(S(1, 2)) = (0, 2, 2, 0, 0)$.

• Operation τ_5 (Lemma 9). Suppose that w is an end strong support vertex. Say $N(w) \cap L(T') = \{u_1, u_2\}$. For each u_i, a tree T is obtained from the tree T' by adding a new vertex t_i and an edge between them or doing nothing. For any $f \in F(T)$, f can be obtained from a $f' \in F(T')$ by defining $f(x) = f'(x)$ for $x \in V(T')$ and $f(x) = 1$ for $x \in V(T) - V(T')$.

• Operation τ_6 (Lemma 10). Suppose that $w \in \mathcal{B}_0^{T'}$, $N(w) = \{x, u\}$ and u is a leaf. A tree T is obtained from the tree T' by adding a P_5 or P_4 and an edge between w and v_3. For any $f \in F(T)$, f can be obtained from a $f' \in F(T')$ with $f'(x) = 0$ by defining $f(z) = f'(z)$ for $z \in V(T')$ and $f(P_5) = (0, 2, 0, 2, 0)$ or $f(P_4) = (0, 2, 0, 1)$.

• Operation τ_7 (Lemma 11, Lemma 12). Suppose that y is a leaf with $y \in \mathcal{B}_2^{T'}$. A tree T is obtained from the tree T' by adding a new vertex x and edge yx, adding a P_5 or P_4 and an edge xv_3. For vertex x, do nothing or add a new vertex w_2 and an edge between them. For any $f \in F(T)$, f can be obtained from a $f' \in F(T')$ with $f'(y) = 2$ by defining $f(z) = f'(z)$ for $z \in V(T')$, $f(x) = 0$ and $f(P_5) = (0, 2, 0, 2, 0)$ or $f(P_4) = (0, 2, 0, 1)$. If x is adjacent to a leaf w_2, then $f(w_2) = 1$.

• Operation τ_8 (Lemma 13, Lemma 14). Suppose that $w \in \mathcal{B}_0^{T'}$ and $N(w) = \{x, u_1, u_2\}$, where u_1 is an end weak support vertex and u_2 is a leaf or an end weak support vertex. A tree T is obtained from the tree T' by adding a P_5 or P_4 and an edge between x and v_3. For any $f \in F(T)$, f can be obtained from a $f' \in F(T')$ with $f'(w) = 0$ by defining $f(z) = f'(z)$ for $z \in V(T')$ and $f(P_5) = (0, 2, 0, 2, 0)$ or $f(P_4) = (0, 2, 0, 1)$.

• Operation τ_9 (Lemma 4). Suppose that $x \in V(T')$ with $x \in \mathcal{B}_0^{T'} \cup \mathcal{B}_1^{T'}$. A tree T is obtained from the tree T' by adding a P_3 and an edge between x and v_3. For any $f \in F(T)$, f can be obtained from a $f' \in F(T')$ with $f'(x) \leq 1$ by defining $f(z) = f'(z)$ for $z \in V(T')$ and $f(P_3) = (0, 2, 0)$. If $f'(x) = 1$, then f can also defined by $f(z) = f'(z)$ for $z \in V(T') - \{x\}$, $f(x) = 0$ and $f(P_3) = (1, 0, 2)$. Suppose that x is a leaf in T' and y is its neighbor. If $f'(x) = 1$ and $f'(y) = 1$, then f can also defined by $f(z) = f'(z)$ for $z \in V(T') - \{x, y\}$, $f(x) = 2$, $f(y) = 0$ and $f(P_3) \in \{(1, 1, 0), (0, 2, 0), (2, 0, 0)\}$.

Let $\mathcal{F}$ be the family of trees consisting of $\{P_1, P_2, P_3\} \cup \{T : T$ is a tree obtained from P_1, P_2, P_3 by a finite sequence of operations τ_i for $i \in \{1, 2, \ldots, 9\}\}$.

We show first that each tree in the family $\mathcal{F}$ has equal strong Roman domination number and Roman domination number.

Theorem 1. *If T belongs to the family $\mathcal{F}$, then T is a (γ_R, γ_{StR})-tree.*

Proof. We proceed by induction on the number of operations $h(T)$ required to construct the tree T. If $h(T) = 0$, then $T \in \{P_1, P_2, P_3\}$ and clearly T is a (γ_R, γ_{StR})-tree. Assume now that T is a tree with $h(T) = k$ for some positive integer k and each tree $T' \in \mathcal{F}$ with $h(T') < k$ is a (γ_R, γ_{StR})-tree. Then T can be obtained from a tree T' belonging to $\mathcal{F}$ by operation τ_i for $i \in \{1, 2, \ldots, 9\}$. By Lemmas 4-14, T is a (γ_R, γ_{StR})-tree. □

We show next that every (γ_R, γ_{StR})-tree belongs to the family $\mathcal{F}$.

Theorem 2. *If T is a (γ_R, γ_{StR})-tree, then T belongs to the family $\mathcal{F}$.*

Proof. Let T be a (γ_R, γ_{StR})-tree. If $diam(T) \leq 2$, then T is P_1, P_2 or P_3. It is clear that the statement is true. For this reason, we only consider trees T with $diam(T) \geq 3$.

Let T be a (γ_R, γ_{StR})-tree and assume that the result holds for all trees on $n(T)-1$ and fewer vertices. We proceed by induction on the number of vertices of a (γ_R, γ_{StR})-tree.

Assume that $\gamma_R(T) = \gamma_{StR}(T)$. Let $P = t \cdots yxwuv$ be the longest path in T chosen to maximize $d(u)$. Consider t as a root of T. For any vertex $z \in V$, let T_z denote the subtree of T including z and its descendants. Let f be a StR-function of T. By Lemma 1, $d(u) \leq 3$.

Case 1: $d(u) = 3$. If w is adjacent to the center of another P_3, then let $T' = T - T_u$. By Lemma 5, $\gamma_R(T') = \gamma_{StR}(T')$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_1. Hence, $T \in \mathcal{F}$. Without loss of generality, we can assume that each child of w except u is a leaf or an end weak support vertex. Let s and l denote the number of end weak support vertices and leaves that are adjacent to w, respectively. Since f is a StR-function of T, $f(u) = f(w) = 2$. Hence, $s + l \leq 2$. If $s + l = 0$, then $d(w) = 2$. It is obvious that $\gamma_R(T) \neq \gamma_{StR}(T)$. Hence $1 \leq s + l \leq 2$. If $s + l = 2$, say $N(w) \cap (L(T) \cup S(T)) = \{t_1, t_2, u\}$, then define $T' = T - \{((N(t_1) \cup N(t_2)) \cap L(T)) \cup (N[u] - \{w\})\}$. By Lemma 6, $\gamma_R(T') = \gamma_{StR}(T')$. Hence, T' is a (γ_R, γ_{StR})-tree and by induction hypothesis, $T' \in \mathcal{F}$. So T is obtained from T' by operation τ_2. Hence, $T \in \mathcal{F}$. If $s = 1$ and $l = 0$, define $T' = T - T_u$. By Lemma 7, $\gamma_R(T') = \gamma_{StR}(T')$ and there exists a γ_{StR}-function f' of T' such that $f'(w) = 2$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_3. So, $T \in \mathcal{F}$. If $l = 1$ and $s = 0$, define $T' = T - T_w$. By Lemma 8, $\gamma_R(T') = \gamma_{StR}(T')$ and there exists a γ_{StR}-function f' of T' such that $f'(x) = 1$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_4. So, $T \in \mathcal{F}$.

Case 2: $d(u) = 2$. Let s and l denote the number of end weak support vertices and leaves that are adjacent to w, respectively. By Lemma 3, we can assume that $f(w) = 2$ or $f(w) = 0$.

Case 2.1: $f(w) = 2$. Then $s + l \leq 2$. If $s = 2$, then assume that $N(w) \cap S(T) = \{u, u'\}$ and $N(\{u, u'\}) \cap L(T) = \{v_1, v_2\}$. Let $T' = T - \{v_1, v_2\}$. By Lemma 9, $\gamma_R(T') = \gamma_{StR}(T')$. Hence, T' is a (γ_R, γ_{StR})-tree and by induction hypothesis, $T' \in \mathcal{F}$. So T is obtained from T' by operation τ_5. Hence, $T \in \mathcal{F}$.

If $s = l = 1$, then let $T' = T - \{v\}$. By Lemma 9, $\gamma_R(T') = \gamma_{StR}(T')$. Hence, T' is a (γ_R, γ_{StR})-tree and by induction hypothesis, $T' \in \mathcal{F}$. So T is obtained from T' by operation τ_5. Hence, $T \in \mathcal{F}$.

If $s = 1$ and $l = 0$, then let $T' = T - T_w$. By Lemma 4, $\gamma_R(T') = \gamma_{StR}(T')$ and there exists a γ_{StR}-function f' of T' such that $f'(x) \leq 1$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_9. So, $T \in \mathcal{F}$.

Case 2.2: $f(w) = 0$. Then $s + l \leq 2$ with $s \geq 1$. If $s = 2$, then $l = 0$. If $s = 1$, then $l \leq 1$. If $s = 1$ and $l = 0$, then let $T' = T - T_w$. By Lemma 4, $\gamma_R(T') = \gamma_{StR}(T')$ and there exists a γ_{StR}-function f' of T' such that $f'(x) \leq 1$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_9. So, $T \in \mathcal{F}$.

Without loss of generality, we may assume that $s = 2$ and $l = 0$ or $s = l = 1$. Since $\gamma_R(T) = \gamma_{StR}(T)$, it follows that without loss of generality, we may assume that $s = 2$ and $l = 0$ or $s = l = 1$. Hence, w is the central vertex of P_5 or w is the support vertex of P_4. Suppose that x is adjacent to an end strong support vertex t. By Lemma 2, $f(t) = 2$ and $f(x) = 2$. Since f is a γ_{StR}-function of T and $f(w) = 0$, there exists at most one vertex $w_1 \in N_T(x) - \{w, t\}$ such that $f(w_1) = 0$. Define f_1 on $V(T)$ by $f_1(z) = f(z)$ for $z \in V(T) - \{x, w_1\}$, $f_1(x) = 0$ and $f_1(w_1) = 1$ if there exists vertex w_1. It is obvious that f_1 is a Roman dominating function of T. So $\gamma_R(T) \leq f_1(V(T)) < f(V(T)) = \gamma_{StR}(T) = \gamma_R(T)$, which is a contradiction. Hence, x is not adjacent to an end strong support vertex. So, among descendant of x, x is only adjacent to a leaf, an end weak support vertex, or the vertex v_3 of P_4, P_5 or P_3.

If x is adjacent to the vertex v_3 of P_4 or P_5, then let $T' = T - V(P_4)$ or $T' = T - V(P_5)$. By Lemma 13 and Lemma 14, $\gamma_R(T') = \gamma_{StR}(T')$ and there exists a γ_{StR}-function f' of T' such that $f'(w) = 0$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_8. Hence, $T \in \mathcal{F}$. If x is adjacent to the vertex v_3 of P_3, then let $T' = T - V(P_3)$. By Lemma 4, $\gamma_R(T') = \gamma_{StR}(T')$ and there exists a γ_{StR}-function f' of T' such that $f'(x) \leq 1$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_9. Hence, $T \in \mathcal{F}$. Without loss of generality, we may assume that x is not adjacent to the vertex v_3 of P_4,P_5 and P_3.

If x is adjacent to an end weak support vertex $x^{'}$, then let $T' = T - T_w$. By Lemma 10, $\gamma_R(T') = \gamma_{StR}(T')$ and there exists a γ_{StR}-function f' of T' such that $f'(x) = 0$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_6. Hence, $T \in \mathcal{F}$.

Without loss of generality, we may assume that x is adjacent to an leaf $x^{'}$. Then x is adjacent to exactly one leaf. Say $y \in N(x) - \{w, x'\}$. By a similar reason in the proof of Lemma 11, $f(y) = 2$ and $d(y) = 2$. Let $T' = T - T_x$. By

Lemma 11, $\gamma_R(T') = \gamma_{StR}(T')$ and there exists a γ_{StR}-function f' of T' such that $f'(y) = 2$ and $d_{T'}(y) = 1$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_7. Hence, $T \in \mathcal{F}$.

If $d(x) = 2$, say $y \in N(x) - \{w\}$, then $f(y) = 2$ and $d(y) = 2$. Let $T' = T - T_x$. By Lemma 12, $\gamma_R(T') = \gamma_{StR}(T')$ and there exists a γ_{StR}-function f' of T' such that $f'(y) = 2$ and $d_{T'}(y) = 1$. By induction hypothesis, $T' \in \mathcal{F}$. Hence T is obtained from T' by operation τ_7. So, $T \in \mathcal{F}$. □

As an immediate consequence of Theorems 1 and 2 we have the following characterization of (γ_R, γ_{StR})-trees.

Theorem 3. *A tree T is a (γ_R, γ_{StR})-tree if and only if T belongs to the family $\mathcal{F}$.*

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