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ON FULLY FILIAL TORSION RINGS

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Abstract. Rings in which all accessible subrings are ideals are called filial. A ring R is called fully filial if all its subrings are filial (that is rings in which the relation of being an ideal is transitive). The present paper is devoted to the study of fully filial torsion rings. We prove a classification theorem for semiprime fully filial torsion rings.

1. Introduction and preliminaries

All rings considered are associative and do not necessarily have unity. To denote that I is an ideal of a ring R we write $I \lhd R$. A ring R is called *fully filial* if $A \lhd B$ and $B \lhd C$ imply $A \lhd C$ for all subrings A, B, C of R. A ring R is called *filial* if $A \lhd B$ and $B \lhd R$ imply $A \lhd R$ for all subrings A, B of R. Obviously a fully filial ring is a ring in which every subring is filial. Filial rings and related topics were studied in many papers (cf. [1, 2, 4, 8–10, 13]).

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{P}$ stand for the set of natural numbers, the set of integers and the set of all prime numbers, respectively. The prime field of characteristic p is denoted by $\mathbb{Z}_p$. In the current paper, for a subset S of a ring R, we denote by $\langle S\rangle$, $[S]$, S_R the subgroup of R^+ (the additive group of R) generated by S, the subring of R generated by S, the ideal of R generated by S, respectively. Moreover, $T(R) = \{x \in R \ : \ nx = 0 \ \text{ for some } n \in \mathbb{N}\}$ and $R_p = \{x \in R : p^n x = 0 \text{ for some } n \in \mathbb{N}\}$ for any $p \in \mathbb{P}$. If the additive group R^+ of the ring R is a p-group, then we say that R is a p-ring. The order of an element $a \in R^+$ is denoted by $o(a)$.

Note (cf. [4, Theorem 1]) that a ring S is filial if and only if $(a)_S = (a)_S^2 + \langle a\rangle$ for every $a \in S$, where $(a)_S$ denotes the ideal of the ring S generated by a. A ring R is *strongly regular* if $a \in Ra^2$ for every $a \in R$. It is well-known that all strongly regular rings are von Neumann regular, and that for commutative rings these two properties coincide. One can easily check that every strongly regular ring is filial.

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2. Preliminary results

Lemma 2.1. *For a given ring R, the following conditions are equivalent*:

(i) *R is fully filial,*
(ii) *for any $a, b \in R$ the subring $[a, b]$ is filial.*

Proof. (ii)$\Rightarrow$(i). Let S be any subring of R and let $a \in S$. We claim that $(a)_S \subseteq (a)_S^2 + \langle a \rangle$. Let $b \in S$. Then, by filiality of the ring $[a, b]$, we have $[a, b]a + a[a, b] \subseteq (a)_{[a,b]} = (a)^2_{[a,b]} + \langle a \rangle \subseteq (a)_S^2 + \langle a \rangle$, hence $(a)_S \subseteq (a)_S^2 + \langle a \rangle$.

The implication (i)$\Rightarrow$(ii) is obvious. □

Clearly, every subring and every homomorphic image of a fully filial ring is fully filial.

Note that the ring $\mathbb{Q} \oplus \mathbb{Q}$ is not fully filial since its subring $\mathbb{Z} \oplus \mathbb{Z}$ is not filial. Now we state some positive results.

Lemma 2.2. *Let S be an ideal of the ring R such that every subring of S is strongly regular and the ring R/S is fully filial. Then R is fully filial.*

Proof. Let A be any subring of R. Then the ring $(A+S)/S$ is filial as a subring of the fully filial ring R/S. Moreover, $(A + S)/S \cong A/(A \cap S)$ and $A \cap S$ is a subring of S, so $A \cap S$ is strongly regular. From Proposition 3 of [4] we obtain that A is filial. □

Lemma 2.3. *Let R be a torsion ring. Then R is fully filial if and only if the ring R_p is fully filial for every prime integer p.*

Proof. Since R is torsion, $R = \bigoplus_{p \in \mathbb{P}} R_p$. If R is fully filial, then obviously every subring of R is fully filial.

Conversely, let R_p be a fully filial ring for every $p \in \mathbb{P}$ and let K be any subring of R. Then $K = \bigoplus_{p \in \mathbb{P}} (K \cap R_p)$. Let $I \lhd K$ and $J \lhd I$. Since $K \cap R_p$ is fully filial as a subring of R_p, we have $J \cap R_p \lhd K \cap R_p$. Moreover, $(J \cap R_p)(K \cap R_q) = \{0\}$ for every $p \neq q$. Hence $J = \bigoplus_{p \in \mathbb{P}} (J \cap R_p) \lhd K$. □

By the above Lemma, the description of torsion fully filial rings is reduced to the description of fully filial p-rings.

3. Reduced torsion fully filial rings

A ring R is called *reduced* if it has no nonzero nilpotent elements; that is $x \in R$ and $x^2 = 0$ implies $x = 0$. An element x of a ring S is called *potent* if $x^n = x$ for some positive integer $n > 1$. A ring S in which every element is potent is called *a J-ring.* It is well-known that every J-ring is strongly regular and commutative (cf. [12]). The structure of J-rings was studied by many authors (cf. [6, 7, 11]) and the description of J-rings which are p-rings can be found, for instance, in [5, Theorem 6.2] or in [14, Corollary B.3.5].

Proposition 3.1. *Let p be a prime integer and let R be a reduced p-ring. Then the following conditions are equivalent*:

(i) *R is fully filial,*
(ii) *for any $a \in R$ the ring $[a]$ is isomorphic to some direct sum of finite fields of characteristic p,*
(iii) *for any $a \in R$ there exists $n \in \mathbb{N}$ such that $a^{p^n} = a$,*
(iv) *for any $a \in R$ the ring $[a]$ is strongly regular,*
(v) *every subring of R is strongly regular.*

Proof. (i)⇒(ii). From the assumptions it follows that, for any $a \in R$, the ring $[a]$ is commutative, torsion, reduced and filial. Moreover, $[a]$ is finitely generated and from Theorem 2.4 of [3] we have that $[a] \cong \bigoplus_{j=1}^{k} F_j$ for some finite fields $F_1, F_2, \ldots, F_k$.

The implications (ii)⇒(iii), (iii)⇒(iv) and (iv)⇒(v) are obvious.

(v)⇒(i). Let A be any subring of R and let $x \in A$. Then $x \in [x]x^2 \subseteq Ax^2$ and consequently A is strongly regular and hence filial. □

From Proposition 3.1 and Lemma 2.3 we immediately get the following corollary.

Corollary 3.2. *A reduced torsion ring is fully filial if and only if it is a* J*-ring.*

4. Main examples

Lemma 4.1. *Let p be any prime integer and let R be a non filial ring such that $pR = \{0\}$ and $|R| = p^3$. Then there exist elements $x, y \in R$ that are linearly independent over $\mathbb{Z}_p$ such that $x^2 = y^2 = xy = yx = 0$.*

Proof. Clearly, $R^+ \cong \mathbb{Z}_p^+ \times \mathbb{Z}_p^+ \times \mathbb{Z}_p^+$. Since R is not filial, there exist subrings A, B of R such that $A \lhd B \lhd R$, but A is not an ideal of R. But $|R| = p^3$ and $A \neq B$, so $|B| = p^2$ and $|A| = p$. Hence $A^2 = A$ or $A^2 = \{0\}$. If $A^2 = A$, then $RA = RA^2 \subseteq RBA \subseteq BA \subseteq A$ (similarly, $AR \subseteq A$) and consequently $A \lhd R$, a contradiction. Hence $A^2 = \{0\}$. Moreover, $A = \langle x \rangle$ for some $x \in A$. Note that $x^2 = 0$ and $o(x) = p$. Next $A_R \subseteq B$ and by Andrunakievich's Lemma, $A_R^3 \subseteq A$. But $|A| = p$, so $A_R^3 = A$ or $A_R^3 = \{0\}$. If $A_R^3 = A$, then $A \lhd R$, a contradiction. Thus $A_R^3 = \{0\}$. Moreover, $A \subsetneq A_R \subseteq B$, so since $|B| = p^2$, we have $A_R = B$ and consequently $\langle x \rangle_R = B$. If $A_R^2 \neq \{0\}$, then since $A_R^2 \neq A_R$, $|A_R^2| = p$. Also, it cannot happen that $x \in A_R^2$ because that would imply that $\langle x \rangle = A_R^2$ and $\langle x \rangle \lhd R$, a contradiction. Hence $\langle x \rangle \oplus A_R^2 = A_R$ and $A \oplus A_R^2 = A_R$. But, then $A_R^2 = A^2 \oplus A_R^4 = \{0\}$, which is a contradiction. It follows that $A_R^2 = \{0\}$. Thus there exists $y \in A_R$ such that $A_R = \langle x \rangle \oplus \langle y \rangle$ and $x^2 = y^2 = xy = yx = 0$. □

For a natural number n and any ring R, let $M_n(R)$ denote the ring of $n \times n$ matrices over R. An $n \times n$ matrix whose entries are all zeros is denoted by 0_n.

Lemma 4.2. *Let K be any field and let $X, Y \in M_2(K)$. If $X^2 = Y^2 = XY = YX = 0_2$, then X and Y are linearly dependent over K.*

Proof. Let $X, Y \in M_2(K)$ be such that $X^2 = Y^2 = XY = YX = 0_2$ and suppose that X and Y are linearly independent over K. Then the linear subspace

S generated by X and Y is of dimension two over K. Moreover, S is not an ideal of the ring $M_2(K)$ since $M_2(K)$ is a simple ring and $S^2 = \{0_2\}$. Then there exists a matrix $A \in M_2(K)$ such that $AX \notin S$ or $XA \notin S$ or $AY \notin S$ or $YA \notin S$. Without loss of generality, assume that $AX \notin S$. Then vectors X, Y, AX are linearly independent over K. We claim that vectors X, Y, AX, A are linearly independent over K. Assume that $aX + bY + cAX + dA = 0_2$ for some $a, b, c, d \in K$. Multipling this equality by X we get that $dAX = 0_2$ and $d = 0$, consequently by linear independence of X, Y, AX we have $a = b = c = 0$. Therefore vectors X, Y, AX, A are linearly independent over K and form a basis of the vector space $M_2(K)$ over K. Hence $aX + bY + cAX + dA = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for some $a, b, c, d \in K$ and multipling this equality by X we get that $dAX = X$, which is a contradiction, since X, Y, AX, A are linearly independent. □

Example 4.3. For every prime integer p the ring $M_2(\mathbb{Z}_p)$ is fully filial. We will show that every subring A of $M_2(\mathbb{Z}_p)$ is filial. If $A = M_2(\mathbb{Z}_p)$, then A is filial as a simple ring. If $|A| < p^3$, then the statement is obvious. Finally, if $|A| = p^3$, then A is filial by Lemmas 4.1 and 4.2.

Note that if $[a]$ is a ring generated by a single element a such that $a^2 = 0$ and $o(a) = p$, then the ring $S = M_2(\mathbb{Z}_p) \times [a]$ is not fully filial since $A = \langle ((\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), a)\rangle \lhd \left(\begin{smallmatrix} 0 & \mathbb{Z}_p \\ 0 & 0 \end{smallmatrix}\right) \times [a] \lhd \left(\begin{smallmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ 0 & 0 \end{smallmatrix}\right) \times [a] = S$ and A is not an ideal of S because $((\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), 0) \cdot ((\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), a) \notin A$. In particular, the ring $M_2(\mathbb{Z}_p) \times M_2(\mathbb{Z}_p)$ is not fully filial.

Lemma 4.4. *Let R be any ring. Assume that every subgroup A of R^+ is an ideal of R and $RA = AR = A^3$. Then $R^2 = \{0\}$ or R is torsion and $R_p^2 = \{0\}$ or $R_p \cong \mathbb{Z}_p$ for every prime integer p.*

Proof. Let A and B be any subgroups of R^+ such that $A \subseteq B$. Then $BA \subseteq RA = A^3 \subseteq A^2 \subseteq BA$, so $BA = A^3 = A^2$. Similarly, one can show that $AB = A^2 = A^3$. Hence, by simple induction, $BA = AB = A^n$ for every $n = 2, 3, \ldots$. Assume that $T(R) \neq R$. We will show that $R^2 = \{0\}$. Assume that $R^2 \neq \{0\}$ and take any $x \in R$ such that $o(x) = \infty$. If $x^2 \neq 0$, then since $x^2 \in \langle x \rangle$, we have $x^2 = kx$ for some $k \in \mathbb{Z} \setminus \{0\}$. Moreover, $\langle x \rangle \cdot \langle 2x \rangle = \langle 2x \rangle^3$, so $\langle 2x^2 \rangle = \langle 8x^3 \rangle$. Moreover, $x^3 = kx^2 = k^2 x$ and hence $2kx = t \cdot 8k^2 x$ for some $t \in \mathbb{Z}$. Thus since $o(x) = \infty$ and $k \neq 0$ we get that $2 = 8kt$, a contradiction. Consequently $x^2 = 0$. But $R\langle x \rangle = \langle x \rangle R = \langle x \rangle^2$, so $Rx = xR = \{0\}$. Since $R^2 \neq \{0\}$, there exist $a, b \in T(R)$ such that $ab \neq 0$. Moreover, $ab \in \langle a \rangle^2 = \langle a^2 \rangle$, so $a^2 \neq 0$. Obviously $o(x+a) = \infty$, so $0 = (x+a)a = xa + a^2 = 0 + a^2 = a^2 \neq 0$, a contradiction. Hence, if $T(R) \neq R$, then $R^2 = \{0\}$.

Now, assume that $R = T(R)$ and fix any prime integer p. Assume that $R_p^2 \neq \{0\}$. Then there exist $a, b \in R_p$ such that $ab \neq 0$. But $ab \in \langle a \rangle^2$, so $a^2 \neq 0$. Moreover, for any natural number $n \geq 2$ we have $\langle a^2 \rangle = \langle a \rangle^2 = \langle a \rangle^n = \langle a^n \rangle$, so $a^n \neq 0$. But $a^2 \in \langle a \rangle$, so $a^2 = Ka$ for some $K \in \mathbb{Z}$. Hence, by a simple induction argument $a^n = K^{n-1} a$ for every $n = 2, 3, \ldots$. Since $o(a) = p^s$ for some $s \in \mathbb{N}$ and as was proved, element a is not nilpotent, p does not divide

K. Hence there exists $L \in \mathbb{Z}$ such that $KL \equiv 1 \pmod p$. Then $e = La$ is an idempotent and $o(e) = o(a) = p^s$. Thus $\langle e\rangle \cdot \langle pe\rangle = \langle pe\rangle^s = \{0\}$, hence $pe^2 = 0$ and $pe = 0$. Consequently $\langle e\rangle = [e] \cong \mathbb{Z}_p$ is an ideal of R_p and $[e]$ has an identity element. Therefore $[e] \oplus I = R_p$ for some $I \lhd R_p$. Take any $i \in I$. Then $e(e+i) \in \langle e+i\rangle$, so $e(e+i) = U(e+i)$ for some $U \in \mathbb{Z}$. But $ei = 0$, so $e(e+i) = e^2 + ei = e$ and $e = Ue + Ui$. Thus $e = Ue$ and $Ui = 0$. Hence p does not divide U and $i = 0$. Finally, $I = \{0\}$ and $R_p = [e] \cong \mathbb{Z}_p$. □

Theorem 4.5. *Let $n > 1$ be any natural number and let R be any ring. The ring $M_n(R)$ is fully filial if and only if*

(i) *$n > 2$, $R^2 = \{0\}$, or*

(ii) *$n = 2$ and $R^2 = \{0\}$ or $R = \bigoplus_{p\in\mathbb{P}} R_p$ and $R_p^2 = \{0\}$ or $R_p \cong \mathbb{Z}_p$ for every prime integer p.*

Proof. (i) Let $n \geq 3$ and assume that the ring $M_n(R)$ is fully filial. Since the ring $M_3(R)$ embeds in $M_n(R)$, it follows that $M_3(R)$ is fully filial. But $\begin{pmatrix} 0 & R & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \lhd \begin{pmatrix} 0 & R & R\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \lhd \begin{pmatrix} 0 & R & R\\ 0 & 0 & R\\ 0 & 0 & 0\end{pmatrix}$, so $\begin{pmatrix} 0 & R & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \lhd \begin{pmatrix} 0 & R & R\\ 0 & 0 & R\\ 0 & 0 & 0\end{pmatrix}$, and hence $R^2 = \{0\}$. The converse is obvious.

(ii) Assume that the ring $M_2(R)$ is fully filial and let A be any subgroup of R^+. Then $\left(\begin{smallmatrix} 0 & A\\ 0 & 0\end{smallmatrix}\right) \lhd \left(\begin{smallmatrix} 0 & R\\ 0 & 0\end{smallmatrix}\right) \lhd \left(\begin{smallmatrix} R & R\\ 0 & 0\end{smallmatrix}\right)$, so $\left(\begin{smallmatrix} 0 & A\\ 0 & 0\end{smallmatrix}\right) \lhd \left(\begin{smallmatrix} R & R\\ 0 & 0\end{smallmatrix}\right)$, hence $RA \subseteq A$. Similarly, $\left(\begin{smallmatrix} 0 & A\\ 0 & 0\end{smallmatrix}\right) \lhd \left(\begin{smallmatrix} 0 & R\\ 0 & 0\end{smallmatrix}\right) \lhd \left(\begin{smallmatrix} 0 & R\\ 0 & R\end{smallmatrix}\right)$, which implies that $AR \subseteq A$. Thus $A \lhd R$. Hence, and by filiality of $M_2(R)$ and Theorem 14 of [10] we have $RA = AR = A^3$. By the above and Lemma 4.4 we obtain that $R^2 = \{0\}$ or R is torsion and, for every prime integer p, $R_p^2 = \{0\}$ or $R_p \cong \mathbb{Z}_p$.

The converse implication is obvious when $R^2 = \{0\}$. If not, then $R = \bigoplus_{p\in\mathbb{P}} R_p$ and $R_p^2 = \{0\}$ or $R_p \cong \mathbb{Z}_p$ for every prime integer p. Hence $M_2(R) \cong \bigoplus_{p\in\mathbb{P}} M_2(R_p)$ and by Lemma 2.3 it is enough to show that for every prime integer p the ring $M_2(R_p)$ is fully filial. If $R_p^2 = \{0\}$, then this is obvious, and if $R_p \cong \mathbb{Z}_p$, then it follows from Example 4.3. □

5. Semiprime torsion fully filial rings

An ideal I of a ring R is *semiprime* if it is an intersection of prime ideals. A ring R is *semiprime* in case $\{0\}$ is a semiprime ideal of R. It is well-known that a ring is semiprime if and only if it has no non-zero nilpotent ideals or equivalently, $xRx = \{0\}$ implies $x = 0$ for all $x \in R$.

Theorem 5.1. *For every semiprime p-ring R the following conditions are equivalent*:

(i) *R is fully filial,*

(ii) *R is a* J*-ring or $R \cong J \times M_2(\mathbb{Z}_p)$, where J is a* J*-ring such that $pJ = \{0\}$.*

Proof. (i)⇒(ii). Since R is semiprime, $pR = \{0\}$. If R is reduced, then by Corollary 3.2, R is a J-ring. Assume that R is not reduced. Then there exists a nonzero $a \in R$ such that $a^2 = 0$. Then $[a] = \langle a \rangle$, $|[a]| = p$ and, since R is semiprime, $aRa \neq \{0\}$. Moreover, $[a] \lhd [a] + aRa \lhd [a] + Ra$. Since R is fully filial, we have $[a] \lhd [a] + Ra$. Hence $aRa \subseteq [a]$ and $|[a]| = p \in \mathbb{P}$ implies $aRa = [a] = \langle a \rangle$. Thus $a = aba$ for some $b \in R$. Denote: $e = ab$ and $f = ba$. Then $a = ea = fa = eaf$, $ae = fa = fe = 0$, $e = e^2$, $f = f^2$, and $e, f \neq 0$ because $a \neq 0$. Moreover, $eRe = abRab \subseteq (aRa)b = \langle a \rangle b = \langle ab \rangle = \langle e \rangle$ and $e = e^3 \in eRe$, so $eRe = \langle e \rangle$. But $e = e^2 \neq 0$ and $pe = 0$, so $[e] = \langle e \rangle \cong \mathbb{Z}_p$. From above, eRe is a division ring and therefore, since R is semiprime, we obtain that Re is a minimal left ideal of R.

Similarly, one can show that $fRf = \langle f \rangle = [f] \cong \mathbb{Z}_p$, so Rf is a minimal left ideal of R.

Since e and f are idempotents, $Re \oplus l_R(e) = R$ and $Rf \oplus l_R(f) = R$. But $fe = 0$, so $Rf \subseteq l_R(e)$ and, by the modularity law for the lattice of subgroups of R^+, we obtain $l_R(e) = Rf \oplus [l_R(e) \cap l_R(f)]$ and this implies that $R = Re \oplus Rf \oplus [l_R(e) \cap l_R(f)]$ which is a direct sum of left ideals of a ring R. Denote: $L = l_R(e) \cap l_R(f)$. Then $Le = Lf = \{0\}$. But $a = ea$, so $La = \{0\}$. Hence, $[a] \lhd [a] + aL \lhd [a] + L$. Since R is fully filial, $[a] \lhd [a] + L$ and $aL \subseteq [a]$. But $a = af \in Rf$, so $aL \subseteq Rf \cap L = \{0\}$ and therefore $aL = \{0\}$. Hence, $fL = baL = \{0\}$ and $eL = abL \subseteq aL = \{0\}$. Next, LRe is a left ideal of R and $LRe \subseteq Re$, by minimality of Re. If $LRe \neq \{0\}$, then $LRe = Re$. Therefore $0 = eLRe = eRe = [e] \neq \{0\}$, a contradiction. Thus $LRe = \{0\}$. Similarly, $LRf = \{0\}$. But $Re \oplus Rf \oplus L = R$, so $L \lhd R$ and $S = Re \oplus Rf \lhd R$. Hence S is semiprime, fully filial and every left ideal of S is a left ideal of R. Thus Re and Rf are minimal ideals of S. Recall that $a \in S$, so S is not reduced. From the Wedderburn-Artin Theorem it follows that, $S \cong M_2(D)$ for some division p-ring D. Theorem 4.5 implies that $S \cong M_2(\mathbb{Z}_p)$.

If $L = \{0\}$, then $R \cong M_2(\mathbb{Z}_p)$. If $L \neq \{0\}$, then L is semiprime and fully filial. If L is not reduced, then by the first part of the proof, L has no ideal $J \cong M_2(\mathbb{Z}_p)$. Therefore $S \oplus J \cong M_2(\mathbb{Z}_p) \times M_2(\mathbb{Z}_p)$ and the ring $S \oplus J$ is fully filial which contradicts Example 4.3. Finally, L is reduced and by Corollary 3.2, L is a J-ring.

(ii)$\Rightarrow$(i). If R is a J-ring, then the conclusion follows from Corollary 3.2. If $R \cong J \times M_2(\mathbb{Z}_p)$, where J is a J-ring, then it follows from Example 4.3 that $R/J \cong M_2(\mathbb{Z}_p)$ is fully filial. Moreover, by Proposition 3.1 every subring of J is strongly regular. Combining these observations, we conclude from Lemma 2.2 that R is fully filial. □

Finally, the next theorem is a direct consequence of the theorem above and its proof, and Proposition 2.3.

Theorem 5.2. *A torsion ring R is fully filial and semiprime if and only if $R = \bigoplus_{p \in \mathbb{P}} R_p$, where R_p is a* J*-ring or $R_p \cong J \times M_2(\mathbb{Z}_p)$ for some* J*-ring such that $pJ = \{0\}$ for every prime integer p.*

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