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SECOND COHOMOLOGY OF $\mathfrak{aff}(1)$ ACTING ON n-ARY DIFFERENTIAL OPERATORS

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Abstract. We compute the second cohomology of the affine Lie algebra $\mathfrak{aff}(1)$ on the dimensional real space with coefficients in the space $\mathcal{D}^n_{\underline{\lambda},\mu}$ of n-ary linear differential operators acting on weighted densities where $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$. We explicitly give 2-cocycles spanning these cohomology.

1. Introduction

In mathematical deformation theory one studies how an object in a certain category of spaces can be varied in dependence on the points of a parameter space. In other words, deformation theory thus deals with the structure of families of objects like varieties, singularities, vector bundles, coherent sheaves, algebras or differentiable maps. Deformation problems appear in various areas of mathematics, in particular in algebra, algebraic and analytic geometry, and mathematical physics. Cohomology is a useful tool in Poisson Geometry, plays an important role in Deformation and Quantization Theory, and attracts more and more interest among algebraists.

In the theory of Lie groups, Lie algebras and their representation theory, a Lie algebra extension is an enlargement of a given Lie algebra $\mathfrak{g}$ by another Lie algebra $\mathfrak{h}$. Extensions arise in several ways. There is the trivial extension obtained by taking a direct sum of two Lie algebras. Other types are the split extension and the central extension. Extensions may arise naturally, for instance, when forming a Lie algebra from projective group representations. Such a Lie algebra will contain central charges.

Starting with a polynomial loop algebra over finite-dimensional simple Lie algebra and performing two extensions, a central extension and an extension by a derivation, one obtains a Lie algebra which is isomorphic with an untwisted affine Kac-Moody algebra. Using the centrally extended loop algebra one may construct a current algebra in two spacetime dimensions. The Virasoro algebra is the universal central extension of the *Witt* algebra ([2]).

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Central extensions are needed in physics, because the symmetry group of a quantized system usually is a central extension of the classical symmetry group, and in the same way the corresponding symmetry Lie algebra of the quantum system is, in general, a central extension of the classical symmetry algebra (see [3]).

In 1967, Victor Kac and Robert Moody independently generalized the notion of classical Lie algebras, resulting in a new theory of infinite-dimensional Lie algebras, now called Kac-Moody algebras (see [9, 12]). They generalize the finite-dimensional simple Lie algebras and can often concretely be constructed as extensions. Kac-Moody algebras have been conjectured to be a symmetry groups of a unified superstring theory (see [8]). The centrally extended Lie algebras play a dominant role in quantum field theory, particularly in conformal field theory, string theory and in M-theory (see [6]). Central extensions are the simplest extensions of Lie algebras. They appear both in geometry and in physics. Thus, they play an important role in symplectic geometry [10, 13] and in various versions of quantization (see also [11]).

A large portion towards the end is devoted to background material for applications of Lie algebra extensions, both in mathematics and in physics, in areas where they are actually useful. A parenthetical link, (background material), is provided where it might be beneficial.

Lie algebra extensions are most interesting and useful for infinite-dimensional Lie algebras. The theory of group extensions and their interpretation in terms of cohomology is well known, see, e.g., [7]. Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. The second space $\mathrm{H}^2(\mathfrak{g}, M)$ classify the nontrivial extensions of the Lie algebra $\mathfrak{g}$ by the module M:

$$0 \longrightarrow M \longrightarrow \cdot \longrightarrow \mathfrak{g} \longrightarrow 0,$$

the Lie structure on $\mathfrak{g} \oplus M$ being given by

$$[(g_1, \alpha),\ (g_2, \beta)]_{\mathfrak{g}\oplus M} = ([g_1, g_2],\ g_1.\beta - g_2.\alpha + \Omega(g_1, g_2)),$$

where Ω is a 2-cocycle with values in M.

In this paper we consider a natural class of "non-central" extensions of $\mathfrak{aff}(1)$, namely extensions by the modules $\mathcal{D}_{\underline{\lambda},\mu}$ of n-ary linear differential operators acting on weighted densities. The result is quite surprising: there exists a C^k_{n+k-1} extensions if and only if $\mu = \lambda_1 + \cdots + \lambda_n + k$, where $k \in \mathbb{N}$.

We consider the one-parameter action of the Lie algebra of vector fields $\mathrm{Vect}(\mathbb{R})$ by the *Lie derivative* on the space $\mathcal{C}^\infty(\mathbb{R})$ of smooth functions on $\mathbb{R}$ defined by:

$$L^\lambda_{X\frac{d}{dx}}(f) = Xf' + \lambda f X', \tag{1.1}$$

where $f \in \mathcal{C}^\infty(\mathbb{R})$ and $X\frac{d}{dx} \in \mathrm{Vect}(\mathbb{R})$ and where the superscript $'$ stands for $\frac{d}{dx}$. We denote by $\mathcal{F}_\lambda$ the $\mathrm{Vect}(\mathbb{R})$-module structure on $\mathcal{C}^\infty(\mathbb{R})$ defined by the action (1.1). Geometrically, $\mathcal{F}_\lambda$ is the space weighted densities of weight λ on

$\mathbb{R}$:

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda,\ f \in \mathcal{C}^\infty(\mathbb{R}) \right\}.$$

Any differential operator A on $\mathbb{R}$ can be viewed as the linear mapping $f(dx)^\lambda \mapsto (Af)(dx)^\mu$ from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ ($\lambda,\ \mu$ in $\mathbb{R}$). Thus the space of differential operators is a Vect($\mathbb{R}$)-module, denoted $\mathcal{D}^1_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$. The Vect($\mathbb{R}$) action is:

$$L_X^{\lambda,\mu}(A) = L_X^\mu \circ A - A \circ L_X^\lambda. \tag{1.2}$$

More generally we consider the structures Vect($\mathbb{R}$)-modules on the space $\mathcal{D}^n_{\underline{\lambda},\mu}$ of n-linear differential operators: $A : \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \to \mathcal{F}_\mu$. The Lie algebra Vect($\mathbb{R}$) acting on the space $\mathcal{D}^n_{\underline{\lambda},\mu}$ of n-ary Linear differential operators by:

$$L_X^{(\lambda_1,\ldots,\lambda_n);\mu}(A) = L_X^\mu \circ A - A \circ L_X^{(\lambda_1,\ldots,\lambda_n)}, \tag{1.3}$$

where $L_X^{(\lambda_1,\ldots,\lambda_n)}$ is the Lie derivative defined by the Leibniz rule:

$$\begin{aligned} & L_X^{(\lambda_1,\ldots,\lambda_n)}(\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_n) \\ &= L_X^{\lambda_1}(\Phi_1) \otimes \Phi_2 \otimes \cdots \otimes \Phi_n + \cdots + \Phi_1 \otimes \cdots \otimes \Phi_{n-1} \otimes L_X^{\lambda_n}(\Phi_n). \end{aligned} \tag{1.4}$$

2. Definitions and notations

In this section, we recall the main definitions and facts related to the geometry of the space $\mathbb{R}$. We also recall some fundamental concepts from cohomology theory (see, e.g., [5, 7]).

2.1. Cohomology theory

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [4, 5, 7, 14]). Let $\mathfrak{g}$ be a Lie algebra acting on a vector space V. The space of n-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g}, V) := \mathrm{Hom}(\Lambda^n \mathfrak{g}; V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g}, V) \longrightarrow C^{n+1}(\mathfrak{g}, V)$ is a $\mathfrak{g}$-map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n, denoted $Z^n(\mathfrak{g}, V)$, is the space of *n-cocycles*, among them, the elements in the range of δ_{n-1} are called *n-coboundaries*. We denote $B^n(\mathfrak{g}, V)$ the space of n-coboundaries.

By definition, the n^{th} cohomolgy space is the quotient space

$$\mathrm{H}^n(\mathfrak{g}, V) = Z^n(\mathfrak{g}, V)/B^n(\mathfrak{g}, V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0, 1 and 2: for $\Xi \in C^0(\mathfrak{g},\ V) = V$, $\delta\Xi(x) := x \cdot \Xi$, and for $\Lambda \in C^1(\mathfrak{g}, V)$,

$$\delta(\Lambda)(x,\ y) := g \cdot \Lambda(y) - y \cdot \Lambda(x) - \Lambda([x,\ y]) \quad \text{for any} \quad x, y \in \mathfrak{g}, \tag{2.5}$$

for $\Omega \in C^2(\mathfrak{g}, V)$,

$$\delta(\Omega)(x, y, z) := x \cdot \Omega(y, z) - y \cdot \Omega(x, z) + z \cdot \Omega(x, y) \tag{2.6}$$

$$- \Omega([x,\ y], z) + \Omega([x,\ z], y) - \Omega([y,\ z], x),$$

where $x,\ y,\ z\ \in \mathfrak{g}$.

2.2. Lie algebra $\mathfrak{aff}(1)$

The Lie algebra $\mathfrak{aff}(1)$ is realized as a subalgebra of the Lie algebra $\mathrm{Vect}(\mathbb{R})$ (see [1]):

$$\mathfrak{aff}(1) = Span(X_1 = \frac{d}{dx}, X_x = x\frac{d}{dx}).$$

The commutation relations are

$$[X_1, X_x] = X_1, \qquad [X_x, X_x] = 0, \qquad [X_1, X_1] = 0.$$

2.3. The space of tensor densities on $\mathbb{R}$

The Lie algebra, $\mathrm{Vect}(\mathbb{R})$, of vector fields on $\mathbb{R}$ naturally acts, by the Lie derivative, on the space

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda \ :\ f \in C^\infty(\mathbb{R}) \right\},$$

of weighted densities of degree λ. The Lie derivative L_X^λ of the space $\mathcal{F}_\lambda$ along the vector field $X\frac{d}{dx}$ is defined by

$$L_X^\lambda = X\partial_x + \lambda X', \tag{2.7}$$

where $X, f \in C^\infty(\mathbb{R})$ and $X' := \frac{dX}{dx}$. More precisely, for all $f dx^\lambda \in \mathcal{F}_\lambda$, we have

$$L_X^\lambda(f dx^\lambda) = (Xf' + \lambda f X')dx^\lambda.$$

In the paper, we restrict ourselves to the Lie algebra $\mathfrak{aff}(1)$ which is isomorphic to the Lie subalgebra of $\mathrm{Vect}(\mathbb{R})$ spanned by

$$\{X_1, X_x\}.$$

2.4. The space of n-ary linear differential operators as a $\mathfrak{aff}(1)$-module

The space of n-ary linear differential operators is a $\mathrm{Vect}(\mathbb{R})$-module, denoted

$$\mathcal{D}^n_{\underline{\lambda},\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}, \mathcal{F}_\mu).$$

The $\mathrm{Vect}(\mathbb{R})$ action is:

$$L_X^{\underline{\lambda},\mu}(A) = L_X^\mu \circ A - A \circ L_X^{\underline{\lambda}}, \tag{2.8}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $L_X^{(\lambda_1,\ldots,\lambda_n)}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ defined by the Leibnitz rule:

$$\begin{aligned} &L_X^{\underline{\lambda}}(f_1 dx^{\lambda_1} \otimes \cdots \otimes f_n dx^{\lambda_n}) \\ &= L_X^{\lambda_1}(f_1) \otimes \cdots \otimes f_n dx^{\lambda_n} + \cdots + f_1 dx^{\lambda_1} \otimes \cdots \otimes L_X^{\lambda_n}(f_n dx^{\lambda_n}). \end{aligned}$$

3. The space $\mathrm{H}^2_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}^3_{\underline{\lambda},\mu})$

The first main result of paper is the following.

Theorem 3.1. *The space* $\mathrm{H}^2_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}^3_{\underline{\lambda},\mu})$ *has the following structure*:

$$\mathrm{H}^2_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}^3_{\underline{\lambda},\mu}) \simeq \begin{cases} \mathbb{R}^{C^k_{k+2}} & \text{if } \mu = \lambda_1 + \lambda_2 + \lambda_3 + k, \\ 0 & \text{otherwise}, \end{cases} \tag{3.9}$$

where $C^p_q = \frac{q!}{(q-p)!p!}$.

The following 2*-cocycles span the corresponding cohomology spaces*:

$$\begin{aligned} &\mathcal{C}(X\frac{d}{dx}, Y\frac{d}{dx})(f_1 dx^{\lambda_1}, f_2 dx^{\lambda_2}, f_3 dx^{\lambda_3}) \\ &= \sum_{i+j=0}^{k-1} c_{i,j}(XY' - X'Y) f_1^{(i)} f_2^{(j)} f_3^{(i+j-k)} dx^{\lambda_1+\lambda_2+\lambda_3+k}, \end{aligned}$$

where $c_{i,j}$ *are constants,* $f_1 dx^{\lambda_1} \in \mathcal{F}_{\lambda_1}$*,* $f_2 dx^{\lambda_2} \in \mathcal{F}_{\lambda_2}$*,* $f_3 dx^{\lambda_3} \in \mathcal{F}_{\lambda_3}$ *and* $X\frac{d}{dx}$, $Y\frac{d}{dx} \in \mathfrak{aff}(1)$.

We need the following lemma.

Lemma 3.2. *Let* $b \in C^1(\mathfrak{aff}(1), \mathcal{D}^3_{\underline{\lambda},\lambda_1+\lambda_2+\lambda_3+k})$ *be defined as follows*: *for* $X\frac{d}{dx} \in \mathfrak{aff}(1)$*,* $f_1 dx^{\lambda_1} \in \mathcal{F}_{\lambda_1}$*,* $f_2 dx^{\lambda_2} \in \mathcal{F}_{\lambda_2}$ *and* $f_3 dx^{\lambda_3} \in \mathcal{F}_{\lambda_3}$

$$\begin{aligned} b(X)(f_1, f_2, f_3) = &\sum_{i+j+n=k} c_{i,j,n} X f_1^{(i)} f_2^{(j)} f_3^{(n)} dx^{\mu} \\ &+ \sum_{i+j+n=k-1} \alpha_{i,j,n} X' f_1^{(i)} f_2^{(j)} f_3^{(n)} dx^{\mu}, \end{aligned} \tag{3.10}$$

where the coefficients $c_{i,j,n}$ *and* $\alpha_{i,j,n}$ *are constants.*

Then the map $\delta b : \mathfrak{aff}(1) \times \mathfrak{aff}(1) \to \mathcal{D}^3_{\underline{\lambda},\lambda_1+\lambda_2+\lambda_3+k}$ *is given by*

$$\begin{aligned} &\delta b(X, Y)(f_1, f_2, f_3) \\ &= L_X^{\underline{\lambda},\lambda_1+\lambda_2+\lambda_3+k} b(Y)(f_1 dx^{\lambda_1}, f_2 dx^{\lambda_2}, f_3 dx^{\lambda_3}) \\ &\quad - L_Y^{\underline{\lambda},\lambda_1+\lambda_2+\lambda_3+k} b(X)(f_1 dx^{\lambda_1}, f_2 dx^{\lambda_2}, f_3 dx^{\lambda_3}) \\ &\quad - b([X,Y])(f_1 dx^{\lambda_1}, f_2 dx^{\lambda_2}, f_3 dx^{\lambda_3}) \\ &= \sum_{i+j+n=k} c_{i,j,n}(\mu - \lambda_1 - \lambda_2 - \lambda_3 - k)(X'Y - XY') f_1^{(i)} f_2^{(j)} f_3^{(n)}. \end{aligned} \tag{3.11}$$

3.1. Proof of theorem

Any 2-cocycle $\mathcal{C} \in Z^1_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}^3_{\underline{\lambda},\mu})$ should be retain the following general form:

$$\mathcal{C}(X, Y)(f_1, f_2, f_3) = \sum_{i+j+n=k-1} c_{i,j,n}(XY' - X'Y) f_1^{(i)} f_2^{(j)} f_3^{(n)} dx^{\mu}, \tag{3.12}$$

where $c_{i,j,n}$ are constants.

The 2-cocycle condition reads as follows: for all $f_1dx^{\lambda_1}\in\mathcal{F}_{\lambda_1}$, $f_2dx^{\lambda_2}\in\mathcal{F}_{\lambda_2}$, $f_3dx^{\lambda_3}\in\mathcal{F}_{\lambda_3}$ and for all $X,Y\in\mathfrak{aff}(1)$, we have

$$\begin{aligned}&L_X^{\lambda_1,\lambda_2,\lambda_3,\mu}\mathcal{C}(Y,Z)(f_1dx^{\lambda_1},f_2dx^{\lambda_2},f_3dx^{\lambda_3})\\&-\mathcal{C}([X,Y],Z)+\circlearrowleft(X,Y,Z)=0,\end{aligned}\tag{3.13}$$

where $\circlearrowleft(X,Y,Z)$ denotes summation over the cyclic permutation on X, Y, Z.

A direct computation, proves that we have

$$\begin{aligned}\Phi_1&=L_X^{\lambda_1,\lambda_2,\lambda_3,\mu}\mathcal{C}(Y,Z)(f_1,f_2,f_3)\\&=L_X^{\mu}\mathcal{C}(Y,Z)(f_1,f_2,f_3)-\mathcal{C}(Y,Z)(L_X^{\lambda_1}(f_1),f_2,f_3)\\&\quad-\mathcal{C}(Y,Z)(f_1,L_X^{\lambda_2}(f_2),f_3)-\mathcal{C}(Y,Z)(f_1,f_2,L_X^{\lambda_3}(f_3))\\&=\sum_{i+j+n=k-1}c'_{i,j,n}X(YZ'-Y'Z)f_1^{(i)}f_2^{(j)}f_3^{(n)}\\&\quad+\sum_{i+j+n=k-1}c_{i,j,n}(\mu-\lambda_1-\lambda_2-\lambda_3-k+1)X'(YZ'-Y'Z)f_1^{(i)}f_2^{(j)}f_3^{(n)},\end{aligned}$$

$$\begin{aligned}\Phi_2&=L_Y^{\lambda_1,\lambda_2,\lambda_3,\mu}\mathcal{C}(X,Z)(f_1,f_2,f_3)\\&=\sum_{i+j+n=k-1}c'_{i,j,n}Y(XZ'-X'Z)f_1^{(i)}f_2^{(j)}f_3^{(n)}\\&\quad+\sum_{i+j+n=k-1}c_{i,j,n}(\mu-\lambda_1-\lambda_2-\lambda_3-k+1)Y'(XZ'-X'Z)f_1^{(i)}f_2^{(j)}f_3^{(n)},\end{aligned}$$

$$\begin{aligned}\Phi_3&=L_Z^{\lambda_1,\lambda_2,\lambda_3,\mu}\mathcal{C}(X,Y)(f_1,f_2,f_3)\\&=\sum_{i+j+n=k-1}c'_{i,j,n}Z(XY'-X'Y)f_1^{(i)}f_2^{(j)}f_3^{(n)}\\&\quad+\sum_{i+j+n=k-1}c_{i,j,n}(\mu-\lambda_1-\lambda_2-\lambda_3-k+1)Z'(XY'-X'Y)f_1^{(i)}f_2^{(j)}f_3^{(n)},\end{aligned}$$

$$\begin{aligned}\Psi_1&=\mathcal{C}([X,Y],Z)=\sum_{i+j+n=k-1}c_{i,j,n}Z'(XY'-X'Y)f_1^{(i)}f_2^{(j)}f_3^{(n)},\\\Psi_2&=\mathcal{C}([X,Z],Y)=\sum_{i+j+n=k-1}c_{i,j,n}Y'(XZ'-X'Z)f_1^{(i)}f_2^{(j)}f_3^{(n)},\\\Psi_3&=\mathcal{C}([Y,Z],X)=\sum_{i+j+n=k-1}c_{i,j,n}X'(YZ'-Y'Z)f_1^{(i)}f_2^{(j)}f_3^{(n)}.\end{aligned}$$

Then, by considering the equation (3.13), we can write

$$L_X^{\lambda_1,\lambda_2,\lambda_3,\mu}\mathcal{C}(Y,Z)(f_1dx^{\lambda_1},f_2dx^{\lambda_2},f_3dx^{\lambda_3})\tag{3.14}$$

$$-\mathcal{C}([X,Y],Z)+\circlearrowleft(X,Y,Z)=\sum_{t=1}^{3}(\Phi_t-\Psi_t)=0.$$

We show that the equation (3.14) is satisfied only if $c'_{i,j,n}=0$ and $c_{i,j,n}(\mu-\lambda_1-\lambda_2-\lambda_3-k)=0$.

Any trivial 2-cocycle should be of the form

$$L_X^{\underline{\lambda},\mu}b(Y)(f_1dx^{\lambda_1},f_2dx^{\lambda_2},f_3dx^{\lambda_3})-L_Y^{\underline{\lambda},\mu}b(X)(f_1dx^{\lambda_1},f_2dx^{\lambda_2},f_3dx^{\lambda_3})$$
$$-b([X,Y])(f_1dx^{\lambda_1},f_2dx^{\lambda_2},f_3dx^{\lambda_3}).$$

By using Lemma 3.2, we have

$$\begin{aligned}(3.15)\qquad &\delta b(X,Y)(f_1,f_2,f_3)\\ &=L_X^{\underline{\lambda},\lambda_1+\lambda_2+\lambda_3+k}b(Y)(f_1dx^{\lambda_1},f_2dx^{\lambda_2},f_3dx^{\lambda_3})\\ &\quad-L_Y^{\underline{\lambda},\lambda_1+\lambda_2+\lambda_3+k}b(X)(f_1dx^{\lambda_1},f_2dx^{\lambda_2},f_3dx^{\lambda_3})\\ &\quad-b([X,Y])(f_1dx^{\lambda_1},f_2dx^{\lambda_2},f_3dx^{\lambda_3})\\ &=\sum_{i+j+n=k}c_{i,j,n}(\mu-\lambda_1-\lambda_2-\lambda_3-k)(X'Y-XY')f_1^{(i)}f_2^{(j)}f_3^{(n)}.\end{aligned}$$

4. The space $\mathrm{H}^2_{\mathrm{diff}}(\mathfrak{aff}(1),\mathcal{D}^n_{\underline{\lambda},\mu})$

The second main result of paper is the following.

Theorem 4.1. *The space* $\mathrm{H}^2_{\mathrm{diff}}(\mathfrak{aff}(1),\mathcal{D}^n_{\underline{\lambda},\mu})$ *has the following structure*:

$$(4.16)\qquad \mathrm{H}^2_{\mathrm{diff}}(\mathfrak{aff}(1),\mathcal{D}^n_{\underline{\lambda},\mu})\simeq\begin{cases}\mathbb{R}^{C^k_{n+k-1}} & \text{if } \mu=\lambda_1+\cdots+\lambda_n+k,\\ 0 & \text{otherwise.}\end{cases}$$

The following 2*-cocycles span the corresponding cohomology spaces*:

$$\mathcal{C}(X\frac{d}{dx},Y\frac{d}{dx})(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n})$$
$$=\sum_{i_1+\cdots+i_n=0}^{k-1}c_{i_1,\dots,i_n}(XY'-X'Y)f_1^{(i_1)}\cdots f_n^{(i_n)}dx^{\lambda_1+\cdots+\lambda_n+k},$$

where $c_{i_1,\dots,i_n}$ *are constants,* $f_idx^{\lambda_i}\in\mathcal{F}_{\lambda_i}$ *and* $X\frac{d}{dx}$, $Y\frac{d}{dx}\in\mathfrak{aff}(1)$.

We need the following lemma.

Lemma 4.2. *Let* $b\in C^1(\mathfrak{aff}(1),\mathcal{D}^n_{\underline{\lambda},\lambda_1+\cdots+\lambda_n+k})$ *be defined as follows*: *for* $X\frac{d}{dx}\in\mathfrak{aff}(1)$, $f_idx^{\lambda_i}\in\mathcal{F}_{\lambda_i}$.

$$\begin{aligned}(4.17)\qquad b(X)(f_1,f_2,\dots,f_n)=&\sum_{i_1+\cdots+i_n=k}c_{i_1,\dots,i_n}Xf_1^{(i_1)}\cdots f_n^{(i_n)}dx^{\mu}\\ &+\sum_{i_1+\cdots+i_n=k-1}\alpha_{i_1,\dots,i_n}X'f_1^{(i_1)}\cdots f_n^{(i_n)}dx^{\mu},\end{aligned}$$

where the coefficients $c_{i_1,\dots,i_n}$ and $\alpha_{i_1,\dots,i_n}$ are constants.
Then the map $\delta b : \mathfrak{aff}(1)^{\otimes 2} \to \mathcal{D}^n_{\underline{\lambda},\lambda_1+\cdots+\lambda_n+k}$ is given by

$$
\begin{aligned}
(4.18)\quad & \delta b(X,Y)(f_1,\dots,f_n) \\
&= L_X^{\lambda_1,\dots,\lambda_n,\lambda_1+\cdots+\lambda_n+k} b(Y)(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}) \\
&\quad - L_Y^{\lambda_1,\dots,\lambda_n,\lambda_1+\cdots+\lambda_n+k} b(X)(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}) \\
&\quad - b([X,Y])(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}) \\
&= \sum_{i_1+\cdots+i_n=k} c_{i_1,\dots,i_n}(\mu-\lambda_1-\cdots-\lambda_n-k)(X'Y-XY')f_1^{(i_1)}\cdots f_n^{(i_n)}.
\end{aligned}
$$

4.1. Proof of theorem

Any 2-cocycle $\mathcal{C} \in Z^1_{\mathrm{diff}}(\mathfrak{aff}(1), \mathcal{D}^n_{\underline{\lambda},\lambda_1+\cdots+\lambda_n+k})$ should be retain the following general form:

$$
\begin{aligned}
(4.19)\quad & \mathcal{C}(X,Y)(f_1,\dots,f_n) \\
&= \sum_{i_1+\cdots+i_n=k-1} c_{i_1,\dots,i_n}(XY'-X'Y)f_1^{(i_1)}\cdots f_n^{(i_n)}dx^{\mu},
\end{aligned}
$$

where $c_{i_1,\dots,i_n}$ are constants.

The 2-cocycle condition reads as follows: for all $f_idx^{\lambda_i} \in \mathcal{F}_{\lambda_i}$, and for all $X, Y \in \mathfrak{aff}(1)$, we have

$$L_X^{\lambda_1,\dots,\lambda_n;\mu}\mathcal{C}(Y,Z)(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}) - \mathcal{C}([X,Y],Z) + \circlearrowleft (X,Y,Z) = 0.$$

A direct computation, proves that the coefficient of the component $f_1^{(i_1)}\cdots f_n^{(i_n)}$ in the 2-cocycle condition above is equal to

$$\mu - \lambda_1 - \cdots - \lambda_n - k = 0. \tag{4.20}$$

Any trivial 2-cocycle should be of the form

$$
\begin{aligned}
& L_X^{\lambda_1,\dots,\lambda_n,\mu} b(Y)(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}) - L_Y^{\lambda_1,\dots,\lambda_n,\mu} b(X)(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}) \\
& - b([X,Y])(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}).
\end{aligned}
$$

By using Lemma 4.2, we have

$$
\begin{aligned}
(4.21)\quad & \delta b(X,Y)(f_1,\dots,f_n) \\
&= L_X^{\lambda_1,\dots,\lambda_n;\lambda_1+\cdots+\lambda_n+k} b(Y)(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}) \\
&\quad - L_Y^{\lambda_1,\dots\lambda_n;\lambda_1+\cdots+\lambda_n+k} b(X)(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}) \\
&\quad - b([X,Y])(f_1dx^{\lambda_1},\dots,f_ndx^{\lambda_n}) \\
&= \sum_{i_1+\cdots+i_n=k} c_{i_1,\dots,i_n}(\mu-\lambda_1-\cdots-\lambda_n-k)(X'Y-XY')f_1^{(i_1)}\cdots f_n^{(i_n)}.
\end{aligned}
$$

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