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THE FRACTIONAL WEAK DISCREPANCY OF $(M, 2)$-FREE POSETS

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Abstract. For a finite poset $P = (X, \preceq)$ the *fractional weak discrepancy* of P, denoted $wd_F(P)$, is the minimum value t for which there is a function $f : X \longrightarrow \mathbb{R}$ satisfying (1) $f(x) + 1 \leq f(y)$ whenever $x \prec y$ and (2) $|f(x) - f(y)| \leq t$ whenever $x \| y$. In this paper, we determine the range of the fractional weak discrepancy of $(M, 2)$-free posets for $M \geq 5$, which is a problem asked in [9]. More precisely, we showed that (1) the range of the fractional weak discrepancy of $(M, 2)$-free interval orders is $W = \{\frac{r}{r+1} : r \in \mathbb{N} \cup \{0\}\} \cup \{t \in \mathbb{Q} : 1 \leq t < M - 3\}$ and (2) the range of the fractional weak discrepancy of $(M, 2)$-free non-interval orders is $\{t \in \mathbb{Q} : 1 \leq t < M - 3\}$. The result is a generalization of a well-known result for semiorders and the main result for split semiorders of [9] since the family of semiorders is the family of $(4, 2)$-free posets.

1. Introduction

In this paper we consider only finite posets $P = (X, \preceq)$. Two elements x and y in P are *comparable* if either $x \preceq y$ or $y \preceq x$. Otherwise they are *incomparable*, denoted $x||y$. The notation $x \prec y$ means $x \preceq y$ and $x \neq y$. A subposet $P' = (X', \preceq)$ of P is a poset with X' as a subset of X and the inherited comparability from P within the elements of X'.

A discrepancy of a poset is a difference between incomparable elements in an order-preserving labelling on the poset. There have been a lot of researches done and going on variations of discrepancies with different constraints since their introductions ([3, 4, 10, 11]).

In this paper we focus on a particular kind of a discrepancy measuring "weakness" of posets.

Definition 1.1. The *weak discrepancy* of a poset $P = (X, \preceq)$, denoted $wd(P)$ is the smallest integer t such that there is a function $f : X \longrightarrow \mathbb{Z}$ satisfying

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(1) $f(x) < f(y)$ whenever $x \prec y$ and
(2) $|f(x) - f(y)| \leq t$ whenever $x \| y$.

Such a labelling is called a *t-weak labelling.* A t_0-weak labelling is an *optimal weak labelling* if $wd(P) = t_0$.

We refine the labelling by allowing any real values.

Definition 1.2. The *fractional weak discrepancy* of a poset $P = (X, \preceq)$, denoted $wd_F(P)$ is the smallest real number t such that there is a function $f : X \longrightarrow \mathbb{R}$ satisfying

(1) $f(x) + 1 \leq f(y)$ whenever $x \prec y$ and
(2) $|f(x) - f(y)| \leq t$ whenever $x \| y$.

Such a labelling is called a *fractional t-weak labelling.* A fractional t_0-weak labelling is an *optimal fractional weak labelling* if $wd_F(P) = t_0$.

It is known that fractional weak discrepancy is always a rational number. Moreover, $wd(P) = \lceil wd_F(P) \rceil$. In some sense, fractional weak discrepancy is a refinement of weak discrepancy.

A natural question regarding fractional weak discrepancy is to classify posets allowing large/small fractional weak discrepancy. In other words, it has been asked which structures could force certain value(s) of fractional weak discrepancy or vice versa. Some results related to these questions can be found in [1], [7], and [8].

It is trivial to see that $wd_F(P) \leq wd_F(P')$ if P is a subposet of P'. Therefore, containing a subposet having a large fractional weak discrepancy certainly forces a large fractional weak discrepancy. However, the converse is not obvious anymore.

A *total order* or a *chain* is a poset any two elements of which are comparable. A chain with n elements is denoted by $\mathbf{n}$. The *length* of $\mathbf{n}$ is $n-1$ and denoted $l(\mathbf{n})$. The fractional weak discrepancy of any chain is zero by the definition. More generally, a *weak order* can be described as a poset whose fractional weak discrepancy is zero. Traditionally a weak order is described in terms of forbidden subposets, which is free of $\mathbf{2}+\mathbf{1}$.

Some well-known families of posets have forbidden characterizations with disjoint union of only two chains, and in this regard they can be important families to study.

An *interval order* is a poset $P = (X, \preceq)$ with a corresponding (closed) interval assignment $[l(x), r(x)]$ for every element $x \in X$ such that $y \prec z$ if and only if $r(y) < l(z)$. A *semiorder* is an interval order with an interval representation in which every interval has the same length. (For this reason it is also called a *unit interval order.*) It is widely known that P is an interval order if and only if P is $\mathbf{2}+\mathbf{2}$-free. Also, Q is a semiorder if and only if Q is a $\mathbf{3}+\mathbf{1}$-free interval order. A poset is called *$(M,2)$-free* if $\{\mathbf{r}+\mathbf{s} \colon r+s = M, r, s \geq 1\}$ is the set of forbidden subposets. Hence, the semiorder is the same as the $(4,2)$-free order. A $(5,2)$-free order is called a *subsemiorder.*

There are a number of results about the range of fractional weak discrepancy of various families including the families mentioned above.

Theorem 1.1 ([5])**.** *A poset P is a semiorder if and only if $wd_F(P)$ is in* $\{\frac{r}{r+1} : \; r \geq 0, r \in \mathbb{Z}\}$.

Theorem 1.2 ([7])**.** *The range of wd_F for interval orders that are $\mathbf{n+1}$-free is $\{\frac{r}{s} : 0 \leq s-1 \leq r < (n-2)s\}$ for $n \geq 3$.*

For general posets, in fact Trenk [12] and Trenk et al. [8] showed that a value for wd_F requires containing a structure of $\mathbf{n+1}$ with n almost as big as wd_F. More precisely,

Theorem 1.3 ([8,12])**.** *Every poset P with $wd_F(P) > n-2$ contains an $\mathbf{n+1}$ as a subposet.*

In [9], the authors determine the range of fractional weak discrepancy for split semiorders. The family of split semiorders contains semiorders and is a subfamily of the family of $(5, 2)$-free posets. In their paper, they present an open question asking the range of $wd_F(P)$ for subsemiorders. Also, more generally they ask what the range of $wd_F(P)$ for $(M, 2)$-free posets is, $M \geq 5$.

2. Main results: The range of the fractional weak discrepancy of $(M, 2)$-free order

In this section, we determine all the possible values for the fractional weak discrepancy of $(M, 2)$-free posets, for each $M \geq 5$.

We use *forcing cycles* introduced in [2] and [3] as the main tool to calculate the values of $wd_F(P)$.

Definition 2.1 ([3])**.** A forcing cycle C of a poset $P = (X, \preceq)$ is a sequence $C : x_0, x_1, \ldots, x_m = x_0$ of $m \geq 2$ elements of X for which $x_i \prec x_{i+1}$ or $x_i \| x_{i+1}$ for each $i : 0 \leq i < m$. If C is a forcing cycle, we write $\text{up}(C) = |\{i : x_i \prec x_{i+1}\}|$ and $\text{side}(C) = |\{i : x_i \| x_{i+1}\}|$.

Forcing cycles are used to obtain a lower bound.

Theorem 2.1 ([5,6])**.** *Let $P = (X, \preceq)$ be a poset that is not a chain. Then $wd_F(P) = \max_C \frac{up(C)}{side(C)}$, where the maximum is taken over all forcing cycles C in P.* (*See* [6].)

In fact, if $C : x_0, x_1, \ldots, x_m = x_0$ is a forcing cycle of P and satisfying $t_0 = wd_F(P) = \frac{up(C)}{side(C)}$. Let $f : P \longrightarrow \mathbb{R}$ be an optimal fractional weak labeling of P. Then for each i,

1. *if $x_i \prec x_{i+1}$, then $f(x_{i+1}) = f(x_i) + 1$.*
2. *if $x_i \| x_{i+1}$, then $f(x_i) - f(x_{i+1}) = t_0$.* (*See* [5].)

A forcing cycle C_0 is called an *optimal forcing cycle* in P if $wd_F(P) = \frac{\text{up}(C_0)}{\text{side}(C_0)}$.

Theorem 2.2. *For every $(M, 2)$-free poset P and $M \geq 4$, $wd_F(P) < M - 3$.*

Proof. Note that $wd_F(P) \leq M-3$ by Theorem 1.3 since P has no $(\mathbf{M-1})+\mathbf{1}$.

Suppose that there is an $(M, 2)$-free poset P' with $wd_F(P') = M-3$. Among optimal forcing cycles we let C have the smallest side(C). In other words, if $r = \text{up}(C)$ and $q = \text{side}(C)$, then $\frac{r}{q} = M - 3$ with minimum possible value for q. Note that $q \geq 3$. Otherwise, C consists of two incomparable chains in P', C_1 and C_2 with $l(C_1) + l(C_2) = 2(M - 3)$. Hence, $C_1 + C_2$ becomes $\mathbf{m} + \mathbf{n}$, where $(m - 1) + (n - 1) = 2(M - 3)$. Then $m + n = M + M - 4 \geq M$. This contradicts to the $(M, 2)$-free condition for P'.

Claim 1. Let $C_1, C_2, \dots, C_q$ be the maximal chains of C in order, where C consists of $x_0, x_1, \dots, x_{m-1}, x_m = x_0$ as in the definition. For any i, $l(C_i) \neq M - 3$.

Proof. Suppose to the contrary that there exists C_i such that $l(C_i) = M - 3$. Let x be the highest (or last) element in C_{i-1} and let y be the lowest (or first) element in C_{i+1}.

If $x \| y$, then $C' = C - \{C_i\}$ forms a forcing cycle since $q \geq 3$. Now $\frac{\text{up}(C')}{\text{side}(C')} = \frac{q(M-3)-(M-3)}{q-1} = M - 3$ and therefore C' is optimal with side(C') smaller than q, which is a contradiction.

If $q = 3$ and $x \prec y$, then the lowest element in C_{i-1} is less than the highest element in C_{i+1}, which is a contradiction. Hence, $x \not\prec y$.

If $q \geq 4$ and $x \prec y$, then we reduce the number of cycles by combining C_{i-1} and C_{i+1} and by deleting C_i from C. This new forcing cycle, say C', has $\frac{\text{up}(C')}{\text{side}(C')} = \frac{q(M-3)-l(C_i)+1}{q-2} > \frac{\text{up}(C)}{\text{side}(C)}$, which is a contradiction.

Hence the only possibility is that $y \prec x$. If y is comparable to an element in C_i, say w, then by the definition of a forcing cycle $y \not\prec w$. Then $w \prec y$. We concatenate C_{i+1} right after w by combining the lower part of C_i to w and C_{i+1}. Let this chain be C'_{i+1}. We obtain a new forcing cycle consisting of $(C - \{C_i, C_{i+1}\}) \cup \{C'_{i+1}\}$. This has a larger fraction of "up"s and "side"s, which is a contradiction. Similarly, if $x \prec z$ for some z in C_i, then we combine the part of C_{i-1} to x and the part of C_i from z. The same contradiction occurs as before. Therefore, x and y are incomparable to every element in C_i. Since $C_i = (\mathbf{M-2})$ together with $y \prec x$ now we see that P allows $(\mathbf{M-2}) + \mathbf{2}$ and this is a contradiction. □

Now there exists a chain C_j such that $l(C_j) \leq M - 4$ and $l(C_{j-1}) > M - 4$. (Therefore, $l(C_{j-1}) \geq M - 2$.) Note first that $q \neq 2$. To see this, if $q = 2$, then $C_j \| C_{j-1}$ and $l(C_j) + l(C_{j-1}) = 2(M - 3) \geq M$, which is a contradiction.

Claim 2. Let y be the first element in C_{j+1}. Then $y \| C_j$, i.e., $y \| x'$ for all x' in C_j.

Proof. First of all, $y \not\prec x'$ for any $x' \in C_j$. Otherwise, $y \prec x' \preceq \overline{x}$, where $\overline{x}$ is the top element in C_j. This is a contradiction. Now suppose that there exists x'' in C_j such that $x'' \prec y$, where $\overline{x} \neq x''$ for the top element $\overline{x}$ in C_j. Then, now combine the part of C_j up to x'' and C_{j+1}. This forms a chain. Consider $\widetilde{C} = C - C'$, where C' is the part of C_j from the next element of x'' in C_j to $\overline{x}$ which forms a forcing cycle consisting of $q-1$ chains. Therefore $\text{side}(\widetilde{C}) = q-1$, $\text{up}(\widetilde{C}) = \text{up}(C) - l(C') = (M-3)q - l(C')$, and

$$\frac{\text{up}(\widetilde{C})}{\text{side}(\widetilde{C})} = \frac{(M-3)q - l(C')}{q-1} \geq \frac{(M-3)q-(M-5)}{q-1}$$
$$= \frac{(q-1)(M-3)+2}{q-1} > M-3,$$

which is a contradiction. □

Let x be the top element in C_{j-1}. Note first that $y \prec x$. Otherwise, we can delete C_j from the optimal forcing cycle and the remaining part also becomes a forcing cycle with either a smaller fraction or $M-3$ with smaller q, contradiction. Let z be the lowest element in C_{j-1} such that $y \prec z$. Let C'' be the part of C_{j-1} from z to x.

Case 1) If $l(C'') + l(C_j) \leq M-3$, then $C_{j-1} - C'' \neq \emptyset$. Now we delete C'' and C_j from C so that the remaining parts form a forcing cycle with either a larger fraction or $M-3$ with smaller q, and this is a contradiction.

Case 2) If $l(C'') + l(C_j) > M-3$, then we claim that $C'' \| C_j$. To see this, note first that any element in C'' cannot be less than any element in C_j. (Otherwise, suppose that $y' \prec y''$, where $y' \in C''$ and $y'' \in C_j$, then $y \prec y''$.) If there exist an element x' in C_j and z' in C'' such that $x' \prec z'$, then (*the bottom element in* C_j) $\prec x' \prec z' \prec x$, and this is contradiction.

Finally, C'' and C_j form $\mathbf{m}+\mathbf{k}$, where $m = l(C'')+1$ and $k = l(C_j)+1$. Therefore, $m+k = l(C'') + l(C_j) + 2 \geq M-2+2 = M$, and this is a contradiction. □

3. The range of the fractional weak discrepancy of $(M,2)$-free non-interval order

It is relatively easy to determine the range of the fractional weak discrepancy of $(M,2)$-free interval orders according to M. In this section we present results on the range of the fractional weak discrepancy of $(M,2)$-free non-interval orders as well.

Theorem 3.1. *For any $M \geq 4$, the range of the fractional weak discrepancy of $(M,2)$-free interval orders is $W = \{\frac{r}{r+1} : r \in \mathbb{N} \cup \{0\}\} \cup \{t \in \mathbb{Q} : 1 \leq t < M-3\}$.*

Proof. Note first that since an interval order is $\mathbf{2}+\mathbf{2}$-free, it does not contain any $\mathbf{a}+\mathbf{b}$ with $a, b \geq 2$. Also, no $(M,2)$-free poset contains an $(\mathbf{M}-\mathbf{1})+\mathbf{1}$. In other words, the intersection of the $(M,2)$-free posets and the interval orders is the set consisting of $(\mathbf{M}-\mathbf{1})+\mathbf{1}$-free interval orders.

Pick a value $t \in W$. By Theorem 1.2 for any nonnegative integers r and s with $0 \leq s-1 \leq r < s(M-3)$ there is a corresponding $(\mathbf{M-1})+\mathbf{1}$-free interval order $P_{(r,s)}$ such that $wd_F(P_{(r,s)}) = \frac{r}{s}$. We let $\frac{r'}{s'} = t$, where $\gcd(r', s') = 1$. Since $t \in W$, either $r' = s' - 1$ or $t \geq 1$. In other words, either $r' = s' - 1$ or $r' \geq s'$. Now, we have $r' = ts' < (M-3)s'$. Therefore, $wd_F(P_{(r',s')}) = t$. □

Now we figure out possible values for $(M, 2)$-free non-interval orders. Note first that a poset whose fractional weak discrepancy is less than 1 must be a semiorder which is always an interval order. Therefore every value of wd_F for a non-interval order must be a rational number at least 1. Moreover, every non-interval order contains $\mathbf{2+2}$, and $wd_F(\mathbf{2+2}) = 1$. The poset $\mathbf{2+2}$ is $(M, 2)$-free for $M \geq 5$. By Theorem 2.2 we see that the range of the fractional weak discrepancy of $(M, 2)$-free non-interval orders is a subset of $A = \{t \in \mathbb{Q} \colon 1 \leq t < M-3\}$.

When $M \geq 6$ for each value $t > 1$ in A we construct a corresponding $(M, 2)$-free non-interval order P whose fractional weak discrepancy is t. We use two copies of an interval order for the construction. This construction requires $M \geq 6$ to achieve all the conditions we need. For the remaining case, i.e., for $(5, 2)$-free non-interval orders, we give a different construction.

Construction 1: When $M \geq 6$ for any rational number s, $1 < s < M-3$, let P be an interval order that is guaranteed in Theorem 1.2 with $wd_F(P) = s$. Let $C_1, C_2, \ldots, C_q$ be the chains of the forcing cycle (in a circular order) from the construction in the proof of Theorem 1.2 ([7]). Now it is easy to see the following result.

Proposition 3.2. *Let P be an interval order with $wd_F(P) > 1$ which satisfies the description in Construction 1. Let $\mathcal{C}$ be an optimal forcing cycle with chains $C_1, C_2, \ldots, C_q$ whose union become the set of elements of P. Let f be an optimal weak labeling on P. If there is an element x in P with the minimum value* (*maximum value, resp.*) *of f, then x must be the first* (*last, resp.*) *element in C_i for some i and $l(C_i) \geq 2$.*

Proof. Let f be an optimal labeling assigned on an optimal forcing cycle in P. Let $L_1 = \min_{x \in P} f(x)$ and $L_2 = \max_{x \in P} f(x)$. By the second part of Theorem 2.1 ([5]) we see that f on this optimal forcing cycle is determined uniquely up to an initial value, say $f(x_0)$ for some x_0 in the forcing cycle.

Among C_is, let C_j be a chain whose top element has the maximum value of f. If C_j consists of only one element, say x, then $f(x) = f(z) - wd_F(P)$, where z is the top element of C_{j-1}. It means that $f(x) < f(z)$, which is a contradiction.

If C_j consists of only two elements, say $y \prec x$, then $f(x) = f(y) + 1 = f(z) - wd_F(P) + 1$ and $f(z) > f(x)$, which is a contradiction.

Similarly, for a chain containing an element with the minimum value of f, we consider the bottom element of the next chain of C. We apply the same argument. □

Let S_1 (S_2, resp.) be the set of elements in P with the maximum (minimum, resp.) value of f, which is defined in the proof of Proposition 3.2. For each element $x \in S_1$ (S_2, resp.), we consider the previous (next, resp.) element of x in its cycle, which is guaranteed in Proposition 3.2, say $\widehat{x}$. We let $\widehat{S_1}$ ($\widehat{S_2}$, resp.) be the set of such elements. Trivially, there is a natural one-to-one correspondence between S_i and $\widehat{S}_i$, $i = 1, 2$.

Let T_1 (T_2, resp.) be the set of elements x in P such that $L_2 - f(x) < 1$ ($f(x) - L_1 < 1$, resp.).

Construction 2: We construct a new poset P' in the following way:

(1) The elements of P' consists of $\{(x, i) \colon x \in P,\ i = 1, 2\}$.

(2) We assign comparability relation $\preceq_{P'}$ as follows:

$(x, i) \preceq_{P'} (x', i)$	if and only if $x \preceq x'$ in P.
$(x, 1) \preceq_{P'} (y, 2)$	if $x \notin S_1 \bigcup \widehat{S_1} \bigcup T_1$ or $y \notin S_2 \bigcup \widehat{S_2} \bigcup T_2$.
$(x, 1) \| (y, 2)$	if $x \in S_1 \bigcup \widehat{S_1}$ and $y \in S_2 \bigcup \widehat{S_2}$.
$(x, 1) \| (y, 2)$	if (1) $x \in S_1$ and $y \in T_2$ **or** (2) $x \in T_1$ and $y \in S_2$.
$(x, 1) \preceq_{P'} (y, 2)$	if (1) $x \in \widehat{S_1}$ and $y \in T_2$ **or** (2) $x \in T_1$ and $y \in \widehat{S_2}$.
$(x, 1) \| (y, 2)$	if $x \in T_1$ and $y \in T_2$.

Note that the comparability between (x, i) and (x', i) is inherited from P.

Lemma 3.3. *Let P be a poset in Proposition 3.2 and P' the poset constructed from P as in Construction 2. If $wd_F(P) \geq 2$, then $wd_F(P') = wd_F(P)$.*

Proof. Since P is a subposet of P', $wd_F(P) \leq wd_F(P')$.

We consider a labeling g on P' as follows:

- $g((x, 1)) = f(x)$ for all $x \in P$.
- $g((y, 2)) = f(y) + (L_2 - L_1)$.

Claim 1. g is a weak labeling of P'.

Proof. (1) Among any two elements (x, i) and (x', i), the effect of g is exactly the restriction on one copy of P, so g is obtained by shifting f.

(2) Consider $(x, 1)$ and $(y, 2)$, where $(x, 1) \prec (y, 2)$. We have two possibilities.

- case 1) $x \notin S_1 \bigcup \widehat{S_1} \bigcup T_1$ or $y \notin S_2 \bigcup \widehat{S_2} \bigcup T_2$: If $x \notin S_1 \bigcup \widehat{S_1} \bigcup T_1$, then $g((x, 1)) < L_2 - 1$ and $g((y, 2)) \geq L_1 + (L_2 - L_1) = L_2$. So, $g((y, 2)) \geq g((x, 1)) + 1$. We apply the same argument for the case $y \notin S_2 \bigcup \widehat{S_2} \bigcup T_2$.
- case 2) If $x \in \widehat{S_1}$ and $y \in T_2$, then $g((x, 1)) = L_2 - 1$ and $g((y, 2)) = L_2 + \epsilon$, where $0 < \epsilon < 1$. Therefore, $g((y, 2)) - g((x, 1)) \geq \epsilon + 1$. Similarly, if $x \in T_1$ and $y \in \widehat{S_2}$, then $g((x, 1)) = L_2 - \epsilon'$, where $0 < \epsilon' < 1$ and $g((y, 2)) = L_2 + 1$. Therefore, $g((y, 2)) - g((x, 1)) = 1 + \epsilon'$. □

There are four types of incomparable pairs.

- $(x, i) \| (x', i)$ and the maximum difference of their labels is $wd_F(P)$.

- $(x,1)\|(y,2)$ with $x \in S_1 \bigcup \widehat{S_1}$ and $y \in S_2 \bigcup \widehat{S_2}$: Note that $L_2 - 1 \leq g((x,1)) \leq L_2$ and $L_2 \leq g((y,2) \leq L_2 + 1$. $g((y,2)) - g((x,1)) \leq 2$.
- $(x,1)\|(y,2)$ with (1) $x \in S_1$ and $y \in T_2$ **or** (2) $x \in T_1$ and $y \in S_2$: $g((y,2)) - g((x,1)) < 1$.
- $(x,1)\|(y,2)$ with $x \in T_1$ and $y \in T_2$: $g((x,1)) = L_2 - \epsilon_1$ and $g((y,2)) = L_2 + \epsilon_2$ for some $0 < \epsilon_1, \epsilon_2 < 1$. Therefore, $g((y,2)) - g((x,1)) = \epsilon_1 + \epsilon_2 < 2$.

Therefore, if $wd_F(P) \geq 2$, then $wd_F(P') \leq wd_F(P)$. □

Lemma 3.4. *Let P be an $(M,2)$-free interval order. If P' is the poset obtained by Construction 2 using P, then P' contains $\mathbf{2}+\mathbf{2}$. Hence P' is not an interval order. Also, if $wd_F(P) \geq M-4$ and $M \geq 6$, then P' is an $(M,2)$-free poset.*

Proof. First, it is easy to see that P' contains $\mathbf{2}+\mathbf{2}$. We pick an element x in S_1 and its corresponding element $\widehat{x}$ in $\widehat{S_1}$. Similarly, pick an element y in S_2 and its corresponding element $\widehat{y}$ in $\widehat{S_2}$. Now $x, \widehat{x}, y$, and $\widehat{y}$ form $\mathbf{2}+\mathbf{2}$ in P'.

Next we will show that P' does not contain $\mathbf{2}+\mathbf{4}$. This fact implies that P' does not contain $\mathbf{r}+\mathbf{s}$ in P' where $r+s = M$, $r, s \geq 2, M \geq 6$. Suppose to the contrary that P' contains a $\mathbf{2}+\mathbf{4}$. Since the maximum length of a chain in $S_1 \bigcup \widehat{S_1} \bigcup S_2 \bigcup \widehat{S_2} \bigcup T_1 \bigcup T_2$ is 2, $\mathbf{4}$ must contain at least two elements not in $S_1 \bigcup \widehat{S_1} \bigcup S_2 \bigcup \widehat{S_2} \bigcup T_1 \bigcup T_2$. For $x, y \in \left(S_1 \bigcup \widehat{S_1} \bigcup S_2 \bigcup \widehat{S_2} \bigcup T_1 \bigcup T_2\right)^c$, if two of those elements are $(x,1)$ and $(y,2)$, then one of them must be comparable to an element in the $\mathbf{2}$, which is a contradiction. Without loss of generality let those two elements be $(x,1)$ and $(y,1)$. Then any element in the other chain $\mathbf{2}$ cannot be $(z,2)$. Therefore the elements in $\mathbf{2}$ must be $(z_1,1)$ and $(z_2,1)$. Then, $(x,1),(y,1),(z_1,1)$, and $(z_2,1)$ form a $\mathbf{2}+\mathbf{2}$ in P, which is a contradiction.

Now assume that P' contains an $(\mathbf{M}-\mathbf{1})+\mathbf{1}$. If the element a in $\mathbf{1}$ is not in $S_1 \bigcup \widehat{S_1} \bigcup S_2 \bigcup \widehat{S_2} \bigcup T_1 \bigcup T_2$, say (x,i), then every element in $(\mathbf{M}-\mathbf{1})$ must be (y,i) form. This is a contradiction since a $(\mathbf{M}-\mathbf{1})+\mathbf{1}$ is contained in P. Hence, the element in $\mathbf{1}$ is in $S_1 \bigcup \widehat{S_1} \bigcup S_2 \bigcup \widehat{S_2} \bigcup T_1 \bigcup T_2$.

There are two cases for $a = (x,i)$.

Case 1: (x,i) with $x \in S_i \bigcup \widehat{S}_i,\ i = 1,2$.

Without loss of generality we assume that $a = (x,1)$. Every element in $(\mathbf{M}-\mathbf{1})$ is either $(y,2)$ with $y \in S_2 \bigcup \widehat{S_2}$ or $(z,1)$. Since there are at most two elements with $(y,2)$ form, at least $M-3$ elements are of the form $(z,1)$. Let $(z_1,i) \prec (z_2,i) \prec \cdots \prec (z_k,i)$ be the elements with $i=1$ in $(\mathbf{M}-\mathbf{1})$. Hence, $k \geq M-3$. Note that $g((z_k,1)) \leq L_2$ and $g((z_j,1)) \geq g((z_{j-1},1)) + 1$ for $j = 2,3,\ldots,k$. Since $M \geq 6$, k is at least 3.

- If $a = (x,1)$ where x is in S_1, then we consider $(\widehat{x},1)$. $g((\widehat{x},1)) = g((x,1)) - 1 = L_2 - 1$. If we consider $(z_{k-2},1)$ and $(z_{k-1},1)$, then we know that $g((z_{k-2},1)) \leq L_2 - 2$ and $g((z_{k-1},1)) \leq L_2 - 1$. In this consequence it is not possible that $(\widehat{x},1) \prec (z_{k-2},1)$ or $(\widehat{x},1) \prec (z_{k-1},1)$. Moreover, if $(z_{k-2},1) \prec (\widehat{x},1)$ or $(z_{k-1},1) \prec (\widehat{x},1)$, then it is implied that either $(z_{k-2}, 1) \prec a$ or $(z_{k-1}, 1) \prec a$, contradiction. Now we have that $(x, 1), (\widehat{x}, 1), (z_{k-1}, 1)$ and $(z_{k-2}, 1)$ that are forming a $\mathbf{2} + \mathbf{2}$ (as in the structure of P), which is a contradiction.

- If $a = (x, 1)$ where x is in $\widehat{S_1}$, then let $(x', 1)$ be the element right below from $(x, 1)$ guaranteed by Proposition 3.2. Then $g((x, 1)) = L_2 - 1$ and $g((x', 1)) = L_2 - 2$.
  - If $k \geq M - 2$, then we repeat the same argument as above considering $(z_{k-2}, 1)$ and $(z_{k-3}, 1)$ that is forcing a $\mathbf{2} + \mathbf{2}$ together with $(x, 1)$ and $(x', 1)$.
  - If $k = M - 3$, then $(\mathbf{M} - \mathbf{1})$ contains $(y_1, 2) \prec (y_2, 2)$, where $y_1 \in \widehat{S_2}$ and $y_2 \in S_2$. This implies that $z_k \notin S_1 \bigcup \widehat{S_1}$. Therefore $g((z_k, 1)) \leq L_2 - 1$. Now we consider $(z_{k-1}, 1), (z_{k-2}, 1), (x, 1)$, and $(x', 1)$. By assumption we have that $(x, 1) \| (z_{k-1}, 1), (z_{k-2}, 1)$. Also, $g((z_{k-1}, 1)) \leq L_2 - 2 = g((x', 1))$ and $g((z_{k-2}, 1)) \leq L_2 - 3 = g((x', 1)) - 1$. Therefore, the only possibility for those four elements is to form a $\mathbf{2} + \mathbf{2}$, which is a contradiction.

Case 2: $x \in T_i,\ i = 1, 2$.

Without loss of generality we assume that $x \in T_1$ and let $a = (x, 1)$. It is obvious to see that $(x, 1)$ is comparable to $(y, 2)$ with $y \in \widehat{S_2}$. We will use the same notation as the case $k = M - 3$ in Case 1 above. Let $g((x, 1)) = L_2 - \epsilon$ with $0 < \epsilon < 1$. Since $z_k \notin S_1 \bigcup T_1,\ g((z_k, 1)) \leq L_2 - 1$. Therefore $g((z_1, 1)) \leq L_2 - 1 - (M - 3) = L_2 - 1 - M + 3 = L_2 - M + 2$. Now we have that $wd_F(P') \geq g((x, 1)) - g((z_1, 1)) \geq L_2 - \epsilon - (L_2 - M + 2) = L_2 - \epsilon - L_2 + M - 2 = M - 3 + (1 - \epsilon) > M - 3$, which is a contradiction. □

Construction 3: When $M = 5$ for any rational number t, $1 < t < 2$, we let $t = \frac{r}{q}$, where $r, q \in \mathbb{N}$. For simplicity we assume that r and q are relatively prime. We consider $(r - q)\mathbf{3} + (2q - r)\mathbf{2}$. Note that $r - q > 0$ and $2q - r > 0$ since $1 < \frac{r}{q} < 2$. We label these $r + q$ elements in the following way.

[Labelings of elements]

- The j-th element from the bottom in the i-th chain of length 2: x_i^j, $i = 1, 2, \ldots, r - q$ and $j = 1, 2, 3$.
- The j-th element from the bottom in the i-th chain of length 1: y_i^j, $i = 1, 2, \ldots, 2q - r$ and $j = 1, 2$.

Now we give a labeling f as follows.

$$f(x_1^1) = 0,$$

$$f(x_i^j) = 2(i - 1) - (i - 1)\frac{r}{q} + (j - 1),$$

$$f(y_i^j) = (r - q - 1)(2 - \frac{r}{q}) + 2 - \frac{r}{q} + (i - 1) - (i - 1)\frac{r}{q} + (j - 1).$$

[Comparability]

Now we add more comparability relations to $(r - q)\mathbf{3} + (2q - r)\mathbf{2}$.

(1) For every element is comparable to itself.
(2) $x_i^3 \| x_{i+1}^1$ for $i = 1, 2, \ldots, r-q-1$.
(3) $y_i^2 \| y_{i+1}^1$ for $i = 1, 2, \ldots, 2q-r-1$.
(4) $x_{r-q}^3 \| y_1^1$ and $y_{2q-r}^2 \| x_1^1$.
(5) For the rest, $x \prec y$ if $f(y) \geq f(x) + 1$.
(6) For the rest with $\neq y$, $x \| y$ if $|f(x) - f(y)| < 1$.

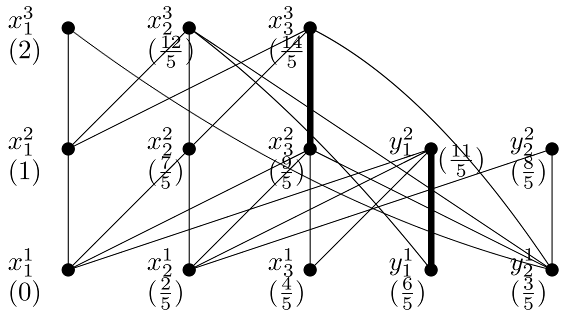

FIGURE 1. A $(5,2)$-free poset P with $wd_F(P) = \frac{8}{5}$. The elements of P form a forcing cycle with $r = 8$ and $q = 5$. The values of f is written in the parenthesis. P contains $\mathbf{2}+\mathbf{2}$ as a subposet but P has no induced $\mathbf{4}+\mathbf{1}$ or $\mathbf{3}+\mathbf{2}$.

Theorem 3.5. *The poset P from Construction 3 is a $(5,2)$-free non-interval order. Moreover, for each given r and q, the fractional weak discrepancy is $\frac{r}{q}$.*

Proof. First we make sure that additionally added comparability with transitivity does not spoil the incomparability in (1), (2), and (3). Let x and y be incomparable and consecutive elements in the above labelling. In fact, if two comparability relations were added through (4) such that there exists z with $f(y) \geq f(z) + 1$ and $f(z) \geq f(x) + 1$, then $2 > \frac{r}{q} = f(y) - f(x) = f(y) - f(z) - (f(z) - f(x)) \geq 2$, which is a contradiction.

It is trivial to see that $x_1^1, x_1^2, x_1^3, x_2^1, \ldots, x_{r-q}^3, y_1^1, y_1^2, \ldots, y_{2q-r}^2, x_1^1$ form a forcing cycle in P. Therefore, $wd_F(P) \geq \frac{r}{q}$.

Note that f satisfies the conditions in Theorem 2.1. In particular, $f(x_2^1) - f(x_1^3) = \frac{r}{q}$ and $x_2^1 \| x_1^3$, and this is a maximum possible gap. Therefore $wd_F(P) = \frac{r}{q}$.

Now we consider $x_{r-q}^2, x_{r-q}^3, y_1^1, y_1^2$. It is easy to see that they form an induced $\mathbf{2}$+$\mathbf{2}$ in P. By construction $x_{r-q}^3 \| y_1^1$. Now note that $x_{r-q}^3 \| y_1^2$ in P since the fact that $1 < f(x_{r-q}^3) - f(y_1^1) = f(x_{r-q}^3) - (f(y_1^2) - 1) < 0$ implies that $0 < f(x_{r-q}^3) - f(y_1^2) < 1$. Similarly, $|f(x_{r-q}^2) - f(y_1^1)| < 1$ and $|f(x_{r-q}^2) - f(y_1^2)| < 1$. P is not an interval order then.

Claim 1. P does not contain $\mathbf{4} + \mathbf{1}$ as a subposet.

Proof. Suppose to the contrary that there is an induced $\mathbf{4}+\mathbf{1}$ in P. Let $a \prec b \prec c \prec d$ be the elements of $\mathbf{4}$ and let e be the element of $\mathbf{1}$. From the construction rules, $f(d) - f(a) \geq 3$. If $f(e) = f(d) + \frac{r}{q}$, then $f(e) - f(a) > \frac{r}{q}$ and this is a contradiction. If $f(e) = f(d) - \frac{r}{q}$, then $f(e) - f(a) \geq 3 - \frac{r}{q} > 1$. Then by the construction a must have been comparable to c. Therefore, $|f(d) - f(e)| < 1$. Then $|f(e) - f(a)| > 2$ so that $a \prec e$, which is a contradiction. □

Claim 2. P does not contain $\mathbf{3} + \mathbf{2}$ as a subposet.

Proof. Suppose to the contrary that there is an induced $\mathbf{3} + \mathbf{2}$ in P. Let $a \prec b \prec c$ be the elements of $\mathbf{3}$ and let $d \prec e$ be the elements of $\mathbf{2}$.

If $f(c) - f(d) = \frac{r}{q}$, then $f(e) - f(a) \geq 3 - \frac{r}{q} > 1$. We get a contradiction.

If $f(e) - f(a) = \frac{r}{q}$, then $f(c) - f(d) \geq 3 - \frac{r}{q} > 1$. Now we get a contradiction.

If either $f(a) - f(e) = \frac{r}{q}$ or $f(d) - f(c) = \frac{r}{q}$, then we have a similar contradiction as above.

Then the only possibility is that $|f(c) - f(d)| < 1$. Since $f(c) - f(a) \geq 2$, $f(d) - f(a) > 1$, which is a contradiction. □
□

Theorem 3.6. *For $M \geq 5$ and for every rational number $1 \leq s < M - 3$, there is an $(M, 2)$-free poset which is not an interval order with fractional weak discrepancy s.*

Proof. For $M = 5$, we apply Theorem 3.5.

Let $\mathcal{P}_M$ be the family of $(M, 2)$-free posets. When $M \geq 6$ by Theorems 2.2 and 3.1, for every rational number $2 \leq M - 4 \leq t < M - 3$, there is an interval order P with $wd_F(P) = t$. Now the poset P' constructed as above satisfies the conclusions in Lemmas 3.3 and 3.4, and P' is a poset we want. □

4. Conclusion

In this section we interpret our main results as a generalization of the result on the range of the fractional weak discrepancy for semiorders. We proved that for $M \geq 5$ every $(M, 2)$-free order has fractional weak discrepancy less than $M - 3$. The class of semiorder is the same as the class of $(4, 2)$-free order so the fractional weak discrepancy must be strictly less than 1, and this result is consistent with our generalization.

For every $M \geq 4$, the family of $(M, 2)$-free poset is a subfamily of the family of $(M+1, 2)$-free poset. Let $\mathcal{P}_M$ be the family of $(M, 2)$-free poset. We consider $\mathcal{Q}_M = \mathcal{P}_{M+1} - \mathcal{P}_M$. Every poset P in $\mathcal{Q}_M$ has $M - 3 \leq wd_F(P) < M - 4$. Also, when $M \geq 5$ for every rational number t with $M - 3 \leq t < M - 4$ in $\mathcal{Q}_M$ there are an interval order P and a non-interval order P' such that $wd_F(P) = wd_F(P') = t$.

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