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Some counterexamples of a skew-normal distribution

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Abstract

Counterexamples of a skew-normal distribution are developed to improve our understanding of this distribution. Two examples on bivariate non-skew-normal distribution owning marginal skew-normal distributions are first provided. Sum of dependent skew-normal and normal variables does not follow a skew-normal distribution. Continuous bivariate density with discontinuous marginal density also exists in skew-normal distribution. An example presents that the range of possible correlations for bivariate skew-normal distribution is constrained in a relatively small set. For unified skew-normal variables, an example about converging in law are discussed. Convergence in distribution is involved in two separate examples for skew-normal variables. The point estimation problem, which is not a counterexample, is provided because of its importance in understanding the skew-normal distribution. These materials are useful for undergraduate and/or graduate teaching courses.

Keywords: skew-normal, bivariate distribution, independence, quadratic form

1. Introduction

Counterexamples are crucial to understand the main ideas of probability and statistics. Romano and Siegel (1986) and Stoyanov (2013) neatly introduced many counterexamples in this direction. A skew-normal distribution is well summarized by Azzalini and Capitanio (2014, p.24). The probability density function (pdf) of a random variable $Z \sim SN(0, 1, \alpha)$ is given by

$$f_Z(z; \alpha) = 2\varphi(z)\Phi(\alpha z), \quad \alpha \in \mathbb{R},\ z \in \mathbb{R}, \tag{1.1}$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cumulative distribution function of the standard normal distribution, respectively. This can be further extended to include the location and scale parameters:

$$Y = \mu + \sigma Z, \quad \mu \in \mathbb{R},\ \sigma \in \mathbb{R}^+,$$

and thus we write $Y \sim SN(\mu, \sigma^2, \alpha)$. A multivariate skew-normal pdf (Azzalini and Capitanio, 2014, p.124) is given by

$$f_{\mathbf{Z}}(\mathbf{z}; \boldsymbol{\alpha}) = 2\varphi_n(\mathbf{z}; \boldsymbol{\mu}, \Omega)\Phi\left(\boldsymbol{\alpha}^\top \boldsymbol{\omega}^{-1}(\mathbf{z} - \boldsymbol{\mu})\right), \quad \boldsymbol{\alpha} \in \mathbb{R}^n,\ \mathbf{z} \in \mathbb{R}^n, \tag{1.2}$$

where $\varphi_n(\mathbf{z}; \boldsymbol{\mu}, \Omega)$ is the n-dimensional normal pdf with mean $\boldsymbol{\mu}$ and covariance matrix Ω. Remark that $\boldsymbol{\omega}$ can be written as $\boldsymbol{\omega} = (\Omega \odot \mathbf{I}_n)^{1/2}$, where $\odot$ denotes the entry-wise or Hadamard product. When $\mathbf{Z}$ has the pdf (1.2), we write $\mathbf{Z} \sim SN_n(\boldsymbol{\mu}, \Omega, \boldsymbol{\alpha})$. When $\boldsymbol{\alpha} = \mathbf{0}$, (1.2) reduces to $N_n(\boldsymbol{\mu}, \Omega)$ pdf; hence, the parameter $\boldsymbol{\alpha}$ is referred to as a "skewness (slant) parameter."

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This paper provides some counterexamples related to the skew-normal distribution to enhance our understanding of this distribution. As far as we are concerned, only one paper (Lin and Stoyanov, 2009) has appeared in this direction. Our contribution to this field is useful for theoretical and applied statisticians. The examples suggested are also good teaching materials for undergraduate and/or graduate students.

2. Some counterexamples

We provide some counterexamples related to skew-normal distribution.

Example 1. A bivariate distribution that is not bivariate skew-normal, but has skew-normal marginal distributions.

The marginal distributions are also skew-normal (Azzalini and Capitanio, 2014, p.133) if the joint distribution of (X, Y) is a bivariate skew-normal distribution; however, the converse is false. We provide two examples here.

Let a bivariate density function $g(x, y)$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$, be the product of two independent density functions, $g_1(x)$ and $g_2(y)$, with corresponding medians a and b, respectively. Now let (X, Y) have a joint density given by

$$f(x,y) = \begin{cases} 2g(x,y), & \text{if } \{x > a \text{ and } y > b\} \text{ or } \{x \le a \text{ and } y \le b\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the marginal density functions of X and Y are $g_1(x)$ and $g_2(y)$, respectively. Note that $g_1(x)$ and $g_2(y)$ can be any density function, for continuous cases, normal, skew-normal, t, and skew-t. Even discrete densities are possible for $g_1(x)$ and $g_2(y)$.

For example, let $g(x, y)$ be a product of two independent skew-normal density functions of the distributions $SN(0, 1, \alpha_1)$ and $SN(0, 1, \alpha_2)$, whose medians are a and b, respectively. Then $f(x, y)$ is nonnegative and the integration of $\mathbb{R}^2$ is one. In addition, $f(x, y)$ is a density function, but not of the bivariate skew-normal distribution. Furthermore, the marginal distributions of X and Y are $SN(0, 1, \alpha_1)$ and $SN(0, 1, \alpha_2)$ by simple integration, respectively.

Another example is as follows. Let $(X, Y) \in \mathbb{R}^2$ with the joint density $f(x, y)$ as

$$\frac{2}{\pi} \exp\left\{-\frac{1}{2}\left(x^2 + y^2\right)\right\} \Phi(\alpha x)\Phi(\alpha y) + \frac{1}{\pi e} x^3 y^3 \Phi(\alpha)\Phi(-\alpha) I_{[-1,1]}(x) I_{[-1,1]}(y),$$

where $\alpha \in \mathbb{R}$. Integration over $\mathbb{R}^2$ is one and the minimum values are $f(-1, 1) = f(1, -1) = (1/(\pi e))\Phi(\alpha)\Phi(-\alpha) \to 0$ as $\alpha \to \infty$ or $\alpha \to -\infty$; thus, $f(x, y)$ is nonnegative. Therefore, $f(x, y)$ is a density function. The marginal density of X is

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{2}{\pi} \exp\left\{-\frac{1}{2}\left(x^2 + y^2\right)\right\} \Phi(\alpha x)\Phi(\alpha y) dy + \int_{-1}^{1} \frac{1}{\pi e} x^3 y^3 \Phi(\alpha)\Phi(-\alpha) I_{[-1,1]}(x) dy \\ &= 2\varphi(x)\Phi(\alpha x), \quad \alpha \in \mathbb{R},\ x \in \mathbb{R}, \end{aligned}$$

which is a density function of $SN(0, 1, \alpha)$. Similarly, $Y \sim SN(0, 1, \alpha)$.

Example 2. Sum of dependent normal and skew-normal random variables is not distributed as skew-normal.

It is well known that the sum of independent skew-normal and normal random variables follows a skew-normal distribution (Azzalini and Capitanio, 2014, Proposition 2.3). However, Proposition 2.3 is not working if we replace independence with dependence.

Let X follow $SN(0, 1, \alpha)$. Observe X; then, toss a fair coin and define Y as:

$$Y = \begin{cases} X, & \text{if the toss is "heads"}, \\ -X, & \text{if the toss is "tails"}. \end{cases}$$

This can be defined more rigorously as follows.

$$(Y|Z = 1) \stackrel{d}{=} X \quad \text{and} \quad (Y|Z = 0) \stackrel{d}{=} -X,$$

where $X \sim SN(0, 1, \alpha)$, $Z \sim \text{Ber}(1/2)$ and X and Z are independent. If the toss is "heads," $Y \sim SN(0, 1, \alpha)$, and if the toss is "tails," $Y \sim SN(0, 1, -\alpha)$. Therefore, the unconditional distribution of Y is given by the standard normal distribution as:

$$\begin{aligned} f_Y(y) &= f(y|z = 1)P(Z = 1) + f(y|z = 0)P(Z = 0) \\ &= \varphi(y)\left[\Phi(\alpha y) + \Phi(-\alpha y)\right] = \varphi(y). \end{aligned}$$

However, the sum $X + Y$ has a positive probability of $1/2$ at zero when the toss is tails. However, $X + Y$ is not degenerate when the toss is heads. Such a mixture of a discrete and a continuous distribution cannot be a skew-normal distribution.

Example 3. Continuous bivariate density with discontinuous marginal density.

Let X have an exponential distribution with mean one. Conditional on X, let Y have a skew-normal distribution with location $1/X$, scale 1, and skewness (slant) parameter α, that is, $Y|X \sim SN(1/X, 1, \alpha)$. Then, (X, Y) has a bivariate density function:

$$f(x, y) = \begin{cases} \dfrac{2}{\sqrt{2\pi}} \exp\left\{-x - \dfrac{1}{2}\left(y - \dfrac{1}{x}\right)^2\right\} \Phi(\alpha y), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $(x_n, y_n) \to (x, y)$. If $x \neq 0$, it is clear that $f(x_n, y_n) \to f(x, y)$. If $x = 0$, then $f(x_n, y_n) \to f(x, y) = 0$ regardless of how many x_n values are positive. Therefore, $f(x, y)$ is continuous everywhere in the plane. However, if $x \neq 0$,

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left\{-x - \frac{1}{2}\left(y - \frac{1}{x}\right)^2\right\} \Phi(\alpha y) dy \\ &= e^{-x} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(y - \frac{1}{x}\right)^2\right\} \Phi(\alpha y) dy \\ &= e^{-x} \end{aligned}$$

and $f(x) = 0$ if $x = 0$. That is, the marginal density $f(x) = e^{-x} I(x > 0)$ has a jump at zero. Hence, the marginal density $f(x)$ is not a continuous function.

For the conditional distribution of Y given X, any continuous distribution is possible with location $1/X$ and scale $c > 0$. For example, an elliptical distribution (Fang *et al.*, 1990) is possible. Furthermore, a skew-elliptical distribution (Azzalini and Capitanio, 2014) is also possible. A skew-normal distribution is a special case of a skew-elliptical distribution.

Example 4. A family of bivariate distributions such that the range of possible correlations is a small subset of $[-1, 1]$.

Suppose that X and Y are bivariate skew-normal variables, namely $(X, Y) \sim SN_2(\boldsymbol{\mu}, \boldsymbol{\Omega}, \boldsymbol{\alpha})$ with

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \quad \boldsymbol{\Omega} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \alpha_X \\ \alpha_Y \end{pmatrix}.$$

Define

$$W = \exp(X) \quad \text{and} \quad Z = \exp(Y).$$

This produces a set (W, Z) of bivariate log-skew-normal random variables, where

$$X \sim SN\left(\mu_X, \sigma_X^2, \alpha_{1(2)}\right), \quad Y \sim SN\left(\mu_Y, \sigma_Y^2, \alpha_{2(2)}\right),$$

with

$$\begin{aligned} \alpha_{1(2)} &= \left(1 + \alpha_Y^2\left(1 - \rho^2\right)\right)^{-\frac{1}{2}} (\alpha_X + \rho\alpha_Y), \\ \alpha_{2(2)} &= \left(1 + \alpha_X^2\left(1 - \rho^2\right)\right)^{-\frac{1}{2}} (\alpha_Y + \rho\alpha_X). \end{aligned}$$

Using the moment generating functions (mgfs) of X and Y, we obtain the following moments:

$$\begin{aligned} E(W) &= E\left(e^X\right) = E\left(e^{Xt}|_{t=1}\right) = 2\exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right)\Phi(\delta_X\sigma_X), \\ \mathrm{Var}(W) &= E\left(W^2\right) - (E(W))^2 \\ &= 2\exp\left(2\mu_X + \sigma_X^2\right)\left\{e^{\sigma_X^2}\Phi(2\delta_X\sigma_X) - 2\Phi^2(\delta_X\sigma_X)\right\}, \\ E(Z) &= 2\exp\left(\mu_Y + \frac{1}{2}\sigma_Y^2\right)\Phi(\delta_Y\sigma_Y), \\ \mathrm{Var}(Z) &= 2\exp\left(2\mu_Y + \sigma_Y^2\right)\left\{e^{\sigma_Y^2}\Phi(2\delta_Y\sigma_Y) - 2\Phi^2(\delta_Y\sigma_Y)\right\}, \end{aligned}$$

where

$$\delta_X = \frac{\alpha_{1(2)}}{\sqrt{1 + \alpha_{1(2)}^2}} \quad \text{and} \quad \delta_Y = \frac{\alpha_{2(2)}}{\sqrt{1 + \alpha_{2(2)}^2}}.$$

The expectation of WZ becomes

$$E(WZ) = 2\exp\left(\frac{\sigma_X^2}{2} + \frac{\sigma_Y^2}{2} + \mu_X + \mu_Y + \rho\sigma_X\sigma_Y\right) \times \Phi\left(\frac{\frac{\alpha_X}{\sigma_X}(\mu_X^* - \mu_X) + \frac{\alpha_Y}{\sigma_Y}(\mu_Y^* - \mu_Y)}{\sqrt{1 + \mathbf{h}^\top\boldsymbol{\Sigma}_*\mathbf{h}}}\right)$$

by using the perfect square, a transformation, and Lemma 5.2 (Azzalini and Capitanio, 2014), where

$$\mu_X^* = \mu_X + \rho\sigma_X\sigma_Y + \sigma_X^2, \quad \mu_Y^* = \mu_Y + \rho\sigma_X\sigma_Y + \sigma_Y^2,$$
$$\mathbf{h} = \begin{pmatrix} \frac{\alpha_X}{\sigma_X} \\ \frac{\alpha_Y}{\sigma_Y} \end{pmatrix} \quad \text{and} \quad \mathbf{\Sigma}_* = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}.$$

Therefore, the correlation between W and Z is

$$\operatorname{Corr}(W, Z) = \frac{E(WZ) - E(W)E(Z)}{\sqrt{\operatorname{Var}(W)\operatorname{Var}(Z)}}$$
$$= \frac{e^{\rho\sigma_X\sigma_Y}\Phi\left(\frac{\frac{\alpha_X}{\sigma_X}(\mu_X^*-\mu_X)+\frac{\alpha_Y}{\sigma_Y}(\mu_Y^*-\mu_Y)}{\sqrt{1+\mathbf{h}^\top\mathbf{\Sigma}_*\mathbf{h}}}\right) - 2\Phi(\delta_X\sigma_X)\Phi(\delta_Y\sigma_Y)}{\left[e^{\sigma_X^2}\Phi(2\delta_X\sigma_X) - 2\Phi^2(\delta_X\sigma_X)\right]^{\frac{1}{2}}\left[e^{\sigma_Y^2}\Phi(2\delta_Y\sigma_Y) - 2\Phi^2(\delta_Y\sigma_Y)\right]^{\frac{1}{2}}}.$$

When $\alpha_1 = \alpha_2 \in [1, 50]$ or $\alpha_1 = -\alpha_2 \in [-50, -2]$, the correlation between W and Z is constrained in $[0.0156, -0.0003]$ or $[0.0137, -0.0009]$, respectively. In spite of a near-zero correlation, W and Z are perfectly functionally (but nonlinearly) related.

Example 5. Two independent sequences of random variables, each converging in law to a limit, such that the sequence of term-by-term sums does not converge in law to the sum of the limits.

Consider two independent sequences of random variables

$$X_1, X_2, \ldots \sim \mathrm{SUN}_{1,1}(0, 1, \rho, \tau, 1) \quad \text{and} \quad Y_1, Y_2, \ldots \sim \mathrm{SUN}_{1,1}(0, 1, -\rho, \tau, 1).$$

For a unified skew-normal (SUN) distribution (Azzalini and Capitanio, 2014). Consider two other random variables: Let $X \sim \mathrm{SUN}_{1,1}(0, 1, \rho, \tau, 1)$ and then define $Y = -X$ so that $Y \sim \mathrm{SUN}_{1,1}(0, 1, -\rho, \tau, 1)$ but is not independent of X.

$$X_n \xrightarrow{d} X \quad \text{and} \quad Y_n \xrightarrow{d} Y.$$

However,

$$X_n + Y_n \sim \mathrm{SUN}_{1,2}\left(0, 2, \begin{pmatrix} \frac{1}{\sqrt{2}}\rho \\ -\frac{1}{\sqrt{2}}\rho \end{pmatrix}, \begin{pmatrix} \tau \\ \tau \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

but $X + Y = 0$. Hence, $X_n + Y_n$ does not converge in law to $X + Y$.

To formulate this type of example, it is sufficient that the distributions of $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ have the convolution property, which indicates that the sums of the independent random variables having this particular distribution come from the same distribution family. For example, if $X_i \sim \chi^2(n_i)$ for $i = 1, 2, \ldots, n$ and they are independent, then $\sum_{i=1}^n X_i \sim \chi^2(\sum_{i=1}^n n_i)$.

Example 6. A sequence of absolutely continuous distributions with support equal to the entire plane that converges to a limit in a law degenerate at the origin.

Let X_n and Y_n be independent skew-normal random variables with location zero, scale $1/n$, and slant parameter α/n, namely $SN(0, 1/n, \alpha/n)$ where $\alpha \in \mathbb{R}$. Then, the joint characteristic function (Kim and Genton, 2011) of (X_n, Y_n) is given by

$$\phi(X_n, Y_n)(s,t) = \exp\left(-\frac{s^2+t^2}{2n}\right)\left\{1 + i\mathcal{F}\left(\delta\frac{s}{\sqrt{n}}\right)\right\}\left\{1 + i\mathcal{F}\left(\delta\frac{t}{\sqrt{n}}\right)\right\},$$

where

$$\mathcal{F}(x) = b\int_0^x \exp\left(\frac{u^2}{2}\right)du, \quad b = \sqrt{\frac{2}{\pi}} \quad \text{and} \quad \delta = \frac{\alpha}{n^2+\alpha^2},$$

for all s and t. $\phi_{X_n,Y_n}(s,t)$ converges to one as $n \to \infty$. Hence, (X_n, Y_n) converges to a limit law degenerate at the origin $(0,0)$.

Example 7. A sequence of dependent bivariate random variables that converges in distribution to an independent bivariate random variable.

In general, if X_n and Y_n are independent and (X_n, Y_n) converges in distribution to (X, Y), then (X, Y) are also independent. This property need not hold if we replace independent by dependent. Let

$$(X_n, Y_n)^\top \sim SN_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1/n \\ 1/n & 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}\right).$$

Then, X_n and Y_n are dependent since $1/n$ is not equal to zero. The necessary and sufficient conditions of independence in the bivariate skew-normal distribution are as:

If $\mathbf{Y} \sim SN_2(\mathbf{0}, \Omega, \boldsymbol{\alpha})$ with $\Omega = \{\omega_{ij}\}$, $i, j = 1, 2$ and $\boldsymbol{\alpha} = (\alpha_1\ \alpha_2)^\top$, then they are independent if and only if

(a) $\omega_{12} = \omega_{21} = 0$,
(b) $\alpha_i \neq 0$ for at most one i, $i = 1, 2$.

The joint mgf of (X_n, Y_n) is as:

$$M_{X_n,Y_n}(\mathbf{t}) = 2\exp\left\{\frac{1}{2}\left(t_1^2 + t_2^2 + \frac{2}{n}t_1t_2\right)\right\}\Phi\left(\frac{\alpha_1}{\sqrt{1+\alpha_1^2}}\left(t_1 + \frac{1}{n}t_2\right)\right).$$

The limit of the joint mgf becomes

$$\begin{aligned}\lim_{n\to\infty} M_{X_n,Y_n}(\mathbf{t}) &= 2\exp\left\{\frac{1}{2}\left(t_1^2 + t_2^2\right)\right\}\Phi\left(\frac{\alpha_1}{\sqrt{1+\alpha_1^2}}t_1\right) \\ &= \exp\left(\frac{1}{2}t_2^2\right)2\exp\left(\frac{1}{2}t_1^2\right)\Phi\left(\frac{\alpha_1}{\sqrt{1+\alpha_1^2}}t_1\right).\end{aligned}$$

So,

$$(X_n, Y_n) \xrightarrow{d} (X, Y),$$

where $X \sim SN(0, 1, \alpha_1)$ and $Y \sim N(0, 1)$ independently.

3. Discussion

We developed some counterexamples related to the skew-normal distribution that extend those of the normal distribution in some cases. This material is useful for undergraduate and/or graduate teaching courses. For future research, completeness and/or more properties of the skew-normal distribution are needed to develop further counterexamples.

Appendix

This is not a counterexample, but it is good for understanding a skew-normal distribution. Let $(Z_1, \ldots, Z_n)$ be a random sample from $SN(0, 1, \alpha)$. An unbiased estimator of δ is

$$\hat{\delta}^{\text{u.e.}} = \frac{1}{nb} \sum_{i=1}^{n} Z_i$$

since $E(Z_i) = b\delta$, where $b = \sqrt{2/\pi}$, $\alpha = \delta/\sqrt{1-\delta^2}$, and $\delta = \alpha/\sqrt{1+\alpha^2}$. The likelihood function of δ or α is

$$L(\alpha) = \prod_{i=1}^{n} 2\varphi(z_i)\Phi(\alpha z_i) = 2^n \prod_{i=1}^{n} \varphi(z_i) \prod_{i=1}^{n} \Phi(\alpha z_i).$$

Therefore, a random sample $(Z_1, \ldots, Z_n)$ or the order statistic, $(Z_{(1)}, \ldots, Z_{(n)})$, are sufficient statistics for δ or α.

We show that $(Z_{(1)}, \ldots, Z_{(n)})$ is not a complete statistic for δ. It is obvious that $\sum_{i=1}^{n} Z_{(i)} = \sum_{i=1}^{n} Z_i$ and $\sum_{i=1}^{n} Z_{(i)}^2 = \sum_{i=1}^{n} Z_i^2$, so $E(\bar{Z}) = b\delta$, where $\bar{Z} = \sum_{i=1}^{n} Z_{(i)}/n = \sum_{i=1}^{n} Z_i/n$. From simple algebra, we find that

$$E\left(1 - S_Z^2\right) = b^2\delta^2 \quad \text{and} \quad E\left(\frac{n\bar{Z}^2 - 1}{n-1}\right) = b^2\delta^2,$$

where $S_Z^2 = (1/(n-1)) \sum_{i=1}^{n} (Z_i - \bar{Z})^2$. Therefore, the order statistic is not a complete statistic for δ.

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