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IDENTITIES AND RELATIONS ON THE q-APOSTOL TYPE FROBENIUS-EULER NUMBERS AND POLYNOMIALS

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Abstract. The main purpose of this paper is to investigate the q-Apostol type Frobenius-Euler numbers and polynomials. By using generating functions for these numbers and polynomials, we derive some alternative summation formulas including powers of consecutive q-integers. By using infinite series representation for q-Apostol type Frobenius-Euler numbers and polynomials including their interpolation functions, we not only give some identities and relations for these numbers and polynomials, but also define generating functions for new numbers and polynomials. Further we give remarks and observations on generating functions for these new numbers and polynomials. By using these generating functions, we derive recurrence relations and finite sums related to these numbers and polynomials. Moreover, by applying higher-order derivative to these generating functions, we derive some new formulas including the Hurwitz–Lerch zeta function, the Apostol-Bernoulli numbers and the Apostol-Euler numbers. Finally, for an application of the generating functions, we derive a multiplication formula, which is very important property in the theories of normalized polynomials and Dedekind type sums.

1. Introduction

We need the following notations, definitions and relations: In the following let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, and positive integers, respectively. Let $[x]$ be q-analogue of x which is given by

$$[x] = [x:q] = \begin{cases} x, & q = 1, \\ \frac{1-q^x}{1-q} = 1 + q + q^2 + \cdots + q^{x-1}, & q \neq 1. \end{cases}$$

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This function satisfies the following properties:

$$[x+y]=[x]+q^{x}[y]$$

and

$$[xy]=[x]\,[y:q^{x}] \tag{1}$$

(*cf.* [8, 15, 16, 18, 21, 34]).

Let $\lambda, q \in \mathbb{C}$. The q-Apostol type Frobenius-Euler polynomials $\mathcal{H}_n(x;u;a,b;\lambda;q)$ are given by

$$\begin{aligned} F_{\lambda,q}(x,t;u,a,b) &= \left(1-\frac{a^t}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n b^{[n+x]t} \\ &= \sum_{n=0}^{\infty}\mathcal{H}_n(x;u;a,b;\lambda;q)\frac{t^n}{n!}, \end{aligned} \tag{2}$$

where $a,b\in\mathbb{R}^{+}$ $(a\neq b)$ and $u\in\mathbb{C}\setminus\{1\}$ with $\left|\frac{\lambda}{u}\right|<1$ (*cf.* [21]).

In the special case when $x=0$, these polynomials reduce to the q-Apostol type Frobenius-Euler numbers

$$\mathcal{H}_n(u;a,b;\lambda;q)=\mathcal{H}_n(0;u;a,b;\lambda;q),$$

which are given by the following generating function:

$$\begin{aligned} F_{\lambda,q}(t;u,a,b) &= \left(1-\frac{a^t}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n b^{[n]t} \\ &= \sum_{n=0}^{\infty}\mathcal{H}_n(u;a,b;\lambda;q)\frac{t^n}{n!} \end{aligned} \tag{3}$$

(*cf.* [21]).

It follows from (2) and (3) that

$$F_{\lambda,q}(x,t;u,a,b)=b^{t[x]}F_{\lambda,q}(q^{x}t;u,a,b)$$

(*cf.* [21]).

If $q\to 1$, then equation (2) reduces to the following generating functions for the generalized Eulerian type polynomials as follows:

$$\begin{aligned} \lim_{q\to 1}F_{\lambda,q}(x,t;u,a,b) &= \frac{a^t-u}{\lambda b^t-u}b^{tx} \\ &= \sum_{n=0}^{\infty}\mathcal{H}_n(x;u;a,b;\lambda;1)\frac{t^n}{n!} \end{aligned}$$

(*cf.* [22, Definition 4.1], [23]). In the above equation, if we set $a=1$ and $b=e$, we have the generating functions for the Apostol-type Frobenius–Euler numbers and polynomials, respectively, as follows:

$$\frac{1-u}{\lambda e^t-u}=\sum_{n=0}^{\infty}H_n(\lambda|u)\frac{t^n}{n!} \tag{4}$$

and

$$\frac{1-u}{\lambda e^{t}-u}e^{tx}=\sum_{n=0}^{\infty}H_{n}\left(x;\lambda|u\right)\frac{t^{n}}{n!} \tag{5}$$

(*cf.* [3,22]). That is,

$$H_{n}\left(\lambda|u\right)=\mathcal{H}_{n}\left(u;1,e;\lambda;1\right),$$

and

$$H_{n}\left(x;\lambda|u\right)=\mathcal{H}_{n}\left(x;u;1,e;\lambda;1\right).$$

For $a=1$, $\lambda=1$ and $b=e$, the functions $F_{\lambda,q}\left(t;u,a,b\right)$ and the functions $F_{\lambda,q}\left(x,t;u,a,b\right)$ reduce to the generating functions for the q-Frobenius–Euler numbers and polynomials, respectively as follows:

$$F_{1,q}\left(t;u,1,e\right)=F_{u,q}\left(t\right)=\left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\frac{e^{[n]t}}{u^{n}}=\sum_{n=0}^{\infty}H_{n}\left(u,q\right)\frac{t^{n}}{n!}, \tag{6}$$

and

$$\begin{aligned} F_{1,q}\left(x,t;u,1,e\right)=F_{u,q}\left(t,x\right)&=\left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\frac{e^{[n+x]t}}{u^{n}} \\ &=\sum_{n=0}^{\infty}H_{n}\left(x,u,q\right)\frac{t^{n}}{n!}. \end{aligned} \tag{7}$$

That is,

$$\mathcal{H}_{n}\left(u;1,e;1;q\right)=H_{n}\left(u,q\right), \tag{8}$$

and

$$\mathcal{H}_{n}\left(x;u;1,e;1;q\right)=H_{n}\left(x,u,q\right) \tag{9}$$

(*cf.* [16,18]).

Moreover, in the special case of $u=-1$, (6) and (7) yields the generating functions for the q-Euler numbers and polynomials, respectively, as follows:

$$2\sum_{n=0}^{\infty}\left(-1\right)^{n}e^{[n]t}=\sum_{n=0}^{\infty}E_{n,q}\frac{t^{n}}{n!},$$

and

$$2\sum_{n=0}^{\infty}\left(-1\right)^{n}e^{[n+x]t}=\sum_{n=0}^{\infty}E_{n,q}\left(x\right)\frac{t^{n}}{n!}.$$

Namely,

$$\mathcal{H}_{n}\left(-1;1,e;1;q\right)=E_{n,q}, \tag{10}$$

and

$$\mathcal{H}_{n}\left(x;-1;1,e;1;q\right)=E_{n,q}\left(x\right) \tag{11}$$

(*cf.* [14,21,25,27]).

Note that, for $\lambda = 1$, (4) and (5) reduce to the generating functions for the Frobenius–Euler numbers and polynomials, respectively as follows:

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \tag{12}$$

and

$$\frac{1-u}{e^t - u} e^{tx} = \sum_{n=0}^{\infty} H_n(x;u) \frac{t^n}{n!},$$

that is,

$$H_n(x;u) = \mathcal{H}_n(x;u;1,e;1;1)$$

and

$$H_n(u) = \mathcal{H}_n(u;1,e;1;1)$$

(*cf.* [16,18,21,23,28,32]).

In [31, Eq. (21)], Srivastava et al. gave the following special case of the generalized Bernoulli polynomials

$$F_{\mathcal{B}}(t;x;\lambda;a,b,c) = \frac{tc^{tx}}{\lambda b^t - a^t} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda;a,b,c) \frac{t^n}{n!}, \tag{13}$$

where $a,b,c \in \mathbb{R}^+$ and $a \neq b$.

The polynomials $Y_n(x;\lambda;a)$ are defined as follows (see, e.g., [1,21,22,24])

$$\frac{t}{\lambda a^t - 1} a^{xt} = \sum_{n=0}^{\infty} Y_n(x;\lambda;a) \frac{t^n}{n!}, \qquad (a \geq 1) \tag{14}$$

and for $x = 0$, these polynomials are reduced to the numbers $Y_n(\lambda;a) = Y_n(0;\lambda;a)$. Note that

$$Y_0(x;\lambda;a) = 0,$$
$$Y_1(x;\lambda;a) = \frac{1}{\lambda - 1}.$$

The Hurwitz-Lerch zeta function $\Phi(\lambda, s, a)$ is defined by:

$$\Phi(\lambda, s, a) = \sum_{m=0}^{\infty} \frac{\lambda^m}{(m+a)^s},$$

where $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $s \in \mathbb{C}$ when $|z| < 1$; $\Re(s) > 1$ when $|z| = 1$ and this function interpolates the Apostol-Bernoulli numbers $B_n(\lambda)$ with the following relation:

$$\Phi(\lambda, -v, 0) = -\frac{B_{v+1}(\lambda)}{v+1}, \tag{15}$$

so that, the Apostol-Bernoulli numbers $B_n(\lambda)$ is defined by the following generating function:

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}$$

(*cf.* [2, 6, 7, 12, 29, 30]).

We note that a relation between the Frobenius-Euler numbers and the Apostol Bernoulli numbers is given by

$$H_n\left(u\right)=\frac{1-u}{u\left(n+1\right)}B_{n+1}\left(\frac{1}{u}\right)$$

(*cf.* [12, 17]).

The Apostol-Euler numbers $\mathcal{E}_v\left(\lambda\right)$ are defined by (*cf.* [6, 7, 12, 29, 30]):

$$\frac{2}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{E}_n\left(\lambda\right)\frac{t^n}{n!}. \tag{16}$$

The well-known relation between the Apostol-Bernoulli numbers and the Apostol-Euler numbers is given as follows:

$$\mathcal{E}_n\left(\lambda\right)=-\frac{2B_{n+1}\left(-\lambda\right)}{n+1} \tag{17}$$

(*cf.* [6, 7, 12, 29, 30]).

This paper is organized as follows:

In Section 2, by using the generating functions for the q-Apostol type Frobenius-Euler numbers and polynomials and a method similar to that in [9] and [19], we provide some alternative summation formulas including powers of consecutive q-integers. We give some special cases of the obtained summation formulas. In Section 3, by using infinite series representation for q-Apostol type Frobenius-Euler numbers and polynomials, we provide some identities and relations associated with these numbers and polynomials. In Section 4, in the light of Section 3, we define generating functions for new numbers and polynomials. We give remarks and observations on these generating functions. In Section 5, we give some recurrence formulas, finite sums and relations including not only these numbers and polynomials, but also other special numbers and polynomials. In Section 6, by applying higher-order derivative to these generating functions, we derive some new formulas including the Hurwitz–Lerch zeta function, the Apostol-Bernoulli numbers and the Apostol-Euler numbers. In Section 7, for an application of the generating functions, we derive a multiplication formula, which is very important in the theories of normalized polynomials and Dedekind type sums. Finally, we give a remark and some observations on this formula.

2. Some alternative summation formulas including powers of consecutive q-integers

In this section, by using the same method as that in [9] and [19], we derive some alternative summation formulas including powers of consecutive q-integers arising from the generating functions for the q-Apostol type Frobenius-Euler numbers and polynomials. We also give some special cases of our summation formulas.

From (3) and (2), we have

$$
\begin{aligned}
&F_{\lambda,q}\left(t;u,a,b\right)-\left(\frac{\lambda}{u}\right)^{n}F_{\lambda,q}\left(n,t;u,a,b\right)\\
&=\left(1-\frac{a^{t}}{u}\right)\left(\sum_{m=0}^{\infty}\left(\frac{\lambda}{u}\right)^{m}b^{[m]t}-\left(\frac{\lambda}{u}\right)^{n}\sum_{m=0}^{\infty}\left(\frac{\lambda}{u}\right)^{m}b^{[m+n]t}\right)\\
&=\left(1-\frac{a^{t}}{u}\right)\left(\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^{m}b^{[m]t}\right).
\end{aligned}
$$

It follows from the above equation that

$$
\begin{aligned}
&F_{\lambda,q}\left(t;u,a,b\right)-\left(\frac{\lambda}{u}\right)^{n}F_{\lambda,q}\left(n,t;u,a,b\right)\\
&=\left(1-\frac{e^{t\ln a}}{u}\right)\left(\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^{m}e^{t[m]\ln b}\right).
\end{aligned}
$$

From the above equation, we have

$$
\begin{aligned}
&\sum_{v=0}^{\infty}\left(\mathcal{H}_{v}\left(u;a,b;\lambda;q\right)-\left(\frac{\lambda}{u}\right)^{n}\mathcal{H}_{v}\left(n;u;a,b;\lambda;q\right)\right)\frac{t^{v}}{v!}\\
&=\left(1-\frac{1}{u}\sum_{v=0}^{\infty}\frac{\left(t\ln a\right)^{v}}{v!}\right)\sum_{v=0}^{\infty}\left(\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^{m}\left([m]\ln b\right)^{v}\right)\frac{t^{v}}{v!}.
\end{aligned}
$$

Using the Cauchy product in the right hand side of the above equation yields

$$
\begin{aligned}
&\sum_{v=0}^{\infty}\left(\mathcal{H}_{v}\left(u;a,b;\lambda;q\right)-\left(\frac{\lambda}{u}\right)^{n}\mathcal{H}_{v}\left(n;u;a,b;\lambda;q\right)\right)\frac{t^{v}}{v!}\\
&=\sum_{v=0}^{\infty}\left(\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^{m}\left([m]\ln b\right)^{v}\right)\frac{t^{v}}{v!}\\
&\quad-\frac{1}{u}\sum_{v=0}^{\infty}\left(\sum_{k=0}^{v}\binom{v}{k}\left(\ln a\right)^{v-k}\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^{m}\left([m]\ln b\right)^{k}\right)\frac{t^{v}}{v!}.
\end{aligned}
$$

Comparing the coefficient of $\frac{t^{v}}{v!}$ in the above equation yields the following theorem:

Theorem 2.1. *Let $n\in\mathbb{Z}^{+}$. Then we have*

$$
\begin{aligned}
(18)\qquad &\mathcal{H}_{v}\left(u;a,b;\lambda;q\right)-\left(\frac{\lambda}{u}\right)^{n}\mathcal{H}_{v}\left(n;u;a,b;\lambda;q\right)\\
&=\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^{m}\left([m]\ln b\right)^{v}-\frac{1}{u}\sum_{k=0}^{v}\binom{v}{k}\left(\ln a\right)^{v-k}\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^{m}\left([m]\ln b\right)^{k}.
\end{aligned}
$$

Remark 2.2.

$$\mathcal{H}_v\left(u;a,b;\lambda;q\right)-\left(\frac{\lambda}{u}\right)^n\mathcal{H}_v\left(n;u;a,b;\lambda;q\right)$$
$$=\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^m\left([m]\ln b\right)^v-\frac{1}{u}\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^m\sum_{k=0}^{v}\binom{v}{k}\left(\ln a\right)^{v-k}\left([m]\ln b\right)^k.$$

By combining the above equation with the binomial theorem, we have

$$\mathcal{H}_v\left(u;a,b;\lambda;q\right)-\left(\frac{\lambda}{u}\right)^n\mathcal{H}_v\left(n;u;a,b;\lambda;q\right)$$
$$=\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^m\left(\left([m]\ln b\right)^v-\frac{\left(\ln a+[m]\ln b\right)^v}{u}\right). \tag{19}$$

Substituting $a=1$ into the above equation yields the following corollary:

Corollary 2.3. *Let $n\in\mathbb{Z}^+$. Then we have*

$$\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^m\left([m]\ln b\right)^v=\frac{u\mathcal{H}_v\left(u;1,b;\lambda;q\right)-\lambda^n u^{n-1}\mathcal{H}_v\left(n;u;1,b;\lambda;q\right)}{u-1}. \tag{20}$$

Remark 2.4. If we substitute $\lambda=1$ and $b=e$ into (20) and combining the final equation with (9) and (8), we have a summation formula in terms of the q-Frobenius–Euler numbers and polynomials as follows:

$$\sum_{m=0}^{n-1}\frac{[m]^v}{u^m}=\frac{uH_v\left(u,q\right)-u^{n-1}H_v\left(n,u,q\right)}{u-1} \tag{21}$$

(*cf.* [9–13], [20, Theorem 3, Eq. (2.1)], [26]). In addition, taking $\lambda=1$, $b=e$ and $u=-1$ in (20) and combining the final equation with (11) and (10) yields a summation formula calculated with the q-Euler numbers and polynomials as follows:

$$\sum_{m=0}^{n-1}\left(-1\right)^m[m]^v=\frac{E_{v,q}+\left(-1\right)^{n-1}E_{v,q}\left(n\right)}{2}$$

(*cf.* [9,11]). In the above equation, when $q\to1$, we have

$$\sum_{m=0}^{n-1}\left(-1\right)^m m^v=\frac{E_v+\left(-1\right)^{n-1}E_v\left(n\right)}{2}$$

(*cf.* [9,11]).

Remark 2.5. In the special case when $\lambda=1$, $b=e$ and $q\to1$, (20) reduces to a summation formula expressed in terms of the Frobenius–Euler numbers and polynomials as follows:

$$\sum_{m=0}^{n-1}\frac{m^v}{u^m}=\frac{uH_v\left(u\right)-u^{n-1}H_v\left(n;u\right)}{u-1},$$

where $u \in \mathbb{C} \setminus \{1\}$ (*cf.* [9, 11, 12]). Furthermore, if we take $\lambda = 1$, $b = e$, $q \to 1$ and $u = -\alpha^{-h}$, with $\alpha \in \mathbb{C}$ and $|\alpha| < 1$, in (20), then we have

$$\sum_{m=0}^{n-1} (-1)^m \alpha^{hm} m^v = \frac{E_{v,\alpha}^{(h)} + (-1)^{n-1} \alpha^{-hn} E_{v,\alpha}^{(h)}(n)}{2},$$

where $E_{v,\alpha}^{(h)}$ and $E_{v,\alpha}^{(h)}(n)$ denote the (h, q)-extension of Euler numbers and polynomials, respectively (*cf.* [20, Theorem 3, Eq. (2.1)]).

3. Infinite series representation for q-Apostol type Frobenius-Euler numbers and polynomials including their interpolation functions

In this section, we give infinite series representation for q-Apostol type Frobenius-Euler numbers and polynomials including their interpolation functions.

By (2), we get

$$\begin{aligned}
&\sum_{m=0}^{\infty} \mathcal{H}_m(x; u; a, b; \lambda; q) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^n ([n+x] \ln b)^m \right) \frac{t^m}{m!} \\
&\quad - \frac{1}{u} \sum_{m=0}^{\infty} (\ln a)^m \frac{t^m}{m!} \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^n ([n+x] \ln b)^m \right) \frac{t^m}{m!}.
\end{aligned}$$

A use of the Cauchy product in the above equation yields

$$\begin{aligned}
&\sum_{m=0}^{\infty} \mathcal{H}_m(x; u; a, b; \lambda; q) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^n ([n+x] \ln b)^m \right) \frac{t^m}{m!} \\
&\quad - \frac{1}{u} \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} (\ln a)^{m-k} \sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^n ([n+x] \ln b)^k \right) \frac{t^m}{m!}.
\end{aligned}$$

Comparing the coefficient of $\frac{t^m}{m!}$ in the above equation yields the following theorem:

Theorem 3.1. *Let $|\lambda| < |u|$. Then we have*

$$\begin{aligned}
&\mathcal{H}_m(x; u; a, b; \lambda; q) \\
&= \sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^n ([n+x] \ln b)^m - \frac{1}{u} \sum_{k=0}^{m} \binom{m}{k} (\ln a)^{m-k} \sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^n ([n+x] \ln b)^k.
\end{aligned}$$

Remark 3.2. By substituting $x = 0$ into the above equation, we have

$$\mathcal{H}_m(u; a, b; \lambda; q) \tag{22}$$

$$= \sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}([n]\ln b)^{m} - \frac{1}{u}\sum_{k=0}^{m}\binom{m}{k}(\ln a)^{m-k}\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}([n]\ln b)^{k}.$$

Remark 3.3. Substituting $a=1$ into (22), we have

$$\begin{aligned}\mathcal{H}_m(u;1,b;\lambda;q) &= \sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}[n]^{m}(\ln b)^{m} - \frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}[n]^{m}(\ln b)^{m}\\ &= \left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}([n]\ln b)^{m}. \qquad (23)\end{aligned}$$

Remark 3.4. Substituting $a=1$ and $b=e$ into (22), we have

$$\mathcal{H}_m(u;1,e;\lambda;q) = \left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}[n]^{m}.$$

When we modify (3), we have

$$\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n} b^{[n]t} = \left(\frac{u}{u-a^t}\right)\sum_{m=0}^{\infty}\mathcal{H}_m(u;a,b;\lambda;q)\frac{t^m}{m!}. \qquad (24)$$

Thus, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}([n]\ln b)^{m}\frac{t^m}{m!} = \frac{u}{u-1}\sum_{m=0}^{\infty}\mathcal{H}_m(u;a,b;\lambda;q)\frac{t^m}{m!}\left(\frac{u-1}{u-e^{t\ln a}}\right).$$

Combining the above equation with (12) yields

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}([n]\ln b)^{m}\frac{t^m}{m!} &= \frac{u}{u-1}\sum_{m=0}^{\infty}\mathcal{H}_m(u;a,b;\lambda;q)\frac{t^m}{m!}\\ &\quad\times\sum_{m=0}^{\infty}H_m(u)(\ln a)^{m}\frac{t^m}{m!}.\end{aligned}$$

Using Cauchy product in the above equation yields

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}([n]\ln b)^{m}\frac{t^m}{m!}\\ &= \frac{u}{u-1}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}(\ln a)^{m-k}\mathcal{H}_k(u;a,b;\lambda;q)H_{m-k}(u)\frac{t^m}{m!}.\end{aligned}$$

Comparing the coefficient of $\frac{t^m}{m!}$ in the above equation yields the following theorem:

Theorem 3.5. *Let $|\lambda|<|u|$ and $a\neq 1$. Then we have*

$$\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}\left([n]\frac{\ln b}{\ln a}\right)^{m} = \frac{u}{u-1}\sum_{k=0}^{m}\binom{m}{k}\frac{\mathcal{H}_k(u;a,b;\lambda;q)H_{m-k}(u)}{(\ln a)^k}. \qquad (25)$$

Remark 3.6. We can also write (25) as follows:

$$\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}([n]\ln b)^{m} = \frac{u}{u-1}\sum_{k=0}^{m}\binom{m}{k}(\ln a)^{m-k}\mathcal{H}_{k}(u;a,b;\lambda;q)\,H_{m-k}(u). \tag{26}$$

Substituting $a=1$ into (26), we have

$$\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}([n]\ln b)^{m} = \frac{u}{u-1}\mathcal{H}_{m}(u;1,b;\lambda;q)\,H_{0}(u).$$

Since

$$H_0(u)=1,$$

we have

$$\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}([n]\ln b)^{m} = \frac{u}{u-1}\mathcal{H}_{m}(u;1,b;\lambda;q),$$

as in (23). If $\lambda=1$ and $b=e$ in the above equation, by (8), one can easily find

$$l_q(u,-m)=\sum_{n=0}^{\infty}\frac{[n]^m}{u^n}=\frac{u}{u-1}H_m(u,q),$$

where $l_q(u,s)$ denotes a complex analytic l_q-series which was constructed by Satoh [15] and investigated by Tsumura [33] in p-adic and twisted cases, respectively.

Remark 3.7. Equation (26) gives us another interpolation functions for the numbers $\mathcal{H}_k(u;a,b;\lambda;q)$ and the numbers $H_k(u)$. If we replace m by $-s\in\mathbb{C}$, we get the following l_q-type series

$$l_q(s,u;b;\lambda)=\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}\frac{1}{([n]\ln b)^{s}}. \tag{27}$$

By using (25), for $a\neq 1$, we set

$$\mathcal{I}_m(u;a,b;\lambda;q):=\left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}\left([n]\frac{\ln b}{\ln a}\right)^{m} \tag{28}$$

$$=\sum_{k=0}^{m}\binom{m}{k}\frac{\mathcal{H}_k(u;a,b;\lambda;q)\,H_{m-k}(u)}{(\ln a)^{k}}. \tag{29}$$

We also set an interpolation function for the numbers $\mathcal{I}_m(u;a,b;\lambda;q)$ as follows:

$$l_q(s,u;a,b;\lambda)=\left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{n}\frac{1}{\left([n]\frac{\ln b}{\ln a}\right)^{s}}, \tag{30}$$

where $s \in \mathbb{C}$, $\left|\frac{\lambda}{u}\right| < 1$, $a \in \mathbb{R}^+ \setminus \{1\}$, $b \in \mathbb{R}^+$ and $\ln(z)$ denotes the principal branch of the multi-valued function $\ln(z)$ with the imaginary part $\operatorname{Im}(\ln(z))$ constrained by $-\pi < arg(z) < \pi$ and $|z| > 0$.

Substituting m by $-s \in \mathbb{C}$ into the above equation, we have

$$l_q(-m, u; a, b; \lambda) = \mathcal{I}_m(u; a, b; \lambda; q).$$

Since

$$\frac{u}{u-1}(\ln a)^{-s} l_q(s, u; a, b; \lambda) = l_q(s, u; b; \lambda),$$

substituting m by $-s \in \mathbb{C}$ into (30) yields

$$l_q(-m, u; b; \lambda) = \frac{u}{u-1}\sum_{k=0}^{m}\binom{m}{k}(\ln a)^{m-k}\mathcal{H}_k(u; a, b; \lambda; q) H_{m-k}(u).$$

When $a \to 1$ in (30), we get

$$\lim_{a\to 1} l_q(s, u; a, b; \lambda) = \zeta_{\lambda,q}^{(1)}(s; u, 1, b) = \left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n \frac{1}{([n]\ln b)^s},$$

where $\zeta_{\lambda,q}^{(1)}(s; u, 1, b)$ denotes the following zeta function (*cf.* [4, Definition 5.1]):

$$\begin{aligned}
&\zeta_{\lambda,q}^{(v)}(w; u, a, b)\\
&= \sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}\sum_{n_1,n_2,\ldots,n_v=0}^{\infty}\left(\frac{\lambda^n}{u^{v-j+n}}\right)\frac{1}{((v-j)\ln a + [n]\ln b)^w}
\end{aligned}$$

with $\operatorname{Re}(w) > 0$ and

$$n_1 + n_2 + \cdots + n_v = n.$$

4. Observations on generating functions for the numbers $\mathcal{I}_m(u; a, b; \lambda; q)$ and polynomials $\mathcal{I}_m(x; u; a, b; \lambda; q)$

In the light of previous section, we here give generating functions for the numbers $\mathcal{I}_m(u; a, b; \lambda; q)$. We also define the polynomials $\mathcal{I}_m(x; u; a, b; \lambda; q)$ with their generating functions.

Now, we define generating functions for the numbers $\mathcal{I}_m(u; a, b; \lambda; q)$ as follows:

$$\begin{aligned}
F_{\mathcal{I}}(t, u; a, b; \lambda; q) &= \sum_{m=0}^{\infty}\mathcal{I}_m(u; a, b; \lambda; q)\frac{t^m}{m!}\\
&= \left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n e^{t[n]\frac{\ln b}{\ln a}}, \qquad (31)
\end{aligned}$$

where $a \neq 1$.

Remark 4.1. Upon setting $q \to 1$ in (31) and using (4), we get for $\left|\frac{\lambda}{u}e^{t\frac{\ln b}{\ln a}}\right| < 1$ that

$$\lim_{q\to 1} F_{\mathcal{I}}\left(t, u; a, b; \lambda; q\right) = \frac{1-u}{\lambda e^{t\frac{\ln b}{\ln a}} - u}.$$

That is,

$$\mathcal{I}_m\left(u; a, b; \lambda; 1\right) = \left(\frac{\ln b}{\ln a}\right)^m H_m\left(\lambda | u\right).$$

By using (31), we also define the polynomials $\mathcal{I}_m\left(x; u; a, b; \lambda; q\right)$ with the following generating functions

$$\begin{aligned} G_{\mathcal{I}}\left(x, t; u; a, b; \lambda; q\right) &= \left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n e^{t[n+x]\frac{\ln b}{\ln a}} \\ &= \sum_{m=0}^{\infty}\mathcal{I}_m\left(x; u; a, b; \lambda; q\right)\frac{t^m}{m!}, \end{aligned} \tag{32}$$

where $a \neq 1$. It should be noted that

$$\mathcal{I}_m\left(u; a, b; \lambda; q\right) = \mathcal{I}_m\left(0; u; a, b; \lambda; q\right). \tag{33}$$

Combining the above equation with (31) yields the following functional equation:

$$G_{\mathcal{I}}\left(x, t; u; a, b; \lambda; q\right) = e^{t[x]\frac{\ln b}{\ln a}} F_{\mathcal{I}}\left(q^x t, u; a, b; \lambda; q\right). \tag{34}$$

It follows from the above functional equation that

$$\sum_{m=0}^{\infty}\mathcal{I}_m\left(x; u; a, b; \lambda; q\right)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left([x]\frac{\ln b}{\ln a}\right)^m\frac{t^m}{m!}\sum_{m=0}^{\infty}\mathcal{I}_m\left(u; a, b; \lambda; q\right)q^{xm}\frac{t^m}{m!}.$$

Using the Cauchy product in the above equation and comparing the coefficient of $\frac{t^m}{m!}$ yields the following theorem:

Theorem 4.2. *Let $a \neq 1$. Then we have*

$$\mathcal{I}_m\left(x; u; a, b; \lambda; q\right) = \sum_{j=0}^{m}\binom{m}{j}\left([x]\frac{\ln b}{\ln a}\right)^{m-j} q^{xj}\mathcal{I}_j\left(u; a, b; \lambda; q\right). \tag{35}$$

Remark 4.3. Substituting $a = e$ into (31) and (32) respectively yields

$$F_{\lambda,q}\left(t; u, 1, b\right) = F_{\mathcal{I}}\left(t, u; e, b; \lambda; q\right)$$

and

$$F_{\lambda,q}\left(x, t; u, 1, b\right) = G_{\mathcal{I}}\left(x, t; u; e, b; \lambda; q\right).$$

Hence, we have

$$\mathcal{H}_m\left(u; 1, b; \lambda; q\right) = \mathcal{I}_m\left(u; e, b; \lambda; q\right)$$

and

$$\mathcal{H}_m\left(x; u; 1, b; \lambda; q\right) = \mathcal{I}_m\left(x; u; e, b; \lambda; q\right).$$

Moreover, by the above relations, (20) may also be written as follows:

$$\sum_{m=0}^{n-1}\left(\frac{\lambda}{u}\right)^m([m]\ln b)^v=\frac{u\mathcal{I}_v\left(u;e,b;\lambda;q\right)-\lambda^n u^{n-1}\mathcal{I}_v\left(n;u;e,b;\lambda;q\right)}{u-1}.$$

5. Recurrence relations and finite sums

In this section, by using generating functions, we derive recurrence formulas for the polynomials $Y_n(x;\lambda;a)$ and the numbers $\mathcal{I}_m\left(u;a,b;\lambda;q\right)$. Moreover, we give a relation between the polynomials $Y_n(x;\lambda;a)$ and the numbers $\mathcal{I}_m\left(u;a,b;\lambda;q\right)$. Finally, we give finite sums related to the aforementioned numbers and polynomials.

By using (14), we have

$$t\sum_{n=0}^{\infty}(x\ln a)^n\frac{t^n}{n!}=\lambda\sum_{n=0}^{\infty}(\ln a)^n\frac{t^n}{n!}\sum_{n=0}^{\infty}Y_n(x;\lambda;a)\frac{t^n}{n!}-\sum_{n=0}^{\infty}Y_n(x;\lambda;a)\frac{t^n}{n!}.$$

Using Cauchy product in the above equation yields

$$\sum_{n=0}^{\infty}n\left(x\ln a\right)^{n-1}\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\lambda\left(\sum_{j=0}^{n}\binom{n}{j}(\ln a)^{n-j}Y_j(x;\lambda;a)\right)-Y_n(x;\lambda;a)\right)\frac{t^n}{n!}.$$

Comparing the coefficient of $\frac{t^n}{n!}$ in the above equation yields a recurrence formula for the polynomials $Y_n(x;\lambda;a)$ given by the following theorem:

Theorem 5.1. *Let $a\geq 1$. Then we have*

$$n\left(x\ln a\right)^{n-1}+Y_n(x;\lambda;a)=\lambda\sum_{j=0}^{n}\binom{n}{j}(\ln a)^{n-j}Y_j(x;\lambda;a). \tag{36}$$

We modify (36) by the following corollary:

Corollary 5.2.

$$n\left(x\ln a\right)^{n-1}=\lambda\left(Y(x;\lambda;a)+\ln a\right)^n-Y_n(x;\lambda;a),$$

where $Y^n(x;\lambda;a)$ replaced by $Y_n(x;\lambda;a)$.

In order to prove Theorem 5.3, we firstly give the following generating function for the q-Bernoulli type numbers $\mathcal{A}_n\left(\lambda;b,u;q\right)$ associated with the positive real parameter b and the complex parameters u and λ as follows:

$$F_{\mathcal{A}}\left(t;\lambda;b,u;q\right)=\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n b^{[n]t}=\sum_{n=0}^{\infty}\mathcal{A}_n\left(\lambda;b,u;q\right)\frac{t^n}{n!} \tag{37}$$

which, in the special case when $q \to 1$, reduces to

$$\lim_{q\to 1} F_{\mathcal{A}}\left(t;\lambda;b,u;q\right) = -\frac{1}{t}F_{\mathcal{B}}\left(t;0;\frac{\lambda}{u};1,b,c\right).$$

Hence, the relation between the polynomial $\mathcal{B}_n\left(x;\lambda;a,b,c\right)$ and the numbers $\mathcal{A}_n\left(\lambda;b,u;q\right)$ is given by

$$\mathcal{A}_n\left(\lambda;b,u;1\right) = -\frac{\mathcal{B}_{n+1}\left(0;\frac{\lambda}{u};1,b,c\right)}{n+1}.$$

Secondly, by modifying (24), we get

$$-\frac{1}{t}\left(\frac{t}{\frac{1}{u}a^t-1}\right)\sum_{m=0}^{\infty}\mathcal{H}_m\left(u;a,b;\lambda;q\right)\frac{t^m}{m!} = \sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n b^{[n]t}.$$

Combining the above equation with (14) and (37), we have

$$\sum_{m=0}^{\infty}Y_m\left(\frac{1}{u};a\right)\frac{t^m}{m!}\sum_{m=0}^{\infty}\mathcal{H}_m\left(u;a,b;\lambda;q\right)\frac{t^m}{m!} = -t\sum_{m=0}^{\infty}\mathcal{A}_m\left(\lambda;b,u;q\right)\frac{t^m}{m!}.$$

Using Cauchy product in the above equation yields

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}\mathcal{H}_j\left(u;a,b;\lambda;q\right)Y_{m-j}\left(\frac{1}{u};a\right)\right)\frac{t^m}{m!}\\ &= -\sum_{m=0}^{\infty}m\mathcal{A}_{m-1}\left(\lambda;b,u;q\right)\frac{t^m}{m!}.\end{aligned}$$

Therefore, comparing the coefficient of $\frac{t^m}{m!}$ in the above equation yields the following theorem:

Theorem 5.3. *Let $m \in \mathbb{N}$. We have*

$$\sum_{j=0}^{m}\binom{m}{j}\mathcal{H}_j\left(u;a,b;\lambda;q\right)Y_{m-j}\left(\frac{1}{u};a\right) = -m\mathcal{A}_{m-1}\left(\lambda;b,u;q\right). \tag{38}$$

Remark 5.4. When $q \to 1$ and $\lambda = 1$, (38) reduces to Theorem 4.1 in [24].

By modifying (31), for $a \neq 1$, we have

$$\sum_{m=0}^{\infty}\mathcal{I}_m\left(u;a,b;\lambda;q\right)\frac{t^m}{m!} = \left(1-\frac{1}{u}\right)e^{\frac{\ln b}{(1-q)\ln a}t}\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n e^{\frac{q^n\ln b}{(q-1)\ln a}t}.$$

After some calculations in the above equation, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\mathcal{I}_m\left(u;a,b;\lambda;q\right)\frac{t^m}{m!} =&\ (u-1)\left(\sum_{m=0}^{\infty}(-1)^m\left(\frac{\ln b}{(q-1)\ln a}\right)^m\frac{t^m}{m!}\right)\\ &\times\sum_{m=0}^{\infty}\left(\frac{\ln b}{(q-1)\ln a}\right)^m\frac{1}{u-\lambda q^m}\frac{t^m}{m!}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{m=0}^{\infty}\mathcal{I}_m\left(u;a,b;\lambda;q\right)\frac{t^m}{m!}\\ &= (u-1)\sum_{m=0}^{\infty}\left(\frac{\ln b}{(q-1)\ln a}\right)^m\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\frac{1}{u-\lambda q^j}\frac{t^m}{m!}.\end{aligned}$$

Using the Cauchy product in the above equation and comparing the coefficient of $\frac{t^m}{m!}$ in the above equation yields the following theorem:

Theorem 5.5.

$$\mathcal{I}_m\left(u;a,b;\lambda;q\right)=(u-1)\left(\frac{\ln b}{(q-1)\ln a}\right)^m\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\frac{1}{u-\lambda q^j}. \tag{39}$$

Remark 5.6. If we set $\lambda=1$, $a=b=e$ and replace u by u^{-1}, then we have the following equation which was given by Srivastava et al. [32, Eq. (10.3)]:

$$H\left(u^{-1}:q|1\right)=\frac{1-u}{(1-q)^m}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{1}{1-q^j u},$$

where $H\left(u^{-1}:q|1\right)$ denotes the Euler–Barnes' type Daehee q-Euler numbers.

Substituting (39) into (35), we derive an explicit formula for the polynomials $\mathcal{I}_m\left(x;u;a,b;\lambda;q\right)$ by the following corollary:

Corollary 5.7.

$$\begin{aligned}\mathcal{I}_m\left(x;u;a,b;\lambda;q\right)=&\ (u-1)\left(\frac{\ln b}{\ln a}\right)^m\sum_{j=0}^{m}\binom{m}{j}[x]^{m-j}\\ &\times\left(\frac{q^x}{q-1}\right)^j\sum_{v=0}^{j}(-1)^{j-v}\binom{j}{v}\frac{1}{u-\lambda q^v}.\end{aligned} \tag{40}$$

Combining (39) with (29) and using (33) yields

$$\begin{aligned}&(u-1)\left(\frac{\ln b}{q-1}\right)^m\sum_{v=0}^{m}(-1)^{m-v}\binom{m}{v}\frac{1}{u-\lambda q^v}\\ &=\sum_{k=0}^{m}\binom{m}{k}(\ln a)^{m-k}\,\mathcal{H}_k\left(u;a,b;\lambda;q\right)H_{m-k}\left(u\right).\end{aligned}$$

When $q\to -1$ in (19) and after some elementary calculation, we get the following theorem:

Theorem 5.8.

$$\sum_{\substack{m=0\\ m\ odd}}^{n-1}\left(\frac{\lambda}{u}\right)^m=\frac{u}{u\left(\ln b\right)^v-\left(\ln\left(ab\right)\right)^v} \tag{41}$$

$$\times \lim_{q\to -1}\left(\mathcal{H}_v\left(u;a,b;\lambda;q\right)-\left(\frac{\lambda}{u}\right)^n\mathcal{H}_v\left(n;u;a,b;\lambda;q\right)\right).$$

Substituting $a=1$ and $b=e$ into (41) yields the following corollary:

Corollary 5.9.

$$\sum_{\substack{m=0\\ m\ odd}}^{n-1}\left(\frac{\lambda}{u}\right)^m=\frac{u}{u-1}\lim_{q\to -1}\left(\mathcal{H}_v\left(u;1,e;\lambda;q\right)-\left(\frac{\lambda}{u}\right)^n\mathcal{H}_v\left(n;u;1,e;\lambda;q\right)\right).$$

6. Identities derived from differential equation for the function $F_{\mathcal{I}}\left(t,u;a,b;\lambda;q\right)$

In this section, we give a higher-order partial differential equation of the generating function $F_{\mathcal{I}}\left(t,u;a,b;\lambda;q\right)$. By using this equation, we derive some new formulas including the Hurwitz–Lerch zeta function, the Apostol-Bernoulli numbers and the Apostol-Euler numbers.

Differentiating v times both side of (31), with respect to t, yields, for $a\neq 1$,

$$\begin{aligned}\frac{\partial^v}{\partial t^v}F_{\mathcal{I}}\left(t,u;a,b;\lambda;q\right)&=\left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n\left([n]\frac{\ln b}{\ln a}\right)^v e^{t[n]\frac{\ln b}{\ln a}}\\&=\left(\frac{\ln b}{\ln a}\right)^v\left(1-\frac{1}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n\left(\frac{1-q^n}{1-q}\right)^v e^{t[n]\frac{\ln b}{\ln a}}.\end{aligned}\tag{42}$$

Combining the binomial theorem with the above equation, we have

$$\begin{aligned}&\frac{\partial^v}{\partial t^v}F_{\mathcal{I}}\left(t,u;a,b;\lambda;q\right)\\&=\left(\frac{\ln b}{(1-q)\ln a}\right)^v\left(1-\frac{1}{u}\right)\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}q^{v-j}\right)^n e^{t[n]\frac{\ln b}{\ln a}}.\end{aligned}$$

Hence, we get the following partial differential equation

$$\frac{\partial^v}{\partial t^v}F_{\mathcal{I}}\left(t,u;a,b;\lambda;q\right)=\left(\frac{\ln b}{(1-q)\ln a}\right)^v\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}F_{\mathcal{I}}\left(t,u;a,b;\lambda q^{v-j};q\right).$$

Combining the differential equation with (31) yields

$$\begin{aligned}&\sum_{m=0}^{\infty}\mathcal{I}_{m+v}\left(u;a,b;\lambda;q\right)\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left(\left(\frac{\ln b}{(1-q)\ln a}\right)^v\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}\mathcal{I}_m\left(u;a,b;\lambda q^{v-j};q\right)\right)\frac{t^m}{m!}.\end{aligned}$$

Comparing the coefficient of $\frac{t^m}{m!}$ in the above equation yields the following theorem:

Theorem 6.1. *Let $a \neq 1$. Then we have*

$$\mathcal{I}_{m+v}\left(u;a,b;\lambda;q\right)=\left(\frac{\ln b}{(1-q)\ln a}\right)^{v}\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}\mathcal{I}_{m}\left(u;a,b;\lambda q^{v-j};q\right).$$

From (42) and (28), we have

$$\left.\frac{\partial^{v}}{\partial t^{v}}F_{\mathcal{I}}\left(t,u;a,b;\lambda;q\right)\right|_{t=0}=\mathcal{I}_{v}\left(u;a,b;\lambda;q\right).$$

If $q\to 1$ in the above equation, then we have

$$\lim_{q\to 1}\left.\frac{\partial^{v}}{\partial t^{v}}F_{\mathcal{I}}\left(t,u;a,b;\lambda;q\right)\right|_{t=0}=\left(\frac{\ln b}{\ln a}\right)^{v}\left(1-\frac{1}{u}\right)\Phi\left(\frac{\lambda}{u},-v,0\right). \tag{43}$$

Combining (43) and (15) yields the following corollary:

Corollary 6.2.

$$\lim_{q\to 1}\left.\frac{\partial^{v}}{\partial t^{v}}F_{\mathcal{I}}\left(t,u;a,b;\lambda;q\right)\right|_{t=0}=\left(\frac{\ln b}{\ln a}\right)^{v}\left(\frac{1}{u}-1\right)\frac{B_{v+1}\left(\frac{\lambda}{u}\right)}{v+1}. \tag{44}$$

Substituting $a=b=e$ and $u=-1$ into (44) and using (17) yields the following corollary:

Corollary 6.3.

$$\lim_{q\to 1}\left.\frac{\partial^{v}}{\partial t^{v}}F_{\mathcal{I}}\left(t,-1;e,e;\lambda;q\right)\right|_{t=0}=\mathcal{E}_{v}\left(\lambda\right).$$

7. Multiplication formula for the polynomials $\mathcal{I}_m\left(x;u;a,b;\lambda;q\right)$

In this section, by using generating function techniques, we derive a multiplication formula for the polynomials $\mathcal{I}_m\left(x;u;a,b;\lambda;q\right)$. We also give a remark and observations on this formula.

Substituting $n=md+k$, $m=0,1,\ldots,\infty$, $k=0,\ldots,d-1$ into (32) yields

$$G_{\mathcal{I}}\left(x,t;u;a,b;\lambda;q\right)=\left(1-\frac{1}{u}\right)\sum_{m=0}^{\infty}\sum_{k=0}^{d-1}\left(\frac{\lambda}{u}\right)^{md+k}e^{t[md+k+x]\frac{\ln b}{\ln a}}.$$

Using (1) in the above equation and some elementary calculation yields

$$\begin{aligned}&G_{\mathcal{I}}\left(x,t;u;a,b;\lambda;q\right)\\&=\frac{u^{d}}{u^{d}-1}\left(1-\frac{1}{u}\right)\sum_{k=0}^{d-1}\left(\frac{\lambda}{u}\right)^{k}\sum_{m=0}^{\infty}\left(\frac{\lambda^{d}}{u^{d}}\right)^{m}e^{t[d]\left[m+\frac{x+k}{d}:q^{d}\right]\frac{\ln b}{\ln a}}.\end{aligned} \tag{45}$$

By (45), we obtain the following functional equation:

$$\begin{aligned}&G_{\mathcal{I}}\left(x,t;u;a,b;\lambda;q\right)\\&=\left(\frac{u^{d}-u^{d-1}}{u^{d}-1}\right)\sum_{k=0}^{d-1}\left(\frac{\lambda}{u}\right)^{k}G_{\mathcal{I}}\left(\frac{x+k}{d},t\left[d\right];u^{d};a,b;\lambda^{d};q^{d}\right).\end{aligned} \tag{46}$$

Combining (46) with (32), we get

$$\sum_{m=0}^{\infty}\mathcal{I}_m\left(x;u;a,b;\lambda;q\right)\frac{t^m}{m!}=\left(\frac{u^d-u^{d-1}}{u^d-1}\right)\sum_{k=0}^{d-1}\left(\frac{\lambda}{u}\right)^k \times\sum_{m=0}^{\infty}\mathcal{I}_m\left(\frac{x+k}{d};u^d;a,b;\lambda^d;q^d\right)\frac{\left(t\left[d\right]\right)^m}{m!}.$$

Comparing the coefficient of $\frac{t^m}{m!}$ in the above equation yields the following theorem:

Theorem 7.1.

$$\mathcal{I}_m\left(x;u;a,b;\lambda;q\right) = \left(\frac{u^d-u^{d-1}}{u^d-1}\right)[d]^m\sum_{k=0}^{d-1}\left(\frac{\lambda}{u}\right)^k\mathcal{I}_m\left(\frac{x+k}{d};u^d;a,b;\lambda^d;q^d\right). \tag{47}$$

Replacing x by xd in (47), we also get the following multiplication formula for the polynomials $\mathcal{I}_m\left(x;u;a,b;\lambda;q\right)$:

Corollary 7.2.

$$\mathcal{I}_m\left(xd;u;a,b;\lambda;q\right) = \left(\frac{u^d-u^{d-1}}{u^d-1}\right)[d]^m\sum_{k=0}^{d-1}\left(\frac{\lambda}{u}\right)^k\mathcal{I}_m\left(x+\frac{k}{d};u^d;a,b;\lambda^d;q^d\right). \tag{48}$$

Remark 7.3. Substituting $a=b=e$, $\lambda=1$ and $q\to 1$ into (48), we have the following known multiplication formula (see [22, Eq. (27)]):

$$\mathcal{I}_m\left(xd;u;e,e;1;1\right)=\left(\frac{u^d-u^{d-1}}{u^d-1}\right)d^m\sum_{k=0}^{d-1}\left(\frac{1}{u}\right)^k\mathcal{I}_m\left(x+\frac{k}{d};u^d;e,e;1;1\right).$$

Here, if we take $u=-1$ (with odd d integer) into the above equation, one has the multiplication formula for the Euler polynomials. Thus, these polynomials are member of the normalized polynomials which satisfy the following relations:

$$f_n\left(xd\right)=d^{n-1}\sum_{k=0}^{d-1}f_n\left(x+\frac{k}{d}\right),$$

where f_n is a polynomial and such polynomials have many applications in the theory of polynomials in finite fields and also the theory of Dedekind type sums (see, for details, [5] and [22, p. 19]; see also the references cited in each of these earlier works on the subject).

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