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A GRADED MINIMAL FREE RESOLUTION OF THE 2ND ORDER SYMBOLIC POWER OF THE IDEAL OF A STAR CONFIGURATION IN $\mathbb{P}^n$

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Abstract. In [9], Geramita, Harbourne, and Migliore find a graded minimal free resolution of the 2nd order symbolic power of the ideal of a linear star configuration in $\mathbb{P}^n$ of any codimension r. In [8], Geramita, Galetto, Shin, and Van Tuyl extend the result on a general star configuration in $\mathbb{P}^n$ but for codimension 2. In this paper, we find a graded minimal free resolution of the 2nd order symbolic power of the ideal of a general star configuration in $\mathbb{P}^n$ of any codimension r using a matroid configuration in [10]. This generalizes both the result on a *linear* star configuration in $\mathbb{P}^n$ of codimension r in [9] and the result on a general star configuration in $\mathbb{P}^n$ of *codimension* 2 in [8].

1. Introduction

In 2013, Geramita, Harbourne, and Migliore introduce a *star configuration of codimension* r *in* $\mathbb{P}^n$, which is a certain union of linear spaces $V_1, \ldots, V_k$ each of codimension r (see [9]). We call this a *linear star configuration of codimension* r *in* $\mathbb{P}^n$ in this article. The name is inspired by the fact that when $n = r = 2$ and $s = 5$, the placement of the five lines $\{L_1, \ldots, L_5\}$ that define a (linear) star configuration resembles a star. On the other hand, our more general definition of a star configuration in $\mathbb{P}^n$ with $n \geq 2$ follows [10, 14], where the geometric objects are called hypersurface configurations. In particular, the codimension 2 case was studied before the general case (see [1]). Star configurations have been shown to have many nice algebraic and geometric properties (see [10, 14]), but at the same time, can be used to exhibit extremal properties (see [2, 11]). Moreover, star configurations have arisen as objects of study in numerous research projects lately (see [3–7, 11, 13, 15, 16]).

Let $\Bbbk$ be an infinite field of any characteristic and let I be a homogeneous ideal of $R = \Bbbk[x_0, x_1, \ldots, x_n]$. For a positive integer m, let $I^{(m)}$ be the m-th

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symbolic power of I. Then $I^m \subseteq I^{(m)}$ in general. Since a general star configuration $\mathbb{X}$ of codimension r in $\mathbb{P}^n$ is a certain union of distinct hypersurface configurations $V_1, \ldots, V_k$ with none containing any of the others, and each is a complete intersection, the m-th symbolic power of the ideal $I_{\mathbb{X}}$ of the star configuration is $I^{(m)} = I_{V_1}^m \cap \cdots \cap I_{V_k}^m$.

In [14, Theorem 3.4] the authors find a graded minimal free resolution of a general star configuration in $\mathbb{P}^n$, and show that any star configuration in $\mathbb{P}^n$ is an arithmetically Cohen-Macaulay (see [9] for a linear star configuration in $\mathbb{P}^n$). In [9, Theorem 3.2], the authors find a graded minimal free resolution of the 2nd order symbolic power of the ideal of a linear star configuration in $\mathbb{P}^n$ of any codimension r. In [8, Theorem 5.3], the authors extend the result on a general star configuration in $\mathbb{P}^n$ but for codimension 2.

Here, we find a graded minimal free resolution of the 2nd order symbolic power of the ideal of a general star configuration in $\mathbb{P}^n$ of any codimension r using a matroid configuration in [10]. This generalizes both the result on a *linear* star configuration in $\mathbb{P}^n$ of codimension r in [9, Theorem 3.2] and the result on a general star configuration in $\mathbb{P}^n$ of *codimension* 2 in [8, Theorem 5.3].

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2. Preliminaries on star configurations in $\mathbb{P}^n$ and a symbolic power of an ideal

We first introduce the notion of a star configuration in $\mathbb{P}^n$.

Definition 2.1. Let $R = \Bbbk[x_0, x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$. For positive integers r and s with $1 \le r \le \min\{n, s\}$, suppose $F_1, \ldots, F_s$ are general forms in R of degrees $d_1, \ldots, d_s$, respectively. We call the variety $\mathbb{X}$ defined by the ideal

$$\bigcap_{1 \le i_1 < \cdots < i_r \le s} (F_{i_1}, \ldots, F_{i_r})$$

a star configuration in $\mathbb{P}^n$ of type (r, s). We sometimes call it *a general star configuration in* $\mathbb{P}^n$ of codimension r.

Notice that each n-forms $F_{i_1}, \ldots, F_{i_n}$ of s-general forms $F_1, \ldots, F_s$ in R defines $d_{i_1} \cdots d_{i_n}$ points in $\mathbb{P}^n$ for each $1 \le i_1 < \cdots < i_n \le s$. Thus the ideal

$$\bigcap_{1 \le i_1 < \cdots < i_n \le s} (F_{i_1}, \ldots, F_{i_n})$$

defines a finite set $\mathbb{X}$ of points in $\mathbb{P}^n$ with

$$\deg(\mathbb{X}) = \sum_{1 \le i_1 < i_2 < \cdots < i_n \le s} d_{i_1} d_{i_2} \cdots d_{i_n}.$$

Furthermore, if $F_1, \dots, F_s$ are general linear (quadratic, cubic, quartic, quintic, etc) forms in R, we call $\mathbb{X}$ *a linear* (*quadratic, cubic, quartic, quintic, etc*) *star configuration in* $\mathbb{P}^n$ *of type* (r, s), respectively.

Theorem 2.2 ([14, Theorem 2.3]). *Let* $F_1, \dots, F_s$ *be general forms in* $R = \Bbbk[x_0, x_1, \dots, x_n]$ *with* $s \geq 2$ *and* $n \geq 2$. *Then*

$$\bigcap_{1\leq j_1<\cdots<j_r\leq s}(F_{j_1}, \dots, F_{j_r}) = \sum_{1\leq i_1<\cdots<i_{r-1}\leq s}\left(\frac{\prod_{\ell=1}^{s} F_\ell}{F_{i_1}\cdots F_{i_{r-1}}}\right)$$

for $1 \leq r \leq \min\{n, s\}$.

Theorem 2.3 ([14, Theorem 3.4]). *Let* $\mathbb{X}$ *be a star configuration in* $\mathbb{P}^n$ *of type* (r, s) *defined by general forms* $F_1, \dots, F_s$ *in* $R = \Bbbk[x_0, x_1, \dots, x_n]$ *of degrees* $d_1, d_2, \dots, d_s$, *where* $2 \leq r \leq \min\{s, n\}$, *and let* $d = d_1 + \cdots + d_s$. *Then the minimal free resolution of* $I_{\mathbb{X}}$ *is*

$$0 \to \mathbb{F}_r^{(r,s)} \to \mathbb{F}_{r-1}^{(r,s)} \to \cdots \to \mathbb{F}_1^{(r,s)} \to I_{\mathbb{X}} \to 0, \tag{2.1}$$

where

$$\begin{aligned}
\mathbb{F}_r^{(r,s)} &= R^{\alpha_r^{(r,s)}}(-d),\\
\mathbb{F}_{r-1}^{(r,s)} &= \bigoplus_{1\leq i_1\leq s} R^{\alpha_{r-1}^{(r,s)}}(-(d-d_{i_1})),\\
&\vdots\\
\mathbb{F}_\ell^{(r,s)} &= \bigoplus_{1\leq i_1<\cdots<i_{r-\ell}\leq s} R^{\alpha_\ell^{(r,s)}}(-(d-(d_{i_1}+\cdots+d_{i_{r-\ell}}))),\\
&\vdots\\
\mathbb{F}_2^{(r,s)} &= \bigoplus_{1\leq i_1<\cdots<i_{r-2}\leq s} R^{\alpha_2^{(r,s)}}(-(d-(d_{i_1}+\cdots+d_{i_{r-2}}))), \quad \textit{and}\\
\mathbb{F}_1^{(r,s)} &= \bigoplus_{1\leq i_1<\cdots<i_{r-1}\leq s} R^{\alpha_1^{(r,s)}}(-(d-(d_{i_1}+\cdots+d_{i_{r-1}}))),
\end{aligned}$$

with

$$\alpha_\ell^{(r,s)} = \binom{s-r+\ell-1}{\ell-1} \quad \textit{and} \quad \operatorname{rank}\mathbb{F}_\ell^{(r,s)} = \binom{s-r+\ell-1}{\ell-1}\cdot\binom{s}{r-\ell}$$

for $1 \leq \ell \leq r$. *In particular, the last free module* $\mathbb{F}_r^{(r,s)}$ *has only one shift* d, *i.e., a star configuration* $\mathbb{X}$ *in* $\mathbb{P}^n$ *is level. Furthermore, any star configuration* $\mathbb{X}$ *in* $\mathbb{P}^n$ *is arithmetically Cohen-Macaulay.*

We now introduce the definition of symbolic power of an ideal with the notations in the introduction.

Definition 2.4. Let I be a homogeneous ideal of $R = \Bbbk[x_0, x_1, \ldots, x_n]$. The m-th *symbolic power* of I, denoted $I^{(m)}$, is defined to be

$$I^{(m)} = \bigcap_{P \in \mathrm{Ass}(I)} (I^m R_P \cap R),$$

where $\mathrm{Ass}(I)$ denotes the set of associated primes of I and R_P is the ring R localized at the prime ideal P.

Note that $I^m \subseteq I^{(m)}$ in general, but the reverse containment may fail. However, it is well known that if I is a complete intersection ideal in R, then $I^m = I^{(m)}$ for $m \geq 1$ (see [17, Appendix 6, Lemma 5]).

3. A matroid configuration and the main theorem

In this section, we shall find the Betti numbers and the shifts of a graded minimal free resolution of the 2nd order symbolic power of the ideal of a star configuration (not necessarily linear star configuration) in $\mathbb{P}^n$ of type (r, s) defined by s-general forms in $R = \Bbbk[x_0, x_1, \ldots, x_n]$ with $1 \leq r \leq \min\{n, s\}$ and $n \geq 2$.

We first introduce some important results of the 2nd order symbolic power of the ideal of a linear star configuration in $\mathbb{P}^n$ in [9, 10].

Remark 3.1 ([10, Remark 2.11]). Let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^n$ of type (r, s) with $2 \leq r \leq \min\{n, s\}$. By [10, Proposition 2.9], the Artinian reduction of the homogeneous coordinate ring of $\mathbb{X}$ is $\Bbbk[t_1, \ldots, t_r]/\boldsymbol{m}^{s-r+1}$, where $\boldsymbol{m} = (t_1, \ldots, t_r)$. Since $\boldsymbol{m}^{s-r+1}$ is generated by the maximal minor of the $(s - r + 1) \times s$ matrix

$$\begin{bmatrix} t_1 & t_2 & \cdots & t_r & 0 & \cdots & 0 & 0 \\ 0 & t_1 & t_2 & \cdots & t_r & 0 & \cdots & 0 \\ & & & \vdots & & & & \\ 0 & \cdots & 0 & t_1 & t_2 & t_3 & \cdots & t_r \end{bmatrix},$$

the graded Betti numbers of the homogeneous coordinate ring of $\mathbb{X}$ are those given by Eagon-Northcott resolution of the maximal minors of a generic matrix of size $(s - r + 1) \times s$ [12]. Denoting by $\mathbb{E}_\bullet^{(r,s)}$ a graded minimal free resolution of $I_{\mathbb{X}}$, we get that

$$\mathrm{rk}\mathbb{E}_\ell^{(r,s)} = \binom{s}{s - r + \ell} \cdot \binom{s - r + \ell - 1}{\ell - 1}.$$

Theorem 3.2 ([9, Theorem 3.2])**.** *With notation as above, let $\mathbb{X}$ be a linear star configuration in $\mathbb{P}^n$ of type (r, s). Then a graded minimal free resolution of $R/I_{\mathbb{X}}^{(2)}$ is*

$$0 \to \mathbb{F}_r \to \cdots \to \mathbb{F}_1 \to R \to R/I_{\mathbb{X}}^{(2)} \to 0,$$

where

$$\mathbb{F}_\ell = \mathbb{E}_\ell^{(s,r)}(-(s-r+1)) \oplus \mathbb{E}_{\ell-1}^{(s,r-1)}(-(s-r+1)) \oplus \mathbb{E}_\ell^{(s,r-1)}$$

for $\ell \geq 1$. *More precisely,*

$$\mathbb{F}_\ell = R^{m_\ell}(-(2s-2r-\ell-1)) \oplus R^{n_\ell}(-(s-r-\ell-1)),$$

where

$$m_\ell = \begin{cases} \binom{s}{s-r+1}, & \text{if } \ell = 1,\\ \binom{s}{s-r+\ell}\cdot\binom{s-r+\ell-1}{\ell-1} + \binom{s}{s-r+\ell}\cdot\binom{s-r+\ell-1}{\ell-2}, & \text{if } 2 \leq \ell \leq r,\end{cases}$$

and

$$n_\ell = \begin{cases} \binom{s}{s-r+\ell+1}\cdot\binom{s-r+\ell}{\ell-1}, & \text{if } 1 \leq \ell \leq r-1,\\ 0, & \text{if } \ell = r.\end{cases}$$

We recall a few of concepts for simplicial complexes. Define $[s]=\{1,2,\dots,s\}$. A *matroid* Δ on a vertax set $[s]$ is a nonempty collection of subsets of $[s]$ that is closed under inclusion and satisfies the following property. If A, B are in Δ and $|A| > |B|$, then there is some $i \in A$ such that $B \cup \{i\} \in \Delta$. We will consider Δ as a simplicial complex.

Let $S = \Bbbk[t_1,\dots,t_s]$. For a subset $A \subseteq [s]$, we write t_A for the square free monomial $\prod_{i\in A} t_i$. The *Stanley-Reisner* ideal of Δ is $I_\Delta = \langle t_A \mid A \subseteq [s], A \notin \Delta\rangle$ and the corresponding *Stanley-Reisner* ring is $\Bbbk[\Delta] = S/I_\Delta$.

Note that if we look at the minimal free S-resolution of S/I_Δ, then the entries in all the maps are monomials in the y_i. Moreover, replacing each y_i by F_i and each S by R give the minimal free resolution of $R/\varphi_*(I_\Delta)$. So the formula $\mathbb{F} \otimes_S R$ implies the following two meanings.

(a) The variable y_i in $S = \Bbbk[y_1,\dots,y_s]$ moves to a form F_i in $R = \Bbbk[x_0,x_1,\dots,x_n]$, and
(b) an S free module $\mathbb{F}_\ell$ changes to an R free module $\mathbb{F}_\ell \otimes_S R$ for $\ell \geq 1$.

Theorem 3.3 ([10, Theorem 3.3]). *Let Δ be a matroid on $[s]$ of dimension $s-r-1$. Assume $f_1,\dots,f_s \in R = \Bbbk[x_0,x_1,\dots,x_n]$ are homogeneous polynomials such that any subset of at most $r+1$ of them forms an R-regular sequence. Consider the ring homomorphism*

$$\varphi : S = \Bbbk[t_1,\dots,t_s] \to R,\ t_i \mapsto f_i.$$

Let I be an ideal of S. We write $\varphi_*(I)$ to denote the ideal in R generated by $\varphi(I)$. If $\mathbb{F}_{\Bbbk[\Delta]}$ is a graded minimal free resolution of $\Bbbk[\Delta]$ over S, then $\mathbb{F}_{\Bbbk[\Delta]} \otimes_S R$ is a graded minimal free resolution of $R/\varphi_*(I_\Delta)$ over R.

The ideal $\varphi_*(I_\Delta)$ is said to be obtained by *specialization* from the matroid ideal I_Δ. The subscheme of $\mathbb{P}^n$ defined by $\varphi_*(I_\Delta)$ is called a *matroid configuration* [10].

Notice that a linear star configuration in $\mathbb{P}^n$ is one of the matroid configuration, we shall use [10, Theorem 3.3] for the proof of this theorem. So we are now ready to find the Betti numbers and the shifts of a graded minimal free resolution of the 2nd order symbolic power of the ideal of a star configuration in $\mathbb{P}^n$.

Theorem 3.4. *Let $\mathbb{X}$ be a star configuration in $\mathbb{P}^n$ of type (r,s) defined by s-general forms $F_1,\dots,F_s$ in $R=\Bbbk[x_0,x_1,\dots,x_n]$ of degrees $d_1,\dots,d_s$ with $2\le r\le \min\{n,s\}$, and let $d=d_1+\cdots+d_s$. Then a graded minimal free resolution of $R/I_{\mathbb{X}}^{(2)}$ is*

$$0 \to \mathbb{G}_r \to \cdots \to \mathbb{G}_1 \to R \to R/I_{\mathbb{X}}^{(2)} \to 0,$$

where

$$\mathbb{G}_1 = \Big[\bigoplus_{1\le i_1<\cdots<i_{r-1}\le s} R(-2(d-(d_{i_1}+\cdots+d_{i_{r-1}})))\Big] \oplus \Big[\bigoplus_{1\le i_1<\cdots<i_{r-2}\le s} R(-(d-(d_{i_1}+\cdots+d_{i_{r-2}})))\Big],$$

$$\mathbb{G}_\ell = \Big[\bigoplus_{1\le i_1<\cdots<i_{r-\ell}\le s}\Big[\bigoplus_{k_1<\cdots<k_{\ell-1}} R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-\ell}}))-(d_{k_1}+\cdots+d_{k_{\ell-1}}))\Big]\Big] \oplus \Big[\bigoplus_{1\le i_1<\cdots<i_{(r-1)-\ell}\le s} R^{\binom{s-r+\ell}{\ell-1}}(-(d-(d_{i_1}+\cdots+d_{i_{(r-1)-\ell}})))\Big],$$

where $\{k_1,\dots,k_{\ell-1}\}$ runs through $\binom{s-(r-\ell)}{\ell-1}$-times among $\{j_1,\dots,j_{s-(r-\ell)}\}:=\{1,2,\dots,s\}-\{i_1,\dots,i_{r-\ell}\}$, and

$$\mathbb{G}_r = \bigoplus_{1\le i_1<\cdots<i_{r-1}\le s} R(-(2d-(d_{i_1}+\cdots+d_{i_{r-1}}))).$$

Proof. Let $S=\Bbbk[t_1,\dots,t_s]$. Consider the ideal of S

$$I_{(r,s)} = \bigcap_{1\le i_1<i_2<\cdots<i_r\le s} \langle t_{i_1},t_{i_2},\dots,t_{i_r}\rangle,$$

generated by all products of $s-r+1$ distinct variables in $\{t_1,\dots,t_s\}$ (see Theorem 2.2). It is the Stanley-Reisner ideal of a uniform matroid on $[s]$. Recall the map

$$\varphi : S=\Bbbk[y_1,\dots,y_s]\to R,\ y_i\mapsto F_i. \tag{3.1}$$

Then

$$I_{\mathbb{X}}^{(2)} = \varphi_*(I_{(r,s)}).$$

Notice that

$$I_{\mathbb{X}} = \sum_{1\le i_1<\cdots<i_{r-1}\le s}\left(\frac{\prod_{\ell=1}^s F_\ell}{F_{i_1}\cdots F_{i_{r-1}}}\right) \tag{3.2}$$

and the ℓ-th free module of a graded minimal free resolution of the ideal $I^{(2)}_{(r,s)}$ ([10, Theorem 3.2]) is

$$\mathbb{F}_\ell = R^{m_\ell}(-(2s-2r+\ell+1)) \oplus R^{n_\ell}(-(s-r+\ell+1)),$$

where

$$m_\ell = \begin{cases} \binom{s}{s-r+1}, & \text{if } \ell = 1, \\ \binom{s}{s-r+\ell} \cdot \binom{s-r+\ell-1}{\ell-1} + \binom{s}{s-r+\ell} \cdot \binom{s-r+\ell-1}{\ell-2}, & \text{if } 2 \le \ell \le r, \end{cases}$$

and

$$n_\ell = \begin{cases} \binom{s}{s-r+\ell+1} \cdot \binom{s-r+\ell}{\ell-1}, & \text{if } 1 \le \ell \le r-1, \\ 0, & \text{if } \ell = r. \end{cases}$$

By Theorem 3.3, the ℓ-th free module of a graded minimal free resolution of the ideal $R/I^{(2)}_{\mathbb{X}}$ is

$$\mathbb{F}_\ell \otimes_S R.$$

Recall that the maps appeared in the minimal free resolution of S/I_Δ are obtained from Eagon-Northcott resolution and the mapping cone construction from *Basic Double G-Linkage* ([9, Proposition 2.6]). As we mentioned before, the entries in all the maps in the minimal free resolution of S/I_Δ are monomials in the y_i, and replacing each y_i by F_i and each S by R gives the minimal free resolution of $R/\varphi_*(I_\Delta)$. Hence one can conclude that

$$s \overset{\varphi_*}{\mapsto} d, \quad \text{and} \quad 1 \overset{\varphi_*}{\mapsto} d_i.$$

• Let $\ell = 1$. By equation (3.2) and Remark 3.1, we have

$$\begin{aligned} \mathbb{E}_1^{r,s}(-(s-r+1)) \otimes_S R &= \left[S^{\binom{s}{r-1}}(-(s-(r-1)))\right](-(s-(r-1))) \otimes_S R \\ &= S^{\binom{s}{r-1}}(-2(s-(r-1))) \otimes_S R \\ &= \bigoplus_{1 \le i_1 < \cdots < i_{r-1} \le s} S(-2(s-(r-1))) \otimes_S R \\ &= \bigoplus_{1 \le i_1 < \cdots < i_{r-1} \le s} R(-2(d-(d_{i_1}+\cdots+d_{i_{r-1}}))), \text{ and} \end{aligned}$$

$$\begin{aligned} \mathbb{E}_1^{r-1,s} \otimes_S R &= \left[S^{\binom{s}{r-2}}(-(s-(r-2)))\right] \otimes_S R \\ &= S^{\binom{s}{r-2}}(-(s-(r-2))) \otimes_S R \\ &= \bigoplus_{1 \le i_1 < \cdots < i_{r-2} \le s} S(-2(s-(r-2))) \otimes_S R \\ &= \bigoplus_{1 \le i_1 < \cdots < i_{r-2} \le s} R(-(d-(d_1+\cdots+d_{i_{r-2}}))). \end{aligned}$$

Thus

$$\begin{aligned}\mathbb{G}_1 &= \mathbb{F}_1 \otimes_S R\\ &= \mathbb{E}_1^{r,s}(-(s-(r-1))) \otimes_S R \oplus \mathbb{E}_1^{r-1,s} \otimes_S R\\ &= \Big[\bigoplus_{1\le i_1<\cdots<i_{r-1}\le s} R(-2(d-(d_{i_1}+\cdots+d_{i_{r-1}})))\Big]\\ &\quad \oplus \Big[\bigoplus_{1\le i_1<\cdots<i_{r-2}\le s} R(-(d-(d_{i_1}+\cdots+d_{i_{r-2}})))\Big].\end{aligned}$$

- Let $1<\ell<r$. Recall that

$$\begin{aligned}\mathrm{rk}\mathbb{E}_\ell^{(r,s)} &= \binom{s}{s-(r-\ell)} \cdot \binom{s-r+\ell-1}{\ell-1},\\ \mathrm{rk}\mathbb{E}_{\ell-1}^{(r-1,s)} &= \binom{s}{s-(r-\ell)} \cdot \binom{s-r+\ell-1}{\ell-2}, \quad \text{and thus}\\ \mathrm{rk}\mathbb{E}_\ell^{(r,s)} + \mathrm{rk}\mathbb{E}_{\ell-1}^{(r-1,s)} &= \binom{s}{s-(r-\ell)} \cdot \binom{s-(r-\ell)}{\ell-1}.\end{aligned}$$

So

$$\mathbb{E}_\ell^{(r,s)} + \mathbb{E}_{\ell-1}^{(r-1,s)} = S^{\binom{s}{s-(r-\ell)}\cdot\binom{s-(r-\ell)}{\ell-1}}(s-(r-\ell)).$$

Now consider the case $\{d_{i_1},\dots,d_{i_{r-\ell}}\}$ of degrees among $\{d_1,\dots,d_s\}$. Then the complement case of the case $\{d_{i_1},\dots,d_{i_{r-\ell}}\}$ among $\{d_1,\dots,d_s\}$ is $\{d_1,\dots,d_s\}-\{d_{i_1},\dots,d_{i_{r-\ell}}\}$. So there is a one to one correspondence between two cases as

$$\{d_{i_1},\dots,d_{i_{r-\ell}}\} \leftrightarrow \{d_1,\dots,d_s\}-\{d_{i_1},\dots,d_{i_{r-\ell}}\} := \{d_{j_1},\dots,d_{j_{s-(r-\ell)}}\}.$$

Recall the map

$$\varphi : S=\Bbbk[y_1,\dots,y_s] \to R, \quad y_i \mapsto F_i, \quad \text{for every} \quad i=1,\dots,s.$$

Hence the shift $(s-(r-\ell))$ in the ℓ-th free module $\mathbb{F}_\ell$ of a graded minimal free resolution of $S/I_{(r,s)}$ changes to the shift $(d-(d_{i_1}+\cdots+d_{i_{r-\ell}}))=(d_{j_1}+\cdots+d_{d_{s-(r-\ell)}})$ in the ℓ-th free module of a graded minimal free resolution of $R/I_{\mathbb{X}}^{(2)}$. In other words, there is a one to one correspondence between two shifts as

$$\begin{aligned}(s-(r-\ell)) \overset{\varphi_*}{\mapsto}\quad & (d-(d_{i_1}+\cdots+d_{i_{r-\ell}}))\\ &= (d_{j_1}+\cdots+d_{d_{j_{s-(r-\ell)}}}), \text{ and so}\\ S^{\binom{s}{s-(r-\ell)}}(-(s-(r-\ell))) \overset{\varphi_*}{\mapsto}\quad & S^{\binom{s}{s-(r-\ell)}}(-(s-(r-\ell))) \otimes_S R\\ &= \sum_{1\le i_1<\cdots<i_{r-\ell}\le s} R(-(d-(d_{i_1}+\cdots+d_{i_{r-\ell}})))\\ &= \sum_{1\le j_1<\cdots<j_{s-(r-\ell)}\le s} R(-(d_{j_1}+\cdots+d_{j_{s-(r-\ell)}}))).\end{aligned}$$

Note that

$$(s-r+1)=(s-(r-\ell))-(\ell-1),$$

and thus

$$\begin{aligned}(s-(r-\ell))+(s-r+1)&=(s-(r-\ell))+((s-(r-\ell))-(\ell-1))\\&=2(s-(r-\ell))-(\ell-1).\end{aligned}$$

This implies that each $\binom{s-(r-\ell)}{\ell-1}$-times shift $(s-(r-\ell))$ of the ℓ-th free module $\mathbb{F}_\ell$ of a graded minimal free resolution of $S/I^{(2)}_{(r,s)}$ changes to the shifts of the ℓ-th free module $\mathbb{G}_\ell$ of a graded minimal free resolution of $R/I^{(2)}_{\mathbb{X}}$ as

$$\begin{aligned}(s-(r-\ell))+(s-r+1)&=(s-(r-\ell))+((s-(r-\ell))-(\ell-1))\\&=2(s-(r-\ell))-(\ell-1)\\&\overset{\varphi_*}{\mapsto}\quad 2(d-(d_{i_1}+\cdots+d_{i_{r-\ell}}))-(d_{k_1}+\cdots+d_{k_{\ell-1}})\\&=2(d_{j_1}+\cdots+d_{j_{s-(r-\ell)}})-(d_{k_1}+\cdots+d_{k_{\ell-1}}),\end{aligned}$$

where $\{k_1,\dots,k_{\ell-1}\}$ runs through $\binom{s-(r-\ell)}{\ell-1}$-times among

$$\{j_1,\dots,j_{s-(r-\ell)}\}:=\{1,2,\dots,s\}-\{i_1,\dots,i_{r-\ell}\}.$$

So, with notations as above

$$\begin{aligned}(3.3)\quad &\Big[S^{\binom{s}{s-(r-\ell)}\binom{s-(r-\ell)}{\ell-1}}(s-(r-\ell))\Big](-(s-r+1))\\&=S^{\binom{s}{s-(r-\ell)}\binom{s-(r-\ell)}{\ell-1}}(-(2(s-(r-\ell))-(\ell-1)))\\&\overset{\varphi_*}{\mapsto}\bigoplus_{1<i_1<\cdots<i_{r-\ell}\le s}\Big[\bigoplus_{k_1<\cdots<k_{\ell-1}}R(-(2(d-(d_{i_1}+\cdots+d_{i_{r-\ell}}))-(d_{k_1}+\cdots+d_{k_{\ell-1}})))\Big].\end{aligned}$$

Thus,

$$\begin{aligned}&\big[\mathbb{E}^{(r,s)}_\ell+\mathbb{E}^{(r-1,s)}_{\ell-1}\big](-(s-r+1))\otimes_S R\\&=\Big[S^{\binom{s}{s-(r-\ell)}\cdot\binom{s-(r-\ell)}{\ell-1}}(-(s-(r-\ell)))\Big](-(s-r+1))\otimes_S R\\&=\bigoplus_{1\le i_1<\cdots<i_{r-\ell}\le s}\Big[\bigoplus_{k_1<\cdots<k_{\ell-1}}R(-(2(d-(d_{i_1}+\cdots+d_{r-\ell}))-(d_{k_1}+\cdots+d_{k_{\ell-1}})))\Big].\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathbb{E}^{(r-1,s)}_\ell\otimes_S R\\&=\big[S^{\binom{s}{(r-1)-\ell}\cdot\binom{s-(r-1)+\ell-1}{\ell-1}}(-(s-((r-1)-\ell)))\big]\otimes_S R\\&=\Big[\bigoplus_{1\le i_1<\cdots<i_{(r-1)-\ell}\le s}S^{\binom{s-(r-1)+\ell-1}{\ell-1}}(-(s-((r-1)-\ell)))\Big]\otimes_S R\\&=\bigoplus_{1\le i_1<\cdots<i_{(r-1)-\ell}\le s}R^{\binom{s-r+\ell}{\ell-1}}(-(d-(d_1+\cdots+d_{(r-1)-\ell}))).\end{aligned}$$

Hence

$$
\begin{aligned}
\mathbb{G}_\ell &= \mathbb{F}_\ell \otimes_S R \\
&= \Big[\big[\mathbb{E}_\ell^{(r,s)}(-(s-(r-1))) \otimes_S R\big] \oplus \big[\mathbb{E}_{\ell-1}^{(r-1,s)}(-(s-(r-1))) \otimes_S R\big]\Big] \\
&\quad \oplus \big[\mathbb{E}_\ell^{(r-1,s)} \otimes_S R\big] \\
&= \Big[\bigoplus_{1\le i_1<\cdots<i_{r-\ell}\le s} \Big[\bigoplus_{k_1<\cdots<k_{\ell-1}} R(-(2(d-(d_{i_1}+\cdots+d_{r-\ell}))-(d_{k_1}+\cdots+d_{k_{\ell-1}})))\Big]\Big] \\
&\quad \oplus \Big[\bigoplus_{1\le i_1<\cdots<i_{(r-1)-\ell}\le s} R^{\binom{s-r+\ell}{\ell-1}}(-(d-(d_1+\cdots+d_{(r-1)-\ell})))\Big],
\end{aligned}
$$

where $\{k_1,\ldots,k_{\ell-1}\}$ runs through $\binom{s-(r-\ell)}{\ell-1}$-times among

$$\{j_1,\ldots,j_{s-(r-\ell)}\} := \{1,2,\ldots,s\} - \{i_1,\ldots,i_{r-\ell}\}.$$

- Let $\ell = r$. Then

$$
\begin{aligned}
&\mathbb{E}_r^{(r,s)}(-(s-(r-1))) \otimes_S R \\
&= \big[S^{\binom{s-1}{r-1}}(-s)\big](-(s-(r-1))) \otimes_S R \\
&= \big[S^{\binom{s-1}{r-1}}(-(2s-(r-1)))\big] \otimes_S R, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}_{r-1}^{(r-1,s)}(-(s-(r-1))) \otimes_S R \\
&= \big[S^{\binom{s-1}{r-2}}(-s)\big](-(s-(r-1))) \otimes_S R \\
&= \big[S^{\binom{s-1}{r-2}}(-(2s-(r-1))) \otimes_S R.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbb{G}_r &= \mathbb{F}_r \otimes_S R \\
&= \mathbb{E}_r^{(r,s)}(-(s-(r-1))) \otimes_S R \oplus \mathbb{E}_{r-1}^{(r-1,s)}(-(s-(r-1))) \otimes_S R \\
&= \big[S^{\binom{s-1}{r-1}}(-(2s-(r-1))) \otimes_S R\big] \oplus \big[S^{\binom{s-1}{r-2}}(-(2s-(r-1))) \otimes_S R\big] \\
&= S^{\binom{s}{r-1}}(-(2s-(r-1))) \otimes_S R \\
&= \bigoplus_{1\le i_1<\cdots<i_{r-1}\le s} S(-(2s-(r-1))) \otimes_S R \\
&= \bigoplus_{1\le i_1<\cdots<i_{r-1}\le s} R(-(2d-(d_{i_1}+\cdots+d_{i_{r-1}}))),
\end{aligned}
$$

as we wished.

This completes the proof. □

Example 3.5. Consider a star configuration $\mathbb{X}$ in $\mathbb{P}^n$ of type $(3,4)$ defined by general forms in $R = \Bbbk[x_0, x_1, \ldots, x_n]$ of degrees $2, 3, 5$, and 8 with $n \geq 3$. We now calculate the graded Betti numbers and the shifts of a graded minimal free resolution of $R/I_{\mathbb{X}}^{(2)}$. Let

$$d_1 = 2,\ d_2 = 3,\ d_3 = 5,\ d_4 = 8, \quad \text{and} \quad d = d_1 + d_2 + d_3 + d_4 = 18,$$

and let

$$0 \to \mathbb{G}_3 \to \mathbb{G}_2 \to \mathbb{G}_1 \to R \to R/I_{\mathbb{X}}^{(2)} \to 0$$

be a graded minimal free resolution of $R/I_{\mathbb{X}}^{(2)}$.

• First we calculate the graded Betti numbers and the shifts of the first free module $\mathbb{G}_1$. Recall that, by Theorem 3.4,

$$\mathbb{G}_1 = \left[\bigoplus_{1 \leq i_1 < i_2 \leq 4} R(-2(d - (d_{i_1} + d_{i_2}))) \right] \oplus \left[\bigoplus_{1 \leq i \leq s} R(-(d - d_i)) \right],$$

and so we get the shifts of $\mathbb{G}_1$ as follows.

$2(d-(d_{i_1}+d_{i_2}))$	
$2(d-(d_3+d_4))$	10
$2(d-(d_2+d_4))$	14
$2(d-(d_2+d_3))$	20
$2(d-(d_1+d_4))$	16
$2(d-(d_1+d_3))$	22
$2(d-(d_1+d_2))$	26

and

$(d-d_i)$	
$d-d_1$	16
$d-d_2$	15
$d-d_3$	13
$d-d_4$	10

Thus

$$\begin{aligned}\mathbb{G}_1 = {} & R(-10)^2 \oplus R(-13) \oplus R(-14) \oplus R(-15) \oplus R^2(-16) \\ & \oplus R(-20) \oplus R(-22) \oplus R(-26).\end{aligned}$$

• Let $\ell = 2$. By Theorem 3.4,

$$\mathbb{G}_2 = \left[\bigoplus_{1 \leq i \leq 4} \left[\bigoplus_{j \neq i} R(-(2(d - d_i) - d_j)) \right] \right] \oplus R^3(-d).$$

So we have the following shifts in $\mathbb{G}_2$ as

$2(d-d_i)$		$j \neq i$	$2(d-d_i)-d_j$
$2(d-d_4)$	20	d_1, d_2, d_3	$18, 17, 15$
$2(d-d_3)$	26	d_1, d_2, d_4	$24, 23, 18$
$2(d-d_2)$	30	d_1, d_3, d_4	$28, 25, 22$
$2(d-d_1)$	32	d_2, d_3, d_4	$29, 27, 24$

and

d, d, d	$18, 18, 18$

.

Hence we get that

$$\begin{aligned}\mathbb{G}_2 = {} & R(-15) \oplus R(-17) \oplus R(-18)^5 \oplus R(-22) \oplus R(-23) \oplus R(-24)^2 \\ & \oplus R(-25) \oplus R(-27) \oplus R(-28) \oplus R(-29).\end{aligned}$$

• Let $\ell = r = 3$. By Theorem 3.4,

$$\mathbb{G}_3 = \bigoplus_{1\le i_1<i_2\le 4} R(-(2d-(d_{i_1}+d_{i_2}))).$$

So we have the following shifts in $\mathbb{G}_3$ as:

$2d-(d_{i_1}+d_{i_2})$	
$2d-(d_1+d_2)$	31
$2d-(d_1+d_3)$	29
$2d-(d_1+d_4)$	26
$2d-(d_2+d_3)$	28
$2d-(d_2+d_4)$	25
$2d-(d_3+d_4)$	23

Hence we have

$$\mathbb{G}_3 = R(-23)\oplus R(-25)\oplus R(-26)\oplus R(-28)\oplus R(-29)\oplus R(-31).$$

Therefore a graded minimal free resolution of $R/I_{\mathbb{X}}^{(2)}$ is

$$\begin{aligned}
0 &\to R(-23)\oplus R(-25)\oplus R(-26)\oplus R(-28)\oplus R(-29)\oplus R(-31)\\
&\to \big[R(-15)\oplus R(-17)\oplus R(-18)^5\oplus R(-22)\oplus R(-23)\oplus R(-24)^2\\
&\quad\oplus\ R(-25)\oplus R(-27)\oplus R(-28)\oplus R(-29)\big]\\
&\to R(-10)^2\oplus R(-13)\oplus R(-14)\oplus R(-15)\oplus R^2(-16)\oplus R(-20)\\
&\quad\oplus R(-22)\oplus R(-26)\\
&\to R \ \to\ R/I_{\mathbb{X}}^{(2)} \ \to\ 0.
\end{aligned}$$

As a special case of Theorem 3.4 with codimension 2, i.e., $r = 2$, the following corollary is immediate.

Corollary 3.6 ([8, Theorem 5.3]). *Let $\mathbb{X}$ be a star configuration in $\mathbb{P}^n$ of type $(2, s)$ defined by s-general forms in $R = \Bbbk[x_0, x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_s$ with $s \ge 2$, and let $d = d_1 + \cdots + d_s$. Then a graded minimal free resolution of $R/I_{\mathbb{X}}^{(2)}$ is*

$$0 \to \bigoplus_{1\le i\le s} R(-(2d-d_i)) \to R(-d)\oplus\Big[\bigoplus_{1\le i\le s} R(-(2(d-d_i)))\Big] \to R \to R/I_{\mathbb{X}}^{(2)} \to 0.$$

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