THE RICCI CURVATURE ON DIRECTED GRAPHS

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ABSTRACT. In this paper, we consider the Ricci curvature of a *directed* graph, based on Lin-Lu-Yau's definition. We give some properties of the Ricci curvature, including conditions for a directed regular graph to be Ricci-flat. Moreover, we calculate the Ricci curvature of the cartesian product of directed graphs.

1. Introduction

The Ricci curvature is one of the most important concepts in Riemannian geometry. In space physics, Ricci-flat manifolds represent vacuum solutions to an analogue of Einstein's equation for Riemannian manifolds with vanishing cosmological constant. They are used in the theory of general relativity. In mathematics, Calabi-Yau manifolds are Ricci-flat and can be applied to the superstring theory. There are some definitions of generalized Ricci curvature, one of which is Olivier's coarse Ricci curvature (see [5], [6]). It is formulated by the 1-Wasserstein distance on a metric space (X, d) with a random walk $m = \{m_x\}_{x \in X}$, where m_x is a probability measure on X. The coarse Ricci curvature is defined as, for two distinct points $x, y \in X$,

$$\kappa(x, y) := 1 - \frac{W(m_x, m_y)}{d(x, y)},$$

where $W(m_x, m_y)$ is the 1-Wasserstein distance between m_x and m_y.

On the other hand, the graph theory is used to model many types of relations and processes in physical, biological, social and information systems (see [9], [10] and [11]). A graph $G = (V, E)$ is a pair of the set V of vertices and the set E of edges. If each edge is represented as an ordered pair of vertices, G is called a *directed* graph.

In 2010, Lin-Lu-Yau [2] defined the Ricci curvature of an undirected graph by using the coarse Ricci curvature of the lazy random walk, and they studied the Ricci curvature of the product space of graphs and random graphs. They also considered the Ricci-flat graph and classified undirected Ricci-flat graphs

with girth at least five (see [3]). In 2012, Jost and Liu [1] studied the relation between the Ricci curvature and the local clustering efficient. Recently, the Ricci curvature on graphs was applied to cancer network [7], internet topology [4] and so on. Sometimes it seems important to consider directed graphs as networks. However, curvatures on directed graphs have not yet been discussed well because it is much more difficult than the undirected case.

In this paper, we define the Ricci curvature of a *directed* graph based on Lin-Lu-Yau's definition, and state basic properties (§2). For some examples, we calculate the Ricci curvature explicitly (§3). Then giving lower and upper bounds, we obtain conditions for a directed regular graph to be Ricci-flat (Theorem 4.4). Finally, we generalize it to the cartesian product graph (Theorem 4.9).

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2. Definition of Ricci curvature on directed graphs

Throughout the paper, we always assume that a graph $G = (V, E)$ is directed. If not, it will be clearly stated. For $x, y \in V$, we write (x, y) as an edge from x to y. We denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$.

Definition 2.1. (1) A *directed path* from $x \in V(G)$ to vertex $y \in V(G)$ is a sequence of edges $\left\{ (a_i, a_{i+1}) \right\}_{i=0}^{n-1}$, where $a_0 = x$, $a_n = y$. We call n the *length* of the path.
   (2) The *distance* $d(x, y)$ between two vertices $x, y \in V$ is given by the length of a shortest directed path from x to y.

Remark 2.2. If G is *strongly connected* (i.e., there exists a directed path from x to y for any $x, y \in V$), then the distance is finite. The distance function satisfies positivity and triangle inequality, but not necessarily the symmetry.

Definition 2.3. (1) For any $x \in V$, the *out-neighborhood* of x is defined as
$$\Gamma^{\mathrm{out}}(x) := \left\{ y \in V \mid (x, y) \in E \right\}.$$
   (2) For any $x \in V$, the *out-degree* of x, denoted by d_x^{out}, is the number of edges starting from x, i.e., $d_x^{\mathrm{out}} = |\Gamma^{\mathrm{out}}(x)|$.
   (3) We call G *d-regular graph* if every vertex has the same out-degree d.

In this paper, we assume that a directed graph G has the following properties.

   (1) Locally finiteness (every vertex has a finite degree).
   (2) Simpleness (there exist no loops and no multi-edges).
   (3) Strongly connectedness.

Definition 2.4. For any $x \in V(G)$ and any $\alpha \in [0,1]$, we define a probability measure m_x^α on $V(G)$ by

$$m_x^\alpha(v) = \begin{cases} \alpha, & \text{if } v = x, \\ \dfrac{1-\alpha}{d_x^{\text{out}}}, & \text{if } (x,v) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.5. For two probability measures μ and ν on $V(G)$, the 1-Wasserstein distance between μ and ν is given by

$$W(\mu, \nu) = \inf_A \sum_{u,v \in V} A(u,v)d(u,v),$$

where $A : V(G) \times V(G) \to [0,1]$ runs over all maps satisfying

$$(2.1) \qquad \begin{cases} \sum_{v \in V} A(u,v) = \mu(u), \\ \sum_{u \in V} A(u,v) = \nu(v). \end{cases}$$

Such a map A is called a *coupling* between μ and ν.

Remark 2.6. To take the infimum of couplings in the definition of W, we should check that the set of couplings is not empty. If we take two probability measures m_x^α and m_y^α for $x, y \in V(G)$, then there always exists at least one coupling between two probability measures. In fact, we define the coupling $\bar{A}$ between m_x^α and m_y^α by

$$\bar{A}(u,v) = \begin{cases} \alpha, & \text{if } u = x, v = y, \\ \dfrac{m_x^\alpha(u)}{d_y^{\text{out}}}, & \text{if } u \in \Gamma^{\text{out}}(x), v \in \Gamma^{\text{out}}(y), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show that $\bar{A}$ satisfies (2.1).

Remark 2.7. A coupling A that attains the Wasserstein distance, does not necessarily exist since the distance is not symmetry. If it exists, we call it the *optimal coupling*.

Definition 2.8. For any two distinct vertices $x, y \in V$, the α-*Ricci curvature* of x and y is defined as

$$\kappa_\alpha(x,y) = 1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x,y)}.$$

Remark 2.9 ([2]). For any two vertices x and y, $\kappa_\alpha(x,y)$ is concave in $\alpha \in [0,1]$.

Proposition 2.10. *For any two vertices x and y, we have*

$$(2.2) \qquad W(m_x^\alpha, m_y^\alpha) \geq \sup_f \left(\sum_{u \in V} f(u)m_x^\alpha(u) - \sum_{v \in V} f(v)m_y^\alpha(v) \right),$$

where $f : V(G) \to \mathbb{R}$ runs over all functions with $f(u) - f(v) \leq d(u,v)$.

Proof. For any coupling A between m_x^α and m_y^α, we have

$$\sum_{u,v \in V} A(u,v)d(u,v) \geq \sum_{u,v \in V} A(u,v)(f(u) - f(v))$$

$$= \sum_{u \in V} f(u) \sum_{v \in V} A(u,v) - \sum_{v \in V} f(v) \sum_{u \in V} A(u,v)$$

$$= \sum_{u \in V} f(u)m_x^\alpha(u) - \sum_{v \in V} f(v)m_y^\alpha(v).$$

Since the left-hand side is independent of f, and so is the right-hand side of A, the proof is completed. $\square$

If there exists a function satisfied the equality of (2.2), then we call it the *optimal function*.

Remark 2.11. Proposition 2.10 holds for f running over all the 1-Lipshitz functions. In the case of undirected graphs, the equality holds in Proposition 2.10, and we call the proposition the Kantorovich-Rubinstein duality [8].

We would like to obtain the upper bound of $\kappa_\alpha/(1 - \alpha)$. In [2], this is obtained by using the symmetry of the 1-Wasserstein distance. However, in the case of directed graphs, the distance is not symmetry in general, so we use another approach. For any two distinct vertices x, y, we decompose $\Gamma^{\text{out}}(y)$ into the following sets according to their distance from x:

$$\Gamma_x^k(y) = \left\{ v \in \Gamma^{\text{out}}(y) \mid d(x,v) = d(x,y) - k \right\},$$

where $-1 \leq k \leq d(x,y)$, since $0 \leq d(x,v) = d(x,y) - k \leq d(x,y) + d(y,v) = d(x,y) + 1$.

Proposition 2.12. *For any two distinct vertices x, y, we have*

$$(2.3) \qquad \kappa_\alpha(x,y) \leq \frac{1-\alpha}{d(x,y)} \left(1 + \frac{1}{d_y^{\text{out}}} \sum_{k=-1}^{d(x,y)} k|\Gamma_x^k(y)| \right).$$

Proof. For a fixed $x \in V$, define $f(z) := -d(x,z)$ for $z \in V$. Then it follows that

$$f(z) - f(w) = -d(x,z) + d(x,w)$$

$$\leq d(z,w).$$

By Proposition 2.10, we have

$$W(m_x^\alpha, m_y^\alpha) \geq \sum_{z \in V} d(x,z)(m_y^\alpha(z) - m_x^\alpha(z))$$

$$= \alpha d(x,y) + \sum_{k=-1}^{d(x,y)} (d(x,y) - k)|\Gamma_x^k(y)| \frac{1-\alpha}{d_y^{\text{out}}} - (1 - \alpha)$$

$$= \alpha d(x,y) + d(x,y)(1-\alpha) - \frac{1-\alpha}{d_y^{\text{out}}} \sum_{k=-1}^{d(x,y)} k|\Gamma_x^k(y)| - (1-\alpha)$$

$$= d(x,y) - \frac{1-\alpha}{d_y^{\text{out}}} \sum_{k=-1}^{d(x,y)} k|\Gamma_x^k(y)| - (1-\alpha).$$

The proof is completed. □

Corollary 2.13. *If any edge (x,y) satisfies $(y,x) \notin E$ and*

$$\Gamma^{\text{out}}(x) \cap \Gamma^{\text{out}}(y) = \emptyset,$$

then we have

$$\kappa_\alpha(x,y) \leq 0.$$

Proof. If we take any edge (x,y), then we have

$$\kappa_\alpha(x,y) \leq 1 + \frac{|\Gamma_x^1(y)| - |\Gamma_x^{-1}(y)|}{d_y^{\text{out}}},$$

by Proposition 2.12. The first assumption implies $|\Gamma_x^1(y)| = 0$, and the second implies $|\Gamma_x^0(y)| = 0$. So, the out-degree of y is equal to $|\Gamma_x^{-1}(y)|$. Thus, we obtain

$$\kappa_\alpha(x,y) \leq 0. \qquad \square$$

Remark 2.14. Remark 2.9 implies that $h(\alpha) := \kappa_\alpha(x,y)/(1-\alpha)$ is a monotone increasing function in $\alpha \in [0,1)$ (the detail is written in the proof of Lemma 2.1 in [2]). Proposition 2.12 implies that $h(\alpha)$ is bounded. Thus, the limit $\kappa(x,y) = \lim_{\alpha \to 1} \kappa_\alpha(x,y)/(1-\alpha)$ exists.

Definition 2.15. For any two distinct vertices $x, y \in V$, the *Ricci curvature* of x and y is defined as

$$\kappa(x,y) = \lim_{\alpha \to 1} \frac{\kappa_\alpha(x,y)}{1-\alpha}.$$

Whenever we are interested in the lower bound of Ricci curvature, the following lemma implies that it is sufficient to consider the Ricci curvature of the edge, although the Ricci curvature is defined for any pair of vertices.

Proposition 2.16. *If $\kappa(u,v) \geq \kappa_0$ for any edge $(u,v) \in E(G)$, then $\kappa(x,y) \geq \kappa_0$ for any pair of vertices (x,y).*

The proof is similar to the case of undirected graphs [2] and is omitted. If $\kappa(x,y) = r \in \mathbb{R}$ holds for all edges $(x,y) \in E$, then we say that G is a *graph of constant Ricci curvature*, and write $\kappa(G) = r$. If $\kappa(G) = 0$, we say G is *Ricci-flat*.

Remark 2.17. On a Ricci-flat graph, $\kappa(x,y) = 0$ does not necessarily hold for any vertices x, y unless $(x,y) \in E$. Example 3.3 below is one of such examples.

3. Examples

In this section, we calculate the Ricci curvature on some directed graphs.

Definition 3.1. For a finite graph G, let $M = (m_{ij})$ be the matrix defined by the following:

$$m_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E, \\ 0, & \text{if } (v_i, v_j) \notin E, \end{cases}$$

where $V(G) = \{v_1, v_2, \ldots, v_n\}$. We call M the *adjacency matrix* of the graph G. Note that the adjacency matrix is not necessarily symmetric.

Example 3.2 (Complete graph K_{2n+1}). We consider a directed complete graph with the following adjacency matrix M_{2n+1}:

$$\begin{cases} m_{1,j} = 1, & j \in \{2, \ldots, n+1\}, \\ m_{1,j} = 0, & j \in \{1\} \cup \{n+2, \ldots, 2n+1\}, \\ m_{i,j} = 1, & i \in \{2, \ldots, n+1\}, j \in \{1+i, \ldots, n+i\}, \\ m_{i,j} = 0, & i \in \{2, \ldots, n+1\}, j \notin \{1+i, \ldots, n+i\}, \\ m_{i,j} = 1, & i \in \{n+2, \ldots, 2n\}, j \in \{1+i, \ldots, 2n+1\} \cup \{1, \ldots, i-n-1\}, \\ m_{i,j} = 0, & i \in \{n+2, \ldots, 2n\}, j \notin \{1+i, \ldots, 2n+1\} \cup \{1, \ldots, i-n-1\}, \\ m_{2n+1,j} = 1, & j \in \{1, \ldots, n\}, \\ m_{2n+1,j} = 0, & j \in \{n+1, \ldots, 2n+1\}. \end{cases}$$

For instance, M_3, M_5, M_7 are given by

$$M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$M_7 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By the definition of M_{2n+1}, for $j \in \{2, \ldots, n\}$, we have

$$\Gamma^{\text{out}}(v_1) \cap \Gamma^{\text{out}}(v_j) = \{v_{j+1}, \ldots, v_{n+1}\}$$

and $\Gamma^{\text{out}}(v_1) \cap \Gamma^{\text{out}}(v_{n+1}) = \emptyset$. For simplicity, we take the vertex v_1, and calculate the Ricci curvature on the edges from v_1. For $j \in \{2, \ldots, n+1\}$ and

$k \in \{1, \ldots, j-1\}$, we define a coupling A_j between $m_{v_1}^{\alpha}$ and $m_{v_j}^{\alpha}$ by

$$A_j(u,v) = \begin{cases} \alpha, & \text{if } u = v_1, v = v_j, \\ \dfrac{1-\alpha}{n}, & \text{if } u = v \in \Gamma^{\text{out}}(v_1) \cap \Gamma^{\text{out}}(v_j), \\ \dfrac{1-\alpha}{n}, & \text{if } u = v_{1+k}, v = v_{n+1+k}, \\ 0, & \text{otherwise}, \end{cases}$$

and define a function $f_j : \Gamma^{\text{out}}(v_1) \to \Gamma^{\text{out}}(v_j)$ by

$$f_j(u) = \begin{cases} 1, & \text{if } u = v_1, \\ -1, & \text{if } u \in \Gamma^{\text{out}}(v_j) \setminus \Gamma^{\text{out}}(v_1), \\ 0, & \text{otherwise}. \end{cases}$$

By using these coupling and function, for any edge $(x,y) \in E(K_{2n+1})$, the value is either one of the following.

$$\kappa(x,y) \in \left\{ 0, \ \frac{1}{n}, \ \ldots, \ \frac{n-1}{n} \right\}.$$

Example 3.3 (Cycle C_n)**.** We consider a directed cycle as follows. Let

$$V(C_n) = \{x_1, x_2, \ldots, x_n\}.$$

For any $i \in \{1, 2, \ldots, n-1\}$, let $(x_i, x_{i+1}) \in E(C_n)$ and $(x_n, x_1) \in E(C_n)$. This cycle is called a *directed cycle*. Then this is Ricci-flat, namely,

$$\kappa(C_n) = 0.$$

However, in the middle of Figure 1, $\kappa(x_1, x_5) = 5/4$, and is not zero.

Example 3.4 (Tree T)**.** In general, a tree has no strongly connected direction. However, if we consider the directed tree with $d_v^{\text{out}} = 1$ for any $v \in V(T)$, we can calculate the Ricci curvature on any edges. The Ricci curvature is given by

$$\kappa(T) = 0.$$

Example 3.5 (Ladder graph)**.** We consider an infinite graph G, called a *Ladder graph*, that is directed as follows.

Let $V(G) = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots\}$ and

$$\begin{aligned} E(G) = \ & \{(x_1, x_2), (x_4, x_3), (x_6, x_5), \ldots\} \cup \{(x_3, x_1), (x_5, x_3), \ldots\} \\ & \cup \{(x_2, x_4), (x_4, x_6), \ldots\}. \end{aligned}$$

Then, the Ricci curvature is given by

$$\kappa(u, v) \leq 0 \text{ for any } (u, v) \in E(G).$$
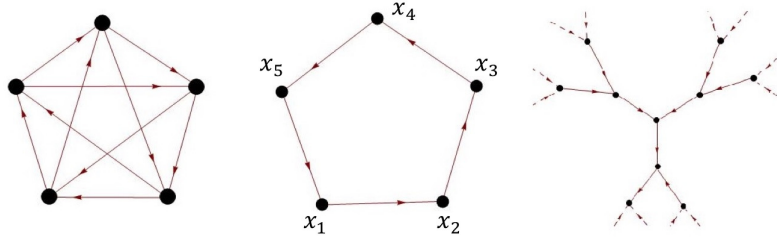
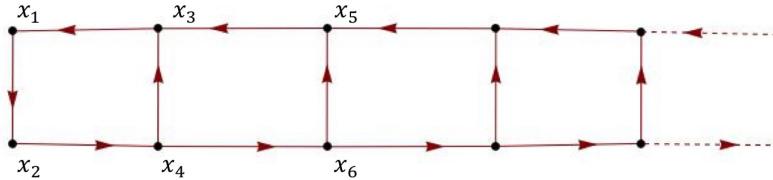FIGURE 1. Complete graph K_5, directed cycle C_5, and regular tree T_3

FIGURE 2. Ladder graph

4. Properties of Ricci curvature on a directed graph

In this section, we prove some properties of the Ricci curvature.

4.1. Conditions to be Ricci-flat graph

Proposition 4.1. *For any edge* $(x, y) \in E(G)$, *we have*

$$\kappa(x, y) \geq (1 - D)\left(1 - \frac{1}{d_x^{\text{out}}}\right),$$

where $D := \max_{u \in \Gamma^{\text{out}}(x), v \in \Gamma^{\text{out}}(y)} d(u, v)$.

Proof. We take any edge (x, y), and calculate Ricci curvature of x and y by the coupling between m_x^α and m_y^α. Our transfer plan moving m_x^α to m_y^α should be as follows:

  (1) Move the mass of α from x to y. The distance is 1.
  (2) Move the mass of $\frac{1-\alpha}{d_x^{\text{out}}}$ from y to $\Gamma^{\text{out}}(y)$. The distance is 1.
  (3) Fill gaps using the mass at $\Gamma^{\text{out}}(x) \setminus \{y\}$.
      The distance is at most $D := \max_{u \in \Gamma^{\text{out}}(x), v \in \Gamma^{\text{out}}(y)} d(u, v)$.

By this transfer plan, calculating the 1-Wasserstein distance between m_x^α and m_y^α, we have

$$W(m_x^\alpha, m_y^\alpha) \leq \alpha + \frac{1-\alpha}{d_x^{\mathrm{out}}} + \frac{D(1-\alpha)}{d_x^{\mathrm{out}}}(d_x^{\mathrm{out}} - 1)$$

$$= \alpha + D(1-\alpha) + \frac{(1-D)(1-\alpha)}{d_x^{\mathrm{out}}}.$$

Then we obtain

$$\kappa_\alpha(x, y) \geq (1-\alpha)(1-D) - \frac{(1-D)(1-\alpha)}{d_x^{\mathrm{out}}},$$

which implies

$$\kappa(x, y) \geq (1-D)\left(1 - \frac{1}{d_x^{\mathrm{out}}}\right). \qquad \square$$

By using Proposition 2.12 and Proposition 4.1, we obtain the following:

Corollary 4.2. *For an edge* $(x, y) \in E$*, we assume that* $d_x^{\mathrm{out}} = 1$ *and* $(y, x) \notin E$*. Then we have*

$$\kappa(x, y) = 0.$$

Proposition 4.3. *If there exists a bijective map* $\phi : \Gamma^{out}(x) \to \Gamma^{out}(y)$ *with* $d(u, \phi(u)) = 1$ *for any edge* (x, y) *and* $u \in \Gamma^{\mathrm{out}}(x)$*, then we have*

$$\kappa(x, y) \geq 0.$$

Proof. We take any edge (x, y), and assume that $|\Gamma^{out}(x)| = d$. By the assumption, G is a d-regular graph. We define a coupling A_0 between m_x^α and m_y^α by

$$A_0(u, v) = \begin{cases} \alpha, & \text{if } u = x, v = y, \\ \dfrac{1-\alpha}{d}, & \text{if } u \in \Gamma^{\mathrm{out}}(x), v = \phi(u), \\ 0, & \text{otherwise.} \end{cases}$$

By using this coupling, calculating the 1-Wasserstein distance between m_x^α and m_y^α, we have

$$W(m_x^\alpha, m_y^\alpha) \leq \alpha + \sum_{i=1}^{d} \frac{1-\alpha}{d}$$

$$= \alpha + (1-\alpha) = 1.$$

Thus we obtain

$$\kappa(x, y) \geq 0. \qquad \square$$

Combining Corollary 2.13 and Proposition 4.3, we obtain the following:

Theorem 4.4. *Assume that any edge* (x, y) *satisfies the following conditions*:

(1) $(y, x) \notin E$, and $\Gamma^{\mathrm{out}}(x) \cap \Gamma^{\mathrm{out}}(y) = \emptyset$.
(2) *There exists a bijective map $\phi : \Gamma^{out}(x) \to \Gamma^{out}(y)$ with $d(u, \phi(u)) = 1$
    for any $u \in \Gamma^{\mathrm{out}}(x)$.*

Then G is Ricci-flat.

Remark 4.5. We cannot replace "any" by "some" in the condition (1) of Theorem 4.4. In fact, under the condition (2), we have examples with $(y, x) \in E$ for some edge, but not all edges (Figure 3(a)), and also, $\Gamma^{\mathrm{out}}(x) \cap \Gamma^{\mathrm{out}}(y) = \emptyset$ for some (x, y), but not all (x, y) (Figure 3(b)). In fact, the Ricci curvature of (x_0, z) in Figure 3(b) is $1/2$, not zero. On the other hand, the graph in Figure 3(a) is Ricci-flat, so there exists a Ricci-flat graph that does not satisfy the conditions of Theorem 4.4.

Remark 4.6. Both conditions of Theorem 4.4 are needed. In fact, the graph in Figure 3(c) satisfies (1),but the distance from $z \in \Gamma^{\mathrm{out}}(x_0)$ to any vertex in $\Gamma^{\mathrm{out}}(y_0)$ is 2. So there do not exist bijective maps satisfying (2). On the other hand, the complete graph K_{2n+1} (Example 3.2) satisfies only the condition (2).
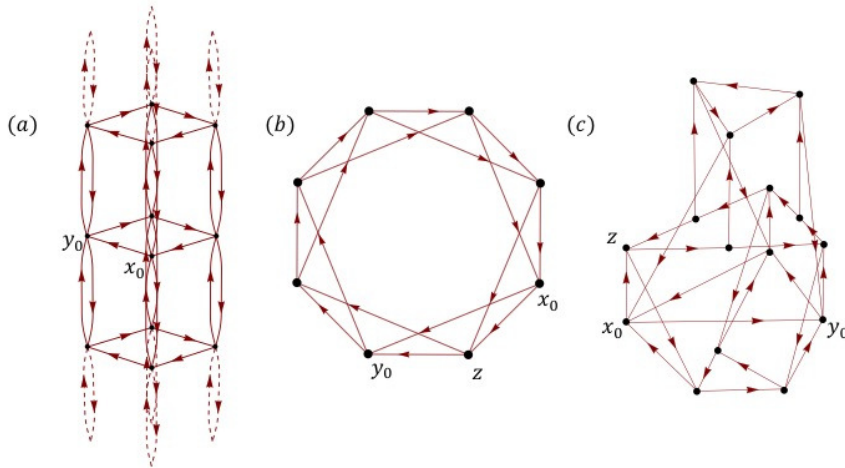

FIGURE 3. Example of Remark 4.5
and Remark 4.6

4.2. Cartesian product graph

Definition 4.7. For two directed graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the *cartesian product graph* of G and H, denoted by $G \times H$, is a directed graph over the vertex set $V(G \times H) = V(G) \times V(H)$, and (x_1, y_1),

$(x_2, y_2) \in V(G \times H)$ are connected if

$$x_1 = x_2 \text{ and } (y_1, y_2) \in E(H),$$

or

$$(x_1, x_2) \in E(G) \text{ and } y_1 = y_2.$$

Remark 4.8. If G is a d_G-regular graph and H is a d_H-regular graph, then $G \times H$ is a $(d_G + d_H)$-regular graph.

For any graph G, the Ricci curvature on G is denoted by κ^G. When we calculate the Ricci curvature on the cartesian product graph, we need to consider an optimal coupling and an optimal function. The details of the calculation are written in Theorem 3.1 in [2].

Theorem 4.9. *Assume that G satisfies the conditions of Theorem 4.4. Then for any d_H-regular graph H, we have*

$$\kappa^{G \times H}((x_1, y), (x_2, y)) = 0$$

for $(x_1, x_2) \in E(G)$ and $y \in V(H)$.

Proof. We take any edge $((x_1, y), (x_2, y)) \in E(G \times H)$, and give the coupling between $m^\alpha_{(x_1, y)}$ and $m^\alpha_{(x_2, y)}$. Since G satisfies the conditions of Theorem 4.4, by Proposition 2.13, there exists the optimal function $f(z) = -d(x_1, z)$ for $z \in V(G)$, and by Proposition 4.3, G is a d_G-regular graph and the optimal coupling A_0 is given in the proof of Proposition 4.3, i.e.,

$$1 = \sum_{z \in V} d(x_1, z)(m^\alpha_{x_2}(z) - m^\alpha_{x_1}(z)) \leq W(m^\alpha_{x_1}, m^\alpha_{x_2}) \leq \sum_{u, v \in V} A_0(u, v) d(u, v) = 1.$$

By using this coupling A_0, a map $B : V(G \times H) \times V(G \times H) \to [0, 1]$ is defined by

$B((u_1, v_1), (u_2, v_2))$

$$= \begin{cases} \dfrac{d_G A_0(x_1, x_2)}{d_G + d_H} + \dfrac{\alpha d_H}{d_G + d_H}, & \text{if } u_1 = x_1, u_2 = x_2, v_1 = v_2 = y, \\ \dfrac{d_G A_0(u_1, u_2)}{d_G + d_H}, & \text{if } u_1 \in \Gamma^{\text{out}}(x_1), u_2 = \phi(u_1), v_1 = v_2 = y, \\ \dfrac{d_H m^\alpha_y(v)}{d_G + d_H}, & \text{if } u_1 = x_1, u_2 = x_2, v_1 = v_2 = v \in \Gamma^{\text{out}}(y), \\ 0, & \text{otherwise}, \end{cases}$$

that is,

$$B((u_1, v_1), (u_2, v_2)) = \begin{cases} \alpha, & \text{if } u_1 = x_1, u_2 = x_2, v_1 = v_2 = y, \\ \dfrac{1 - \alpha}{d_G + d_H}, & \text{if } u_1 \in \Gamma^{\text{out}}(x_1), u_2 = \phi(u_1), v_1 = v_2 = y, \\ \dfrac{1 - \alpha}{d_G + d_H}, & \text{if } u_1 = x_1, u_2 = x_2, v_1 = v_2 = v \in \Gamma^{\text{out}}(y), \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to check that this map is a coupling between $m^\alpha_{(x_1,y)}$ and $m^\alpha_{(x_2,y)}$. So, the 1-Wasserstein distance satisfies

$$W(m^\alpha_{(x_1,y)}, m^\alpha_{(x_2,y)}) \le \frac{d_G}{d_G + d_H}W(m^\alpha_{x_1}, m^\alpha_{x_2}) + \frac{d_H}{d_G + d_H}.$$

Then we have

$$\kappa^{G\times H}_\alpha((x_1,y),(x_2,y)) \ge \frac{d_G}{d_G + d_H}\kappa^G_\alpha(x_1,x_2).$$

Thus, we obtain

(4.1) $$\kappa^{G\times H}_\alpha((x_1,y),(x_2,y)) \ge \frac{d_G}{d_G + d_H}\kappa^G_\alpha(x_1,x_2).$$

On the other hand, by using the optimal function, we define a function F : $V(G \times H) \times V(G \times H) \to \mathbb{R}$ by

$$F(u,v) = \begin{cases} f(u), & \text{if } v = y, \\ \dfrac{f(x_1) + f(x_2) + 1}{2}, & \text{if } u = x_1, v \ne y, \\ \dfrac{f(x_1) + f(x_2) - 1}{2}, & \text{if } u = x_2, v \ne y, \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$F(u,v) = \begin{cases} -d(x_1,u), & \text{if } v = y, \\ -1, & \text{if } u = x_2, v \ne y, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that F satisfies $F(u_1,v_1) - F(u_2,v_2) \le d((u_1,v_1),(u_2,v_2))$. By Proposition 2.10, the 1-Wasserstein distance satisfies

$$W(m^\alpha_{(x_1,y)}, m^\alpha_{(x_2,y)}) \ge \frac{d_G + \alpha d_H}{d_G + d_H}W(m^{\alpha'}_{x_1}, m^{\alpha'}_{x_2}) + (1-\alpha)\frac{d_H}{d_G + d_H},$$

where $\alpha' = \alpha(d_G + d_H)/(d_G + \alpha d_H)$. Then we have

$$\kappa^{G\times H}_\alpha((x_1,y),(x_2,y)) \le \frac{d_G + \alpha d_H}{d_G + d_H}\kappa^G_{\alpha'}(x_1,x_2).$$

Since $(1-\alpha')/(1-\alpha) = d_G/(d_G + \alpha d_H)$, we obtain

$$\kappa^{G\times H}((x_1,y),(x_2,y)) = \lim_{\alpha \to 1}\frac{1}{1-\alpha}\kappa^{G\times H}_\alpha((x_1,y),(x_2,y))$$

(4.2) $$\le \frac{d_G}{d_G + d_H}\kappa^G(x_1,x_2).$$

By (4.1) and (4.2), we have

$$\kappa^{G\times H}((x_1,y),(x_2,y)) = \frac{d_G}{d_G + d_H}\kappa^G(x_1,x_2).$$

Since G is Ricci-flat by Theorem 4.4, the proof is completed. $\qquad\square$

Corollary 4.10. *Assume that H satisfies the conditions of Theorem 4.4. Then for any d_G-regular graph G, we have*

$$\kappa^{G \times H}((x, y_1), (x, y_2)) = 0$$

for $x \in V(G)$ and $(y_1, y_2) \in E(H)$.

Combining Theorem 4.9 and Corollary 4.10, we obtain the following:

Corollary 4.11. *Assume that both G and H satisfy the conditions of Theorem 4.4. Then $G \times H$ is Ricci-flat.*

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