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HEAT KERNEL ESTIMATES FOR DIRICHLET FRACTIONAL LAPLACIAN WITH GRADIENT PERTURBATION

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Abstract. We give a direct proof of the sharp two-sided estimates, recently established in [4, 9], for the Dirichlet heat kernel of the fractional Laplacian with gradient perturbation in $C^{1,1}$ open sets by using Duhamel's formula. We also obtain a gradient estimate for the Dirichlet heat kernel. Our assumption on the open set is slightly weaker in that we only require D to be $C^{1,\theta}$ for some $\theta \in (\alpha/2, 1]$.

1. Introduction and main results

Let $X = (X_t)_{t\geqslant 0}$ be an isotropic α-stable process on $\mathbb{R}^d$ with $d \geqslant 1$ and $\alpha \in (0,2)$. The infinitesimal generator of X is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. For $f \in C_c^2(\mathbb{R}^d)$, the fractional Laplacian $\Delta^{\alpha/2}$ can be written in the following form:

$$\Delta^{\alpha/2} f(x) := \int_{\mathbb{R}^d} \left[f(x+z) - f(x) - 1_{|z|\leqslant 1} z\cdot\nabla f(x)\right] \frac{c_{d,\alpha}}{|z|^{d+\alpha}} \mathrm{d}z,$$

where $c_{d,\alpha}$ is a positive constant. It is well known that the heat kernel $p(t,x,y)$ of $\Delta^{\alpha/2}$ (or equivalently, the transition density of X) has the following estimate:

$$p(t,x,y) \asymp \frac{t}{(|x-y| + t^{1/\alpha})^{d+\alpha}}, \quad \forall (t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d.$$

Here and below, for two non-negative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 such that $c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$ in the common domain of f and g.

In [1], Bogdan and Jakubowski studied the following perturbation of $\Delta^{\alpha/2}$ by a gradient operator

$$\mathscr{L}^b := \Delta^{\alpha/2} + b\cdot\nabla$$

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in the case $d \geq 2$ and $\alpha \in (1,2)$. They assumed that the drift b belongs to the Kato class defined below.

Definition. For any real-valued function f on $\mathbb{R}^d$, define for $r > 0$

$$K_f^\alpha(r) := \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{d+1-\alpha}} \mathrm{d}y,$$

where $B(x,r)$ denotes the open ball centered at $x \in \mathbb{R}^d$ with radius r. Then f is said to belong to the Kato class $\mathbb{K}^{\alpha-1}$ if $\lim_{r\downarrow 0} K_f^\alpha(r) = 0$.

In the remainder of this paper, we will always assume that $d \geqslant 2$ and $\alpha \in (1,2)$, unless explicitly stated otherwise. Intuitively, the heat kernel $p^b(t,x,y)$ of $\mathscr{L}^b$ should satisfy the following Duhamel formula:

$$p^b(t,x,y) = p(t,x,y) + \int_0^t\!\!\int_{\mathbb{R}^d} p^b(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)\mathrm{d}z\mathrm{d}s.$$

Define $p_0^b(t,x,y) := p(t,x,y)$ and for $k \geqslant 1$,

$$p_k^b(t,x,y) := \int_0^t\!\!\int_{\mathbb{R}^d} p_{k-1}^b(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)\mathrm{d}z\mathrm{d}s.$$

The following theorem is the main result of [1].

Theorem 1.1. *Assume that $b \in \mathbb{K}^{\alpha-1}$.*

(1) *There exist $T_0 > 0$ and $C > 1$ depending on b only through the rate at which $K_{|b|}^\alpha(r)$ goes to zero such that $\sum_{k=0}^\infty p_k^b(t,x,y)$ converges locally uniformly on $(0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$ to a positive jointly continuous function $p^b(t,x,y)$ and that on $(0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$C^{-1}p(t,x,y) \leqslant p^b(t,x,y) \leqslant Cp(t,x,y).$$

Moreover, $\int_{\mathbb{R}^d} p^b(t,x,y)\mathrm{d}y = 1$ for every $t \in (0,T_0]$ and $x \in \mathbb{R}^d$.

(2) *The function $p^b(t,x,y)$ can be extended uniquely to a positive jointly continuous function on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ so that for all $s,t \in (0,\infty)$ and $(x,y) \in \mathbb{R}^d\times\mathbb{R}^d$, $\int_{\mathbb{R}^d} p^b(t,x,y)dy = 1$ and*

$$p^b(t+s,x,y) = \int_{\mathbb{R}^d} p^b(t,x,z)p^b(s,z,y)\mathrm{d}z.$$

(3) *If we define*

$$P_t^b f(x) := \int_{\mathbb{R}^d} p^b(t,x,y)f(y)\mathrm{d}y,$$

then for any $f,g \in C_c^\infty(\mathbb{R}^d)$,

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^d} t^{-1}\left(P_t^b f(x) - f(x)\right)g(x)\mathrm{d}x = \int_{\mathbb{R}^d} (\mathscr{L}^b f)(x)g(x)\mathrm{d}x.$$

Thus, $p^b(t,x,y)$ is the fundamental solution of $\mathscr{L}^b$ in the distributional sense.

Using the semigroup property, one can easily check that for any $T > 0$, there exists a constant $C > 1$ such that for all $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$C^{-1}p(t,x,y) \le p^b(t,x,y) \le Cp(t,x,y). \tag{1}$$

It follows from [4, Proposition 2.3] that $\{P_t^b, t \geqslant 0\}$ form a Feller semigroup, so there is a conservative Feller process $X^b := \{X_t^b, t \geqslant 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$ such that $P_t^b f(x) = \mathbb{E}_x[f(X_t^b)]$. The process X^b is nonsymmetric and is called an α-stable process with drift b. See also [2, 13] for the two-sided heat kernel estimates of more general non-local operators in the whole space $\mathbb{R}^d$.

For any open subset $D \subset \mathbb{R}^d$, define $\tau_D^b := \inf\{t > 0 : X_t^b \notin D\}$. We will use $X^{b,D}$ to denote the subprocess of X^b in D; that is, $X^{b,D}(\omega) := X^b(\omega)$ if $t < \tau_D^b(\omega)$ and $X^{b,D}(\omega) := \partial$ if $t \geqslant \tau_D^b(\omega)$, where ∂ is a cemetery state. Throughout this paper, we use the convention that for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. The infinitesimal generator of $X^{b,D}$ is given by $\mathscr{L}^{b,D} := \mathscr{L}^b|_D$, that is, $\mathscr{L}^b$ on D with zero exterior condition. The process $X^{b,D}$ has a joint continuous transition density $p^{b,D}(t,x,y)$ which is also the Dirichlet heat kernel for $\mathscr{L}^{b,D}$. The subprocess of X in D will be denoted by X^D and it is known to have a transition density $p^D(t,x,y)$.

Due to the complication near the boundary, sharp two-sided estimates for the Dirichlet heat kernel are much more difficult to obtain. The first sharp two-sided estimates for the Dirichlet heat kernels of discontinuous Markov processes are due to [3]. To state the related results, we first recall the definition of $C^{1,\theta}$ open sets. For $\theta \in (0, 1]$, an open set D in $\mathbb{R}^d$ is said to be $C^{1,\theta}$ if there exist $r_0 > 0$ and $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a $C^{1,\theta}$-function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \leqslant \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \leqslant \Lambda|x-z|^\theta$ and an orthonormal coordinate system $y = (y_1, \dots, y_{d-1}, y_d) := (\tilde{y}, y_d)$ such that $B(Q, r_0) \cap D = B(Q, r_0) \cap \{y : y_d > \phi(\tilde{y})\}$. The pair (r_0, Λ) is called the characteristics of the $C^{1,\theta}$ open set D. For $t > 0$ and $x, y \in D$, we define

$$q^D(t,x,y) := \Big(1 \wedge \frac{\rho(x)^{\alpha/2}}{\sqrt{t}}\Big)\Big(1 \wedge \frac{\rho(y)^{\alpha/2}}{\sqrt{t}}\Big)p(t,x,y),$$

where $\rho(x)$ denotes the distance between x and D^c. In [3], Chen, Kim and Song proved that for any $d \geqslant 1$, $\alpha \in (0, 2)$ and $T > 0$, when D is a $C^{1,1}$ open set in $\mathbb{R}^d$, there exists a constant $C > 0$ such that

$$C^{-1}q^D(t,x,y) \leqslant p^D(t,x,y) \leqslant Cq^D(t,x,y), \quad (t,x,y) \in (0,T] \times D \times D. \tag{2}$$

The above result has been generalized to $C^{1,\theta}$ open sets with $\theta \in (\alpha/2, 1]$ in [8]. As for the estimates of $p^{b,D}(t,x,y)$, the following result is proved in [4] in the case when D is a bounded $C^{1,1}$ open set. The unbounded case is due to [9]. By using the results of [8], and repeating the arguments in [4] and [9], one can get the following result.

Theorem 1.2. *Let $b \in \mathbb{K}^{\alpha-1}$, $\theta \in (\alpha/2, 1]$ and D be a $C^{1,\theta}$ open set in $\mathbb{R}^d$ with $C^{1,\theta}$ characteristics(r_0, Λ). Then for any $T > 0$, there exists a constant*

$C = C(T, r_0, \Lambda, d, \alpha, \theta, b) > 1$ *which depends on* b *only via the rate at which* $K^{\alpha}_{|b|}(r)$ *tends to zero such that for all* $(t, x, y) \in (0, T] \times D \times D$,

$$C^{-1} q^D(t, x, y) \leqslant p^{b,D}(t, x, y) \leqslant C q^D(t, x, y).$$

One might think that the estimates in Theorem 1.2 can be obtained from the estimates (2) for $p^D(t, x, y)$ using the following Duhamel formula:

$$p^{b,D}(t, x, y) = p^D(t, x, y) + \int_0^t \int_D p^{b,D}(t - s, x, z) b(z) \cdot \nabla_z p^D(s, z, y) \mathrm{d}z \mathrm{d}s. \tag{3}$$

However, unlike the whole space case, there was no good estimate on the gradient $\nabla_z p^D(t, z, y)$ of $p^D(t, z, y)$, so the approach mentioned above could not be carried through. Another obstacle to carrying out the approach above in the present case is that the following form of 3-P inequality: there exists $C > 0$ such that for any $0 < s < t$ and $x, y, z \in D$,

$$\frac{p^D(t - s, x, z) p^D(s, z, y)}{p^D(t, x, y)} \leq C\big(p^D(t - s, x, z) + p^D(s, z, y)\big), \tag{4}$$

does not hold (see [5, Remark 2.3]). A whole space analog of the inequality above played a crucial role in proving the estimates in Theorem 1.1. Partly due to the two reasons mentioned above, Theorem 1.2 was much more difficult to prove than Theorem 1.1. To get around the difficulties mentioned above, [4, 9] used the Duhamel formula for the Green function of $X^{b,D}$ and the probabilistic road-map designed in [3] for establishing the estimates (2).

In the recent paper [10], Kulczycki and Ryznar proved the following gradient estimate for $p^D(t, x, y)$ (see [10, Theorem 1.1 and Corollary 1.2]): there exists a constant $C_1 = C_1(d, \alpha) > 0$ such that for any open set $D \subset \mathbb{R}^d$ and all $(t, x, y) \in (0, 1] \times D \times D$,

$$|\nabla_x p^D(t, x, y)| \leqslant \frac{C_1}{\rho(x) \wedge t^{1/\alpha}} p^D(t, x, y).$$

It follows immediately that for any $T > 0$, there exists a constant $C_2 = C_2(d, \alpha, T) > 0$ such that for any open set $D \subset \mathbb{R}^d$ and all $(t, x, y) \in (0, T] \times D \times D$,

$$|\nabla_x p^D(t, x, y)| \leqslant \frac{C_2}{\rho(x) \wedge t^{1/\alpha}} p^D(t, x, y). \tag{5}$$

In this paper, we will use (5) and the Duhamel formula (3) to give a direct proof of Theorem 1.2. In fact, we will establish two-sided estimates for $p^{b,D}$ with b in a certain local Kato class and D being a $C^{1,\theta}$ open set with $\theta \in (\alpha/2, 1]$ instead of $C^{1,1}$ open set. We also prove a gradient estimate for $p^{b,D}(t, x, y)$, which is of independent interest.

To state our main results, we first introduce the following local Kato class.

Definition. Let D be any open set in $\mathbb{R}^d$. For any real-valued function f defined on D, we define for every $r > 0$,

$$K_f^{\alpha,D}(r) := \sup_{x\in D}\int_{D\cap B(x,r)} \frac{|f(y)|}{|x-y|^{d+1-\alpha}}\mathrm{d}y.$$

Then f is said to belong to the local Kato class $\mathbb{K}_D^{\alpha-1}$ if $\lim_{r\downarrow 0} K_f^{\alpha,D}(r) = 0$.

Remark 1.3. Using the maximum principle (see [7, Theorem 5.2.2]) it is easy to check that a function $b : D \to \mathbb{R}^d$ belongs to $\mathbb{K}_D^{\alpha-1}$ if and only $b1_D$ belongs to $\mathbb{K}^{\alpha-1}$.

In the remainder of this paper, for any $b : D \to \mathbb{R}^d$ belonging to $\mathbb{K}_D^{\alpha-1}$, we will use X^b to denote the α-stable process with drift $b1_D$. For any open set $D \subset \mathbb{R}^d$, we will use $X^{b,D}$ to denote the subprocess of X^b killed upon exiting D.

The following is the main result of this paper.

Theorem 1.4. *Let D be a $C^{1,\theta}$ open subset of $\mathbb{R}^d$ with $\theta \in (\alpha/2, 1]$ and $b : D \to \mathbb{R}^d$ belongs to $\mathbb{K}_D^{\alpha-1}$. Then there exists a unique function $\widetilde{p}^{b,D}(t,x,y)$ on $(0,\infty)\times D\times D$ satisfying* (3) *and the following: for any $T > 0$, there exists a constant $C_1 > 1$ such that for all $t \in (0,T]$ and $x, y \in D$,*

$$C_1^{-1} q^D(t,x,y) \leqslant \widetilde{p}^{b,D}(t,x,y) \leqslant C_1 q^D(t,x,y). \tag{6}$$

Moreover, the following properties hold:

(i) *for any $T > 0$, there exists a constant $C_2 > 0$ such that for all $t \in (0,T]$ and $x, y \in D$,*

$$|\nabla_x \widetilde{p}^{b,D}(t,x,y)| \leqslant \frac{C_2}{\rho(x)\wedge t^{1/\alpha}} p^D(t,x,y), \tag{7}$$

and $\widetilde{p}^{b,D}(t,x,y)$ also satisfies

$$\widetilde{p}^{b,D}(t,x,y) = p^D(t,x,y) + \int_0^t\int_D p^D(t-s,x,z)b(z)\cdot\nabla_z \widetilde{p}^{b,D}(s,z,y)\mathrm{d}z\mathrm{d}s; \tag{8}$$

(ii) *for all $0 < s < t$ and $x, y \in D$, the following Chapman-Kolmogorov's equation holds:*

$$\int_D \widetilde{p}^{b,D}(t-s,x,z)\widetilde{p}^{b,D}(s,z,y)\mathrm{d}z = \widetilde{p}^{b,D}(t,x,y); \tag{9}$$

(iii) *for any $f \in C_c^2(D)$, we have*

$$P_t^{b,D} f(x) = f(x) + \int_0^t P_{t-s}^{b,D}\mathscr{L}^{b,D} f(x)\mathrm{d}s, \tag{10}$$

where $P_t^{b,D} f(x) := \int_D \widetilde{p}^{b,D}(t,x,y)f(y)\mathrm{d}y$;

(iv) *for any uniformly continuous function $f(x)$ with compact supports, we have*

$$\lim_{t\downarrow 0}\|P_t^{b,D} f - f\|_\infty = 0. \tag{11}$$

Remark 1.5. By Lemma 2.2 below and Theorem 1.2, we know that $p^{b,D}(t,x,y)$ also satisfy (3) and (6). Thus, by the uniqueness in Theorem 1.4, we have that $\widetilde{p}^{b,D}(t,x,y)$ is the transition density $X^{b,D}$, that is, $\widetilde{p}^{b,D}(t,x,y) = p^{b,D}(t,x,y)$.

As an application of our heat kernel estimates, we can get the following Harnack inequality on the semigroup $P_t^{b,D}$, which may be used to study the long time behavior of the process, see, for example, [11, 12].

Corollary 1.6. *There exists a constant* C *such that for any non-negative function* $f \in \mathscr{B}_b(D)$*,* $T > 0$ *and* $x, y \in D$*, we have*

$$P_T^{b,D} f(x) \leqslant C \left(1 \vee \frac{\rho(x)}{\rho(y)}\right)^{\alpha/2} \left(1 \vee \frac{|x-y|}{(T \wedge 1)^{1/\alpha}}\right)^{d+\alpha} P_T^{b,D} f(y). \tag{12}$$

The remainder of this paper is organized as follows. In Section 2, we prepare some important inequalities for latter use; the proof of the main result, Theorem 1.4, will be given in Section 3.

We conclude this introduction by spelling out some conventions that will be used throughout this paper. The letter C with or without subscripts will denote an unimportant constant and $f \preceq g$ means that $f \leqslant Cg$ for some $C \geqslant 1$. The letter $\mathbb{N}$ will denote the collection of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will use $:=$ to denote a definition, $\mathscr{B}_b(D)$ to denote the space of all bounded Borel measurable functions on D and we assume that all the functions considered in this paper are Borel measurable.

2. Preliminaries

By combining [1, Corollary 12] with Remark 1.3, we immediately get the following equivalent characterization of $\mathbb{K}_D^{\alpha-1}$, which will be used in the proof of our heat kernel estimates.

Lemma 2.1. *Let* $\beta > \frac{\alpha-1}{\alpha}$*. A function* f *belongs to* $\mathbb{K}_D^{\alpha-1}$ *if and only if*

$$\lim_{t\to 0} \sup_{x\in D} \int_D \left(\frac{1}{|y-x|^{d+1-\alpha}} \wedge \frac{t^\beta}{|y-x|^{d+1-\alpha+\alpha\beta}} \right) |f(y)| \mathrm{d}y = 0.$$

The next result says that if $b \in \mathbb{K}^{\alpha-1}$, then the density $p^{b,D}(t,x,y)$ of $X_t^{b,D}$ does satisfy the Duhamel formula (3). This result will used in the proof of our main result.

Lemma 2.2. *Assume that* $b \in \mathbb{K}^{\alpha-1}$ *and* D *is an open subset of* $\mathbb{R}^d$*. Then the transition density* $p^{b,D}(t,x,y)$ *of* $X_t^{b,D}$ *satisfies* (3).

Proof. Let us choose a function $\phi \in C_c^\infty\left((0,\infty) \times \mathbb{R}^d\right)$ with $\mathrm{Supp}[\phi] \subset (0,1) \times B(0,1)$ and $\int_0^\infty \int_{\mathbb{R}^d} \phi(r,y) \mathrm{d}y \mathrm{d}r = 1$. Fix $t > 0$, for any $\psi \in C_c(D)$, define $f(s,x) := P_{t-s}^D \psi(x)$ and $f_n := \phi_n * f$, where $\phi_n(r,y) = n^{d+1}\phi(nr, ny)$. Let D_j be a sequence of relatively compact open subsets of D such that $D_j \subset \overline{D_j} \subset$

D_{j+1} for all $j \geq 1$ and $D_j \uparrow D$. Let $\tau^b_{D_j} := \inf\{t > 0 : X^b_t \in D^c_j\}$. It follows from [6] that X^b is a weak solution of the stochastic differential equation

$$dX^b_t = dX_t + b(X^b_t)dt.$$

Thus by Itô's formula, we have for sufficiently large n,

$$\begin{aligned}
&\mathbb{E}\big[f_n\big(t \wedge \tau^b_{D_j}, X_{t\wedge\tau^b_{D_j}}\big)\big] - f_n(0,x)\\
&= \int_0^t P^{b,D_j}_s \big[\partial_s f_n + \Delta^{\alpha/2} f_n + b \cdot \nabla f_n\big](s,x)\mathrm{d}s\\
&= \int_0^t P^{b,D_j}_s \big[\phi_n * \big(\partial_s f + \Delta^{\alpha/2} f\big) + b \cdot \phi_n * \nabla f\big](s,x)\mathrm{d}s.
\end{aligned}$$

Since $b \in \mathbb{K}^{\alpha-1}$ and

$$p^{b,D_j}(t,x,y) \leq p^b(t,x,y),$$

applying (5), (1) and letting $n \to \infty$ we get

$$\mathbb{E}\big[f\big(t \wedge \tau^b_{D_j}, X_{t\wedge\tau^b_{D_j}}\big)\big] = f(0,x) + \int_0^t P^{b,D_j}_s \big(b \cdot \nabla f\big)(s,x)\mathrm{d}s.$$

Note that $f(t,x) = \psi(x)$ and $f(0,x) = P^D_t\psi(x)$, taking $j \to \infty$ we arrive at

$$P^{b,D}_t \psi(x) = P^D_t \psi(x) + \int_0^t P^{b,D}_s \big(b \cdot \nabla P^D_{t-s}\psi\big)(x)\mathrm{d}s,$$

which in turn means the desired result. □

For any $\gamma \in \mathbb{R}$, we define for $t > 0$ and $x \in \mathbb{R}^d$,

$$\varrho^\gamma(t,x) := \frac{t^\gamma}{(|x| + t^{1/\alpha})^{d+\alpha}}$$

and

$$\widetilde{q}(t,x) := 1 \wedge \frac{\rho(x)^{\alpha/2}}{\sqrt{t}}.$$

The following easy result will be used several times below.

Lemma 2.3. *For every $-1 < \gamma < d/\alpha$, $t > 0$ and $x \in \mathbb{R}^d$,*

$$\int_0^t \varrho^\gamma(s,x)\mathrm{d}s \preceq \frac{1}{|x|^{d-\alpha\gamma}} \wedge \frac{t^{1+\gamma}}{|x|^{d+\alpha}}. \tag{13}$$

Proof. If $|x| \geqslant t^{1/\alpha}$, we have

$$\int_0^t \varrho^\gamma(s,x)\mathrm{d}s \leqslant \int_0^t \frac{s^\gamma}{|x|^{d+\alpha}}\mathrm{d}s \preceq \frac{t^{1+\gamma}}{|x|^{d+\alpha}}.$$

If $|x| < t^{1/\alpha}$, we have

$$\int_0^t \varrho^\gamma(s,x)\mathrm{d}s \leqslant \int_0^{|x|^\alpha} \frac{s^\gamma}{|x|^{d+\alpha}}\mathrm{d}s + \int_{|x|^\alpha}^\infty s^{\gamma-\frac{d+\alpha}{\alpha}}\mathrm{d}s \preceq \frac{1}{|x|^{d-\alpha\gamma}}.$$

Combining the above computations, we get the desired result. □

In the remainder of this section, we fix an arbitrary $T > 0$ and assume that D is a $C^{1,\theta}$ open set in $\mathbb{R}^d$ with $\theta \in (\alpha/2, 1]$. Recall that $p^D(t,x,y)$ is the transition density of X^D and it holds that (see [8]) for all $(t,x,y) \in (0,T] \times D \times D$,

$$p^D(t,x,y) \asymp \widetilde{q}(t,x)\widetilde{q}(t,y)\varrho^1(t,x-y). \tag{14}$$

Although the classical 3-P inequality of the form (4) does not hold for $p^D(t,x,y)$ we do have the following generalized 3-P type inequality.

Lemma 2.4. *For any $0 < s \le t \le T$ and $x, y, z \in D$, we have*

$$\frac{p^D(t-s,x,z)p^D(s,z,y)}{p^D(t,x,y)} \preceq \rho(z)^\alpha \Big(\varrho^0(t-s,x-z) + \varrho^0(s,z-y)\Big). \tag{15}$$

Proof. Note that

$$\big(|x-y| + t^{1/\alpha}\big)^{d+\alpha} \preceq \big(|x-z| + (t-s)^{1/\alpha}\big)^{d+\alpha} + \big(|z-y| + s^{1/\alpha}\big)^{d+\alpha}.$$

Thus

$$\begin{aligned}\frac{\varrho^1(t-s,x-z)\varrho^1(s,z-y)}{\varrho^1(t,x-y)} &= \frac{(t-s)s}{t} \cdot \frac{\varrho^0(t-s,x-z)\varrho^0(s,z-y)}{\varrho^0(t,x-y)} \\ &\preceq \frac{(t-s)s}{t} \cdot \Big(\varrho^0(t-s,x-z) + \varrho^0(s,z-y)\Big). \end{aligned} \tag{16}$$

It is obvious that

$$\sqrt{s} \cdot \widetilde{q}(s,x) \leqslant \sqrt{t} \cdot \widetilde{q}(t,x).$$

Combining this with (14) we can derive that

$$\begin{aligned} &\frac{p^D(t-s,x,z)p^D(s,z,y)}{p^D(t,x,y)} \\ &\preceq \frac{\rho(z)^\alpha}{\sqrt{(t-s)s}} \cdot \frac{t}{\sqrt{(t-s)s}} \cdot \frac{\varrho^1(t-s,x-z)\varrho^1(s,z-y)}{\varrho^1(t,x-y)} \\ &\preceq \frac{\rho(z)^\alpha}{\sqrt{(t-s)s}} \cdot \frac{t}{\sqrt{(t-s)s}} \cdot \frac{(t-s)s}{t} \cdot \Big(\varrho^0(t-s,x-z) + \varrho^0(s,z-y)\Big) \\ &= \rho(z)^\alpha \Big(\varrho^0(t-s,x-z) + \varrho^0(s,z-y)\Big). \end{aligned}$$

The proof is complete. □

We will also need the following generalized integral inequality.

Lemma 2.5. *For any $t \in (0,T]$ and $y, z \in D$, we have*

$$\widetilde{q}(t,z)\int_0^{t/2} s^{-1/\alpha} p^D(s,z,y)\mathrm{d}s \preceq \widetilde{q}(t,y)\int_0^{t/2} s^{-1/\alpha}\widetilde{q}(s,z)\varrho^1(s,z-y)\mathrm{d}s. \tag{17}$$

Proof. It can be easily checked that (17) holds when $\rho(y) \geqslant (t/2)^{1/\alpha}$ or $\rho(z) \leqslant 2\rho(y)$. So we will assume $\rho(y) < (t/2)^{1/\alpha} \wedge (\rho(z)/2)$ throughout this proof. Note that in this case, we have

$$|z-y| \geqslant \rho(z) - \rho(y) \geqslant \frac{\rho(z)}{2} \geqslant \rho(y).$$

For convenience, we define

$$\mathbf{L} := \widetilde{q}(t,z)\int_0^{t/2} s^{-1/\alpha} p^D(s,z,y)\mathrm{d}s$$

and

$$\mathbf{R} := \widetilde{q}(t,y)\int_0^{t/2} s^{-1/\alpha}\widetilde{q}(s,z)\varrho^1(s,z-y)\mathrm{d}s.$$

We deal with three different cases separately.
Case 1: $(t/2)^{1/\alpha} < \rho(z)/2 < |z-y|$. In this case, we have

$$\begin{aligned}\mathbf{L} &\preceq \int_0^{t/2} s^{-1/\alpha}\left(1\wedge\frac{\rho(y)^{\alpha/2}}{\sqrt{s}}\right)\frac{s}{|z-y|^{d+\alpha}}\mathrm{d}s\\ &= \int_0^{\rho(y)^\alpha} s^{-1/\alpha}\frac{s}{|z-y|^{d+\alpha}}\mathrm{d}s + \int_{\rho(y)^\alpha}^{t/2} s^{-1/\alpha}\frac{\rho(y)^{\alpha/2}}{\sqrt{s}}\cdot\frac{s}{|z-y|^{d+\alpha}}\mathrm{d}s\\ &\asymp \frac{\rho(y)^{\alpha/2}}{|z-y|^{d+\alpha}}t^{3/2-1/\alpha} \qquad (18)\end{aligned}$$

and

$$\mathbf{R} \asymp \frac{\rho(y)^{\alpha/2}}{\sqrt{t}}\int_0^{t/2} s^{-1/\alpha}\frac{s}{|z-y|^{d+\alpha}}\mathrm{d}s \asymp \frac{\rho(y)^{\alpha/2}}{|z-y|^{d+\alpha}}t^{3/2-1/\alpha}.$$

Thus, we have $\mathbf{L}\preceq\mathbf{R}$ in this case.
Case 2: $\rho(z)/2 \leqslant (t/2)^{1/\alpha} < |z-y|$. By using the same argument as in (18), we can get that

$$\begin{aligned}\mathbf{L} &\preceq \frac{\rho(z)^{\alpha/2}}{\sqrt{t}}\int_0^{t/2} s^{-1/\alpha}\left(1\wedge\frac{\rho(y)^{\alpha/2}}{\sqrt{s}}\right)\frac{s}{|z-y|^{d+\alpha}}\mathrm{d}s\\ &\asymp \frac{\rho(z)^{\alpha/2}}{\sqrt{t}}\cdot\frac{\rho(y)^{\alpha/2}}{|z-y|^{d+\alpha}}t^{3/2-1/\alpha}\end{aligned}$$

and

$$\begin{aligned}\mathbf{R} &\asymp \frac{\rho(y)^{\alpha/2}}{\sqrt{t}}\int_0^{t/2} s^{-1/\alpha}\left(1\wedge\frac{\rho(z)^{\alpha/2}}{\sqrt{s}}\right)\frac{s}{|z-y|^{d+\alpha}}\mathrm{d}s\\ &\asymp \frac{\rho(z)^{\alpha/2}}{\sqrt{t}}\cdot\frac{\rho(y)^{\alpha/2}}{|z-y|^{d+\alpha}}t^{3/2-1/\alpha}.\end{aligned}$$

Thus, we also have $\mathbf{L}\preceq\mathbf{R}$ in this case.
Case 3: $\rho(z)/2 \leqslant |z-y| \leqslant (t/2)^{1/\alpha}$. In this case, we have

$$\begin{aligned}\mathbf{L} &\preceq \frac{\rho(z)^{\alpha/2}}{\sqrt{t}}\int_0^{t/2} s^{-1/\alpha}\left(1\wedge\frac{\rho(y)^{\alpha/2}}{\sqrt{s}}\right)\left(s^{-d/\alpha}\wedge\frac{s}{|z-y|^{d+\alpha}}\right)\mathrm{d}s\\ &\asymp \frac{\rho(z)^{\alpha/2}}{\sqrt{t}}\left(\int_0^{\rho(y)^\alpha} s^{-1/\alpha}\frac{s}{|z-y|^{d+\alpha}}\mathrm{d}s + \int_{\rho(y)^\alpha}^{|z-y|^\alpha} s^{-1/\alpha}\frac{\rho(y)^{\alpha/2}}{\sqrt{s}}\right.\end{aligned}$$

$$\times \frac{s}{|z-y|^{d+\alpha}}\mathrm{d}s + \int_{|z-y|^\alpha}^{t/2} s^{-1/\alpha}\frac{\rho(y)^{\alpha/2}}{\sqrt{s}}s^{-d/\alpha}\mathrm{d}s\Big)$$

$$\asymp \frac{\rho(z)^{\alpha/2}\cdot\rho(y)^{\alpha/2}}{\sqrt{t}\cdot|z-y|^{d+1-\alpha/2}} - \frac{\rho(z)^{\alpha/2}\cdot\rho(y)^{\alpha/2}}{t^{(d+1)/\alpha}}.$$

Using the same idea, we can also get

$$\mathbf{R} \asymp \frac{\rho(y)^{\alpha/2}}{\sqrt{t}}\int_0^{t/2} s^{-1/\alpha}\left(1\wedge\frac{\rho(z)^{\alpha/2}}{\sqrt{s}}\right)\left(s^{-d/\alpha}\wedge\frac{s}{|z-y|^{d+\alpha}}\right)\mathrm{d}s$$

$$\asymp \frac{\rho(z)^{\alpha/2}\cdot\rho(y)^{\alpha/2}}{\sqrt{t}\cdot|z-y|^{d+1-\alpha/2}} - \frac{\rho(z)^{\alpha/2}\cdot\rho(y)^{\alpha/2}}{t^{(d+1)/\alpha}}.$$

Thus, $\mathbf{L} \preceq \mathbf{R}$ is true. The proof is now complete. □

3. Proof of Theorem 1.4

Throughout this section, unless specified otherwise, we always assume that $b : D \to \mathbb{R}^d$ belongs to $\mathbb{K}_D^{\alpha-1}$. The following lemma plays an important role in proving our main result.

Lemma 3.1. *Let $T > 0$. For any $t \in (0, T]$, there exists a constant $C(t) = C(t, b) > 0$ such that for all $x, y \in D$, we have*

$$\int_0^t\int_D p^D(t-s,x,z)|b(z)|\cdot|\nabla_z p^D(s,z,y)|\mathrm{d}z\mathrm{d}s \leqslant C(t)p^D(t,x,y),$$

where $C(t)$ is nondecreasing in t and $C(t) \to 0$ as $t \to 0$.

Proof. Define

$$\mathcal{I} := \frac{1}{p^D(t,x,y)}\int_0^t p^D(t-s,x,z)\cdot|\nabla_z p^D(s,z,y)|\mathrm{d}s.$$

Then, by (5), we have

$$\mathcal{I} \preceq \int_0^t \frac{1}{\rho(z)}1_{[\rho(z)<s^{1/\alpha}\wedge(t-s)^{1/\alpha}]}\frac{p^D(t-s,x,z)p^D(s,z,y)}{p^D(t,x,y)}\mathrm{d}s$$

$$+ \int_0^t \frac{1}{\rho(z)\wedge s^{1/\alpha}}1_{[\rho(z)\geqslant s^{1/\alpha}\wedge(t-s)^{1/\alpha}]}\frac{p^D(t-s,x,z)p^D(s,z,y)}{p^D(t,x,y)}\mathrm{d}s$$

$$=: \mathcal{I}_1 + \mathcal{I}_2.$$

On one hand, we have by (15) that

$$\mathcal{I}_1 \preceq \int_0^t \rho(z)^{\alpha-1}1_{[\rho(z)<s^{1/\alpha}\wedge(t-s)^{1/\alpha}]}\Big(\varrho^0(t-s,x-z)+\varrho^0(s,z-y)\Big)\mathrm{d}s$$

$$\leqslant \int_0^t (t-s)^{1-1/\alpha}\varrho^0(t-s,x-z)\mathrm{d}s + \int_0^t s^{1-1/\alpha}\varrho^0(s,z-y)\mathrm{d}s$$

$$= \int_0^t s^{-1/\alpha}\Big(\varrho^1(s,x-z)+\varrho^1(s,z-y)\Big)\mathrm{d}s.$$

We proceed to show that $\mathcal{I}_2$ has the same estimate:

$$\mathcal{I}_2 \preceq \int_0^t s^{-1/\alpha}\Big(\varrho^1(s,x-z)+\varrho^1(s,z-y)\Big)\mathrm{d}s. \tag{19}$$

Since

$$\frac{1}{\rho(z)\wedge s^{1/\alpha}}1_{[\rho(z)\geqslant s^{1/\alpha}\wedge(t-s)^{1/\alpha}]}\leqslant \frac{1}{s^{1/\alpha}}+\frac{1}{(t-s)^{1/\alpha}},$$

we have

$$\mathcal{I}_2 \preceq \int_0^t \left(\frac{1}{s^{1/\alpha}}+\frac{1}{(t-s)^{1/\alpha}}\right)\frac{p^D(t-s,x,z)p^D(s,z,y)}{p^D(t,x,y)}\mathrm{d}s.$$

Using the symmetry in s and $t-s$, we only need to prove that

$$\begin{aligned}\hat{\mathcal{I}}_2 &:= \int_0^t \frac{1}{s^{1/\alpha}}\frac{p^D(t-s,x,z)p^D(s,z,y)}{p^D(t,x,y)}\mathrm{d}s\\ &\preceq \int_0^t \frac{1}{s^{1/\alpha}}\Big(\varrho^1(s,x-z)+\varrho^1(s,z-y)\Big)\mathrm{d}s.\end{aligned}$$

By (17), we have

$$\begin{aligned}\hat{\mathcal{I}}_2 &\preceq \frac{p^D(t,x,z)}{p^D(t,x,y)}\int_0^{t/2} s^{-1/\alpha}p^D(s,z,y)\mathrm{d}s\\ &\quad+\frac{p^D(t,z,y)}{p^D(t,x,y)}\int_{t/2}^t s^{-1/\alpha}p^D(t-s,x,z)\mathrm{d}s\\ &\preceq \frac{\widetilde{q}(t,x)\widetilde{q}(t,y)}{p^D(t,x,y)}\varrho^1(t,x-z)\int_0^{t/2} s^{-1/\alpha}\widetilde{q}(s,z)\varrho^1(s,z-y)\mathrm{d}s\\ &\quad+\frac{\widetilde{q}(t,x)\widetilde{q}(t,y)}{p^D(t,x,y)}\varrho^1(t,z-y)\int_0^{t/2} s^{-1/\alpha}\widetilde{q}(s,z)\varrho^1(s,x-z)\mathrm{d}s\\ &=: \hat{\mathcal{I}}_{21}+\hat{\mathcal{I}}_{22},\end{aligned}$$

where we have used a change of variables and the facts that $\rho^1(t,x-z)\asymp\rho^1(t-s,x-z)$ for $s\in(0,t/2)$. It suffices to take care of one of the two terms of the right hand side above, the other term can be handled in a similar fashion. By (16), we have

$$\begin{aligned}\hat{\mathcal{I}}_{21} &\preceq \frac{\widetilde{q}(t,x)\widetilde{q}(t,y)}{p^D(t,x,y)}\int_0^{t/2} s^{-1/\alpha}\varrho^1(t-s,x-z)\varrho^1(s,z-y)\mathrm{d}s\\ &\preceq \int_0^{t/2} s^{-1/\alpha}\frac{\varrho^1(t-s,x-z)\varrho^1(s,z-y)}{\varrho^1(t,x-y)}\mathrm{d}s\\ &\preceq \int_0^{t/2} s^{-1/\alpha}\left(\frac{s}{\big(|x-z|+(t-s)^{1/\alpha}\big)^{d+\alpha}}+\frac{s}{\big(|z-y|+s^{1/\alpha}\big)^{d+\alpha}}\right)\mathrm{d}s\\ &\leqslant \int_0^{t/2} s^{-1/\alpha}\Big(\varrho^1(s,x-z)+\varrho^1(s,z-y)\Big)\mathrm{d}s.\end{aligned}$$

Combining the four displays above, we get (19). Using (13) with $\gamma = 1 - 1/\alpha$, we arrive at

$$\begin{aligned}\mathcal{I} &\preceq \int_0^t s^{-1/\alpha}\Big(\varrho^1(s, x-z) + \varrho^1(s, z-y)\Big)\mathrm{d}s \\ &\preceq \left(\frac{1}{|x-z|^{d+1-\alpha}} \wedge \frac{t^{2-1/\alpha}}{|x-z|^{d+\alpha}} + \frac{1}{|z-y|^{d+1-\alpha}} \wedge \frac{t^{2-1/\alpha}}{|z-y|^{d+\alpha}}\right). \qquad (20)\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_0^t\int_D p^D(t-s,x,z)|b(z)| \cdot |\nabla_z p^D(s,z,y)|\mathrm{d}z\mathrm{d}s \\ &\preceq \sup_{w\in D}\int_D \left(\frac{1}{|w-z|^{d+1-\alpha}} \wedge \frac{t^{2-1/\alpha}}{|w-z|^{d+\alpha}}\right)|b(z)|\mathrm{d}z \cdot p^D(t,x,y).\end{aligned}$$

The desired conclusion now follows from Lemma 2.1 with $\beta = 2 - 1/\alpha$. ☐

To derive our gradient estimate, we will also need the following result.

Lemma 3.2. *Let $T > 0$. For any $t \in (0, T]$, there exists $\widehat{C}(t) = \widehat{C}(t, b) > 0$ such that for all $x, y \in D$, we have*

$$\int_0^t\int_D \frac{p^D(t-s,x,z)}{\rho(x)\wedge(t-s)^{1/\alpha}}|b(z)| \cdot |\nabla_z p^D(s,z,y)|\mathrm{d}z\mathrm{d}s \leqslant \frac{\widehat{C}(t)}{\rho(x)\wedge t^{1/\alpha}}p^D(t,x,y),$$

where $\widehat{C}(t)$ is nondecreasing in t and $\widehat{C}(t) \to 0$ as $t \to 0$.

Proof. Define

$$\mathcal{Q} := \frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\int_0^t \frac{p^D(t-s,x,z)}{\rho(x)\wedge(t-s)^{1/\alpha}} \cdot |\nabla_z p^D(s,z,y)|\mathrm{d}s.$$

Then, we have that

$$\begin{aligned}\mathcal{Q} &\preceq \frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\int_0^t \frac{p^D(t-s,x,z)}{\rho(x)}1_{[\rho(x)\leqslant(t-s)^{1/\alpha}]} \cdot |\nabla_z p^D(s,z,y)|\mathrm{d}s \\ &\quad + \frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\int_0^t \frac{p^D(t-s,x,z)}{(t-s)^{1/\alpha}}1_{[\rho(x)>(t-s)^{1/\alpha}]} \cdot \frac{p^D(s,z,y)}{\rho(z)\wedge s^{1/\alpha}}\mathrm{d}s \\ &=: \mathcal{Q}_1 + \mathcal{Q}_2.\end{aligned}$$

Using (20) in the second line below, we get that

$$\begin{aligned}\mathcal{Q}_1 &\leqslant \frac{1}{p^D(t,x,y)}\int_0^t p^D(t-s,x,z) \cdot |\nabla_z p^D(s,z,y)|\mathrm{d}s \\ &\preceq \left(\frac{1}{|x-z|^{d+1-\alpha}} \wedge \frac{t^{2-1/\alpha}}{|x-z|^{d+\alpha}} + \frac{1}{|z-y|^{d+1-\alpha}} \wedge \frac{t^{2-1/\alpha}}{|z-y|^{d+\alpha}}\right).\end{aligned}$$

To deal with $\mathcal{Q}_2$, we rewrite it as

$$\mathcal{Q}_2 = \frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\int_0^t \frac{p^D(t-s,x,z)}{(t-s)^{1/\alpha}}1_{[\rho(x)>(t-s)^{1/\alpha}]} \cdot \frac{p^D(s,z,y)}{s^{1/\alpha}}1_{[\rho(z)>s^{1/\alpha})]}\mathrm{d}s$$

$$+\frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\int_0^t \frac{p^D(t-s,x,z)}{(t-s)^{1/\alpha}}1_{[\rho(x)>(t-s)^{1/\alpha}]}\cdot\frac{p^D(s,z,y)}{\rho(z)}1_{[\rho(z)\leqslant s^{1/\alpha})]}\mathrm{d}s$$
$$=: \mathcal{Q}_{21}+\mathcal{Q}_{22}.$$

On one hand, we have by (17) that

$$\begin{aligned}\mathcal{Q}_{21} &\leqslant \frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\left(\int_0^{t/2}+\int_{t/2}^t\right)\frac{p^D(t-s,x,z)}{(t-s)^{1/\alpha}}\cdot\frac{p^D(s,z,y)}{s^{1/\alpha}}\mathrm{d}s\\ &\preceq \frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\frac{\widetilde{q}(t,x)\widetilde{q}(t,y)}{t^{1/\alpha}}\varrho^1(t,x-z)\int_0^{t/2}s^{-1/\alpha}\tilde{q}(s,z)\varrho^1(s,z-y)\mathrm{d}s\\ &\quad+\frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\frac{\widetilde{q}(t,x)\widetilde{q}(t,y)}{t^{1/\alpha}}\varrho^1(t,z-y)\int_0^{t/2}s^{-1/\alpha}\tilde{q}(s,z)\varrho^1(s,x-z)\mathrm{d}s.\end{aligned}$$

Repeating the argument used to estimate $\hat{\mathcal{I}}_{21}$ in the proof of Lemma 3.1, we get that

$$\begin{aligned}\mathcal{Q}_{21} &\leqslant \frac{\widetilde{q}(t,x)\widetilde{q}(t,y)}{p^D(t,x,y)}\int_0^{t/2}s^{-1/\alpha}\varrho^1(t-s,x-z)\varrho^1(s,z-y)\mathrm{d}s\\ &\preceq\left(\frac{1}{|x-z|^{d+1-\alpha}}\wedge\frac{t^{2-1/\alpha}}{|x-z|^{d+\alpha}}+\frac{1}{|z-y|^{d+1-\alpha}}\wedge\frac{t^{2-1/\alpha}}{|z-y|^{d+\alpha}}\right).\end{aligned}$$

To deal with $\mathcal{Q}_{22}$, we write

$$\begin{aligned}\mathcal{Q}_{22} &= \frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\int_0^{t/2}\frac{p^D(t-s,x,z)}{(t-s)^{1/\alpha}}1_{[\rho(x)>(t-s)^{1/\alpha}]}\frac{p^D(s,z,y)}{\rho(z)}1_{[\rho(z)\leqslant s^{1/\alpha})]}\mathrm{d}s\\ &\quad+\frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\int_{t/2}^t\frac{p^D(t-s,x,z)}{(t-s)^{1/\alpha}}1_{[\rho(x)>(t-s)^{1/\alpha}]}\frac{p^D(s,z,y)}{\rho(z)}1_{[\rho(z)\leqslant s^{1/\alpha})]}\mathrm{d}s\\ &=: \hat{\mathcal{Q}}_{21}+\hat{\mathcal{Q}}_{22}.\end{aligned}$$

We can use (15) to deduce that

$$\begin{aligned}\hat{\mathcal{Q}}_{21} &\leqslant \frac{\rho(x)\wedge t^{1/\alpha}}{t^{1/\alpha}}\int_0^{t/2}\rho(z)^{\alpha-1}1_{[\rho(z)\leqslant s^{1/\alpha}]}\Big(\varrho^0(t-s,x-z)+\varrho^0(s,z-y)\Big)\mathrm{d}s\\ &\leqslant \int_0^{t/2}s^{1-1/\alpha}\Big(\varrho^0(s,x-z)+\varrho^0(s,z-y)\Big)\mathrm{d}s\\ &\preceq\left(\frac{1}{|x-z|^{d+1-\alpha}}\wedge\frac{t^{2-1/\alpha}}{|x-z|^{d+\alpha}}+\frac{1}{|z-y|^{d+1-\alpha}}\wedge\frac{t^{2-1/\alpha}}{|z-y|^{d+\alpha}}\right).\end{aligned}$$

We claim that

$$\hat{\mathcal{Q}}_{22}\preceq\left(\frac{1}{|x-z|^{d+1-\alpha}}\wedge\frac{t^{2-2/\alpha}}{|x-z|^{d+\alpha-1}}+\frac{1}{|z-y|^{d+1-\alpha}}\wedge\frac{t^{2-2/\alpha}}{|z-y|^{d+\alpha-1}}\right). \tag{21}$$

To prove this claim, we write

$$\hat{\mathcal{Q}}_{22}=\frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)}\int_{t/2}^t\frac{p^D(t-s,x,z)}{(t-s)^{1/\alpha}}1_{[\rho(x)>(t-s)^{1/\alpha}]}$$

$$\times \frac{p^D(s,z,y)}{\rho(z)} 1_{[(t-s)^{1/\alpha}<\rho(z)\leqslant s^{1/\alpha})]}\mathrm{d}s$$

$$+ \frac{\rho(x)\wedge t^{1/\alpha}}{p^D(t,x,y)} \int_{t/2}^{t} \frac{p^D(t-s,x,z)}{(t-s)^{1/\alpha}} 1_{[\rho(x)>(t-s)^{1/\alpha}]}$$

$$\times \frac{p^D(s,z,y)}{\rho(z)} 1_{[\rho(z)\leqslant(t-s)^{1/\alpha})]}\mathrm{d}s$$

$$=: \tilde{\mathscr{Q}}_1 + \tilde{\mathscr{Q}}_2.$$

If we denote $A := [(t-s)^{1/\alpha} < \rho(z) \leqslant s^{1/\alpha}]$, then we have by (16) that

$$\begin{aligned}
&\tilde{\mathscr{Q}}_1\\
&\preceq \frac{\rho(x)\wedge t^{1/\alpha}}{\sqrt{t}\cdot\tilde{q}(t,x)} \int_{t/2}^{t} (t-s)^{1-1/\alpha}\rho(z)^{\alpha/2-1}\Big(\varrho^0(t-s,x-z)+\varrho^0(s,z-y)\Big)1_A\mathrm{d}s\\
&\leqslant t^{1/\alpha-1/2}\int_{t/2}^{t}(t-s)^{1-1/\alpha}\rho(z)^{\alpha/2-1}\varrho^0(s,z-y)1_A\mathrm{d}s\\
&\quad + \rho(x)^{1-\alpha/2}\int_{t/2}^{t}(t-s)^{1-1/\alpha}\rho(z)^{\alpha/2-1}\varrho^0(t-s,x-z)1_A\mathrm{d}s\\
&=: \tilde{\mathscr{Q}}_{11} + \tilde{\mathscr{Q}}_{12},
\end{aligned}$$

where in the second inequality we used the fact that $\frac{\rho(x)\wedge t^{1/\alpha}}{\tilde{q}(t,x)} = \sqrt{t}\big(\rho(x)\wedge t^{1/\alpha}\big)^{1-\alpha/2}$. One can easily check that

$$\begin{aligned}
\tilde{\mathscr{Q}}_{11} &\preceq t^{1/\alpha-1/2}\varrho^0(t,z-y)\int_{t/2}^{t}(t-s)^{3/2-2/\alpha}\mathrm{d}s\\
&\preceq \frac{1}{|z-y|^{d+1-\alpha}} \wedge \frac{t}{|z-y|^{d+1}}.
\end{aligned}$$

By the fact $\rho(x) \leqslant \rho(z) + |x-z|$, we further have

$$\begin{aligned}
\tilde{\mathscr{Q}}_{12} &\preceq \big(\rho(z)^{1-\alpha/2}+|x-z|^{1-\alpha/2}\big)\int_{t/2}^{t}(t-s)^{-1/\alpha}\rho(z)^{\alpha/2-1}\varrho^1(t-s,x-z)1_A\mathrm{d}s\\
&\leqslant \int_{t/2}^{t}(t-s)^{-1/\alpha}\varrho^1(t-s,x-z)\mathrm{d}s\\
&\quad + |x-z|^{1-\alpha/2}\int_{t/2}^{t}(t-s)^{1/2-2/\alpha}\varrho^1(t-s,x-z)\mathrm{d}s\\
&\preceq \frac{1}{|x-z|^{d+1-\alpha}} \wedge \frac{t^{2-2/\alpha}}{|x-z|^{d+\alpha-1}}.
\end{aligned}$$

To estimate $\tilde{\mathscr{Q}}_2$, we can use (15) to deduce that

$$\tilde{\mathscr{Q}}_2 \preceq \big(\rho(x)\wedge t^{1/\alpha}\big)\int_{t/2}^{t}(t-s)^{-1/\alpha}\rho(z)^{\alpha-1}1_{[\rho(z)\leqslant(t-s)^{1/\alpha}]}$$

$$\times \Big(\varrho^0(t-s,x-z)+\varrho^0(s,z-y)\Big)\mathrm{d}s$$
$$\leqslant t^{1/\alpha}\int_{t/2}^t (t-s)^{-1/\alpha}\rho(z)^{\alpha-1}1_{[\rho(z)\leqslant(t-s)^{1/\alpha}]}\varrho^0(s,z-y)\mathrm{d}s$$
$$+\rho(x)\int_{t/2}^t (t-s)^{-1/\alpha}\rho(z)^{\alpha-1}1_{[\rho(z)\leqslant(t-s)^{1/\alpha}]}\varrho^0(t-s,x-z)\mathrm{d}s.$$

Then by the same argument used for $\tilde{\mathscr{Q}}_1$, one can check that

$$\tilde{\mathscr{Q}}_2 \preceq \frac{1}{|z-y|^{d+1-\alpha}}\wedge\frac{t}{|z-y|^{d+1}}+\frac{1}{|x-z|^{d+1-\alpha}}\wedge\frac{t^{2-2/\alpha}}{|x-z|^{d+\alpha-1}}.$$

Since for any $x,z\in D$, we have

$$\frac{1}{|x-z|^{d+1-\alpha}}\wedge\frac{t^{2-1/\alpha}}{|x-z|^{d+\alpha}}\leqslant\frac{1}{|x-z|^{d+1-\alpha}}\wedge\frac{t}{|x-z|^{d+1}}$$
$$\leqslant\frac{1}{|x-z|^{d+1-\alpha}}\wedge\frac{t^{2-2/\alpha}}{|x-z|^{d+\alpha-1}},\tag{22}$$

combining the displays above, we get (21).

Combining (22), (21) with our estimates for $\mathcal{Q}_1$, $\mathcal{Q}_{21}$, $\hat{\mathcal{Q}}_{21}$, we get

$$\mathcal{Q}\preceq\left(\frac{1}{|x-z|^{d+1-\alpha}}\wedge\frac{t^{2-2/\alpha}}{|x-z|^{d+\alpha-1}}+\frac{1}{|z-y|^{d+1-\alpha}}\wedge\frac{t^{2-2/\alpha}}{|z-y|^{d+\alpha-1}}\right).$$

Hence

$$\int_0^t\int_D |\nabla_x p^D(t-s,x,z)||b(z)|\cdot|\nabla_z p^D(s,z,y)|\mathrm{d}z\mathrm{d}s$$
$$\preceq \sup_{w\in D}\int_D\left(\frac{1}{|w-z|^{d+1-\alpha}}\wedge\frac{t^{2-2/\alpha}}{|w-z|^{d+\alpha-1}}\right)|b(z)|\mathrm{d}z\cdot\frac{1}{\rho(x)\wedge t^{1/\alpha}}p^D(t,x,y).$$

The desired conclusion now follows from Lemma 2.1 with $\beta=2-2/\alpha$. □

We now proceed to solve the integral equation (3). For all $(t,x,y)\in(0,T]\times D\times D$, set $p_0(t,x,y):=p^D(t,x,y)$, and define inductively that for $k\geqslant 1$,

$$p_k(t,x,y):=\int_0^t\int_D p_{k-1}(t-s,x,z)b(z)\cdot\nabla_z p_0(s,z,y)\mathrm{d}z\mathrm{d}s.\tag{23}$$

The following result is an easy consequence of Lemmas 3.1 and 3.2.

Lemma 3.3. *Let $T>0$. For every $k\geqslant 1$ and $x,y\in D$, we have*

$$|p_k(t,x,y)|\leqslant C(t)^k p^D(t,x,y)\tag{24}$$

and

$$|\nabla_x p_k(t,x,y)|\leqslant\frac{\widehat{C}(t)^k}{\rho(x)\wedge t^{1/\alpha}}p^D(t,x,y),\tag{25}$$

where $C(t)$ is the constant in Lemma 3.1 and $\widehat{C}(t)$ is the constant in Lemma 3.2. Moreover, it holds that

$$p_k(t,x,y) = \int_0^t\int_D p_0(t-s,x,z)b(z)\cdot\nabla_z p_{k-1}(s,z,y)\mathrm{d}z\mathrm{d}s. \tag{26}$$

Proof. We first prove (24) by induction. By Lemma 3.1, we know that (24) holds for $k=1$. Now suppose that it holds for $k>1$. Then by definition and using Lemma 3.1 again, we have

$$\begin{aligned}|p_{k+1}(t,x,y)| &\leqslant \int_0^t\int_D |p_k(t-s,x,z)|\cdot|b(z)|\cdot|\nabla_z p_0(s,z,y)|\mathrm{d}z\mathrm{d}s\\ &\leqslant C(t)^k\int_0^t\int_D p^D(t-s,x,z)|b(z)|\cdot|\nabla_z p_0(s,z,y)|\mathrm{d}z\mathrm{d}s\\ &\leqslant C(t)^{k+1}p^D(t,x,y).\end{aligned}$$

Following the same argument with Lemma 3.1 replaced by Lemma 3.2, we can show (25) is true. We proceed to prove (26). It is obvious that (26) holds for $k=1$. Suppose that it is true for certain $k>1$. Then, we have by Fubini's theorem that

$$\begin{aligned}p_{k+1}(t,x,y) &= \int_0^t\int_D p_k(t-s,x,z)b(z)\cdot\nabla_z p_0(s,z,y)\mathrm{d}z\mathrm{d}s\\ &= \int_0^t\int_D\int_0^{t-s}\int_D p_0(t-s-r,x,u)b(u)\cdot\nabla_u p_{k-1}(r,u,z)\mathrm{d}u\mathrm{d}r\\ &\qquad\qquad \times b(z)\cdot\nabla_z p_0(s,z,y)\mathrm{d}z\mathrm{d}s\\ &= \int_0^t\int_D p_0(t-\hat{r},x,u)b(u)\cdot\int_0^{\hat{r}}\int_D \nabla_u p_{k-1}(\hat{r}-s,u,z)\\ &\qquad\qquad \times b(z)\cdot\nabla_z p_0(s,z,y)\mathrm{d}z\mathrm{d}s\mathrm{d}u\mathrm{d}\hat{r}\\ &= \int_0^t\int_D p_0(t-\hat{r},x,u)b(u)\cdot\nabla_u p_k(\hat{r},u,y)\mathrm{d}u\mathrm{d}\hat{r},\end{aligned}$$

here in the third equality, we used the change of variable $\hat{r}=r+s$. The proof is complete. □

Now, we are in the position to give:

Proof of Theorem 1.4. Let p_k be defined by (23). It follows from Lemma 3.1 that there exists $T_0\in(0,1]$ such that $C(T_0)<1/4$. Hence

$$\sum_{k=0}^{\infty}|p_k(t,x,y)| \leqslant \frac{4}{3}p^D(t,x,y), \quad (t,x,y)\in(0,T_0]\times D\times D, \tag{27}$$

which means that the series $\sum_{k=0}^{\infty} p_k(t,x,y)$ is convergent on $(0,T_0]\times D\times D$. Define $\widetilde{p}^{b,D}(t,x,y) := \sum_{k=0}^{\infty} p_k(t,x,y)$ on $(0,T_0]\times D\times D$. By (23), we have

$$\sum_{k=0}^{n+1} p_k(t,x,y) = p_0(t,x,y)+\int_0^t\int_D \sum_{k=0}^{n} p_k(t-s,x,z)b(z)\cdot\nabla_z p_0(s,z,y)\mathrm{d}z\mathrm{d}s. \tag{28}$$

Letting $n\to\infty$ on both sides, we get (3).

The upper bound on $(0,T_0]\times D\times D$ follows from (27). As for the lower bound on $(0,T_0]\times D\times D$, we have

$$\widetilde{p}^{b,D}(t,x,y) \geqslant p^D(t,x,y) - \sum_{k=1}^{\infty}|p_k(t,x,y)| \geqslant \frac{2}{3}p^D(t,x,y).$$

Thus, (6) is valid on $(0,T_0]\times D\times D$.

Now let $\widehat{p}^{b,D}(t,x,y)$ be another solution to (3) satisfying (6), with T replaced by T_0. We claim that for every $k\in\mathbb{N}$ and $t\in(0,T_0], x,y\in D$, there exists a constant C_0 such that

$$|\widetilde{p}^{b,D}(t,x,y) - \widehat{p}^{b,D}(t,x,y)| \leqslant C_0 C(t)^k p^D(t,x,y). \tag{29}$$

Indeed, for $k=1$, using (3), (6) and Lemma 3.1 we have

$$\begin{aligned}
&|\widetilde{p}^{b,D}(t,x,y) - \widehat{p}^{b,D}(t,x,y)|\\
\leqslant\ & \int_0^t\int_D \big(|\widetilde{p}^{b,D}(t-s,x,z)| + |\widehat{p}^{b,D}(t-s,x,z)|\big)\cdot|b(z)|\cdot|\nabla_z p^D(s,z,y)|\mathrm{d}z\mathrm{d}s\\
\leqslant\ & C_0\int_0^t\int_D p^D(t-s,x,z)\cdot|b(z)|\cdot|\nabla_z p^D(s,z,y)|\mathrm{d}z\mathrm{d}s \leqslant C_0C(t)p^D(t,x,y).
\end{aligned}$$

Suppose that (29) holds for some $k\in\mathbb{N}$. By (3), Lemma 3.1 and the induction hypothesis, we have

$$\begin{aligned}
&|\widetilde{p}^{b,D}(t,x,y) - \widehat{p}^{b,D}(t,x,y)|\\
\leqslant\ & \int_0^t\int_D |\widetilde{p}^{b,D}(t-s,x,z) - \widehat{p}^{b,D}(t-s,x,z)|\cdot|b(z)|\cdot|\nabla_z p^D(s,z,y)|\mathrm{d}z\mathrm{d}s\\
\leqslant\ & C_0C(t)^k\int_0^t\int_D p^D(t-s,x,z)\cdot|b(z)|\cdot|\nabla_z p^D(s,z,y)|\mathrm{d}z\mathrm{d}s\\
\leqslant\ & C_0C(t)^{k+1}p^D(t,x,y).
\end{aligned}$$

Since $C(t)<1$, letting $k\to\infty$, we obtain the uniqueness.

(i) By choosing T_0 smaller if necessary, we can assume that $\widehat{C}(T_0)<1$. It then follows from (25) that for every $t\in(0,T_0]$ and $x,y\in D$,

$$\left|\sum_{k=0}^{\infty}\nabla_x p_k(t,x,y)\right| \preceq \frac{1}{\rho(x)\wedge t^{1/\alpha}}p^D(t,x,y),$$

which means that (7) is true. Moreover, by (26) and Fubini's theorem, we have

$$\begin{aligned}\widetilde{p}^{b,D}(t,x,y) &= \sum_{k=0}^{\infty} p_k(t,x,y)\\ &= p^D(t,x,y) + \sum_{k=0}^{\infty}\int_0^t\int_D p_0(t-s,x,z)b(z)\cdot\nabla_z p_k(s,z,y)\mathrm{d}z\mathrm{d}s\\ &= p^D(t,x,y) + \int_0^t\int_D p_0(t-s,x,z)b(z)\cdot\nabla_z \widetilde{p}^{b,D}(s,z,y)\mathrm{d}z\mathrm{d}s,\end{aligned}$$

that is (8).

(ii) By Fubini's theorem, we have for all $0<s<t\le T_0$,

$$\int_D \widetilde{p}^{b,D}(t-s,x,z)\widetilde{p}^{b,D}(s,z,y)\mathrm{d}z = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\int_D p_m(t-s,x,z)p_{n-m}(s,z,y)\mathrm{d}z.$$

Thus, to prove (9) for $0<s<t\le T_0$, it suffices to show that for each $n\in\mathbb{N}_0$,

$$\sum_{m=0}^{n}\int_D p_m(t-s,x,z)p_{n-m}(s,z,y)\mathrm{d}z = p_n(t,x,y). \tag{30}$$

It is clear that the above equality holds for $n=0$. Suppose now that it holds for some $n\in\mathbb{N}$. Write

$$\sum_{m=0}^{n+1}\int_D p_m(t-s,x,z)p_{n+1-m}(s,z,y)\mathrm{d}z = \mathcal{J}_1+\mathcal{J}_2,$$

where

$$\mathcal{J}_1 := \int_D p_{n+1}(t-s,x,z)p_0(s,z,y)\mathrm{d}z$$

and

$$\mathcal{J}_2 := \sum_{m=0}^{n}\int_D p_m(t-s,x,z)p_{n+1-m}(s,z,y)\mathrm{d}z.$$

By (23) and Fubini's theorem, we have

$$\begin{aligned}\mathcal{J}_1 &= \int_D\left(\int_0^{t-s}\int_D p_n(t-s-r,x,u)b(u)\cdot\nabla_u p_0(r,u,z)\mathrm{d}u\mathrm{d}r\right)p_0(s,z,y)\mathrm{d}z\\ &= \int_0^{t-s}\int_D p_n(t-s-r,x,u)b(u)\cdot\left(\int_D \nabla_u p_0(r,u,z)p_0(s,z,y)\mathrm{d}z\right)\mathrm{d}u\mathrm{d}r\\ &= \int_s^t\int_D p_n(t-r,x,u)b(u)\cdot\nabla_u p_0(r,u,y)\mathrm{d}u\mathrm{d}r.\end{aligned}$$

Similarly, by (23) and the induction hypothesis, we have

$$\mathcal{J}_2 = \int_0^s\int_D p_n(t-r,x,u)b(u)\cdot\nabla_u p_0(r,u,y)\mathrm{d}u\mathrm{d}r.$$

Hence,

$$\mathcal{J}_1 + \mathcal{J}_2 = \int_0^t\int_D p_n(t-r,x,u)b(u)\cdot\nabla_u p_0(r,u,y)\mathrm{d}u\mathrm{d}r = p_{n+1}(t,x,y),$$

which gives (30).

We now extend the function $\widetilde{p}^{b,D}(t,x,y)$ from $(0,T]\times D\times D$ to $(0,\infty)\times D\times D$ via the Chapman-Kolmogorov equation. Then it is routine to extend the above assertions (i)-(ii) on $(0,T_0]$ to $(0,T]$ for any $T>0$.

(iii) Let $P_t^D f(x) := \int_D p^D(t,x,y)f(y)\mathrm{d}y$. By (3), we have for all $f\in C_c^2(D)$, $t>0$ and $x\in D$,

$$P_t^{b,D} f(x) = P_t^D f(x) + \int_0^t P_{t-s}^{b,D}(b\cdot\nabla P_s^D f)(x)\mathrm{d}s. \tag{31}$$

It then follows that for all $f\in C_c^2(D)$, $t>0$ and $x\in D$,

$$\begin{aligned} P_t^{b,D} f(x) - f(x) &= P_t^D f(x) - f(x) + \int_0^t P_{t-s}^{b,D}(b\cdot\nabla P_s^D f)(x)\mathrm{d}s \\ &= \int_0^t P_{t-s}^D(\Delta^{\alpha/2}|_D f)(x)\mathrm{d}s + \int_0^t P_{t-s}^{b,D}(b\cdot\nabla P_s^D f)(x)\mathrm{d}s. \end{aligned} \tag{32}$$

Using (31) and Fubini's theorem, we get that for all $f\in C_c^2(D)$, $t>0$ and $x\in D$,

$$\begin{aligned} &\int_0^t P_{t-s}^{b,D}(\Delta^{\alpha/2}|_D f)(x)\mathrm{d}s - \int_0^t P_{t-s}^D(\Delta^{\alpha/2}|_D f)(x)\mathrm{d}s \\ &= \int_0^t\int_0^{t-s} P_{t-s-r}^{b,D}(b\cdot\nabla P_r^D \Delta^{\alpha/2}|_D f)(x)\mathrm{d}r\mathrm{d}s \\ &= \int_0^t\int_s^t P_{t-\hat{r}}^{b,D}(b\cdot\nabla P_{\hat{r}-s}^D \Delta^{\alpha/2}|_D f)(x)\mathrm{d}\hat{r}\mathrm{d}s \\ &= \int_0^t P_{t-\hat{r}}^{b,D}\left(b\cdot\nabla\int_0^{\hat{r}} P_{\hat{r}-s}^D(\Delta^{\alpha/2}|_D f)ds\right)(x)\mathrm{d}\hat{r} \\ &= \int_0^t P_{t-\hat{r}}^{b,D}\left(b\cdot\nabla(P_{\hat{r}}^D f - f)(x)\right)\mathrm{d}\hat{r}. \end{aligned}$$

Combining this with (32), we obtain that for all $f\in C_c^2(D)$, $t>0$ and $x\in D$,

$$P_t^{b,D} f(x) - f(x) = \int_0^t P_{t-s}^{b,D}\left(\Delta^{\alpha/2}|_D + b\cdot\nabla\right) f(x)\mathrm{d}s,$$

which gives (10) for all $f\in C_c^2(D)$, $t>0$ and $x\in D$.

(iv) Since $p^D(t,x,y)$ is the transition density of X^D, for any uniformly continuous function $f(x)$ with compact supports, we have

$$\lim_{t\downarrow 0}\|P_t^D f - f\|_\infty = 0.$$

Meanwhile, by (6) and Lemma 3.1 we have

$$\begin{aligned}
&\left|\int_D\left(\int_0^t\int_D \widetilde{p}^{b,D}(t-s,x,z)b(z)\cdot\nabla_z p^D(s,z,y)\mathrm{d}z\mathrm{d}s\right)f(y)\mathrm{d}y\right|\\
&\preceq \|f\|_\infty\int_D\left(\int_0^t\int_D p^D(t-s,x,z)|b(z)|\cdot|\nabla_z p^D(s,z,y)|\mathrm{d}z\mathrm{d}s\right)\mathrm{d}y\\
&\leqslant C(t)\|f\|_\infty\int_D p^D(t,x,y)\mathrm{d}y\leqslant C(t)\|f\|_\infty,
\end{aligned}$$

where $C(t)\to 0$ as $t\to 0$, which yields (11) by (3). The whole proof is finished. □

Finally, following the idea in [11] we can give:

Proof of Corollary 1.6. By the two-sided heat kernel estimates (6), there exists a constant $C>0$ such that for every $t\in(0,1]$ and $x,y\in D$,

$$\frac{\widetilde{p}^{b,D}(t,x,z)}{\widetilde{p}^{b,D}(t,y,z)}\leqslant C_0\frac{\widetilde{q}(t,x)}{\widetilde{q}(t,y)}\frac{\varrho^1(t,x-z)}{\varrho^1(t,y-z)}\leqslant C\left(1\vee\frac{\rho(x)}{\rho(y)}\right)^{\alpha/2}\left(1\vee\frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}.$$

Therefore, for any non-negative function $f\in\mathscr{B}_b(D)$, $t\in(0,1)$ and $x,y\in D$, we have

$$\begin{aligned}
P_t^{b,D}f(x)&=\int_D\frac{\widetilde{p}^{b,D}(t,x,z)}{\widetilde{p}^{b,D}(t,y,z)}\widetilde{p}^{b,D}(t,y,z)f(z)\mathrm{d}z\\
&\leqslant\left(\sup_{z\in D}\frac{\widetilde{p}^{b,D}(t,x,z)}{\widetilde{p}^{b,D}(t,y,z)}\right)\int_D\widetilde{p}^{b,D}(t,y,z)f(z)\mathrm{d}z\\
&\leqslant C\left(1\vee\frac{\rho(x)}{\rho(y)}\right)^{\alpha/2}\left(1\vee\frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}P_t^{b,D}f(y),
\end{aligned}$$

thus (12) holds for $t\in(0,1]$. For $T>1$, we can write by (30) that

$$P_T^{b,D}f(x)=P_1^{b,D}P_{T-1}^{b,D}f(x).$$

This together with the above inequality yields the desired result. □

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