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THE NUMBER OF REPRESENTATIONS BY A TERNARY SUM OF TRIANGULAR NUMBERS

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Abstract. For positive integers a, b, c, and an integer n, the number of integer solutions $(x, y, z) \in \mathbb{Z}^3$ of $a\frac{x(x-1)}{2} + b\frac{y(y-1)}{2} + c\frac{z(z-1)}{2} = n$ is denoted by $t(a, b, c; n)$. In this article, we prove some relations between $t(a, b, c; n)$ and the numbers of representations of integers by some ternary quadratic forms. In particular, we prove various conjectures given by Z. H. Sun in [6].

1. Introduction

For a positive integer x, a non negative integer of the form $T_x = \frac{x(x-1)}{2}$ is called a *triangular number*. For example, $0, 1, 3, 6, 10, 15, \dots$ are triangular numbers. Since $T_x = T_{1-x}$, T_x is a triangular number for any integer x. For positive integers $a_1, a_2, \dots a_k$, a polynomial of the form

$$\mathcal{T}_{(a_1,\dots,a_k)}(x_1, \dots, x_k) = a_1T_{x_1} + a_2T_{x_2} + \cdots + a_kT_{x_k}$$

is called a k-ary sum of triangular numbers. For a non negative integer n, we define

$$T(a_1, \dots, a_k; n) = \left\{(x_1, \dots, x_k) \in \mathbb{Z}^k : \mathcal{T}_{(a_1,\dots,a_k)}(x_1, \dots, x_k) = n\right\}$$

and $t(a_1, \dots, a_k; n) = |T(a_1, \dots, a_k; n)|$. One may easily show that

$$t(a_1, \dots, a_k; n) = |\{(x_1, \dots, x_k) \in (\mathbb{Z}_o)^k : a_1x_1^2+\cdots+a_kx_k^2 = 8n+a_1+\cdots+a_k\}|,$$

where $\mathbb{Z}_o$ is the set of all odd integers. Hence $t(a_1, \dots, a_k; n)$ is closely related with the number of representations by some diagonal quadratic form of rank k. For example, if $k = 3$ and $a_1 = a_2 = a_3 = 1$, then every integer solution (x, y, z) of $x^2 + y^2 + z^2 = 8n + 3$ is in $(\mathbb{Z}_o)^3$. Therefore, for any positive integer n, we have

$$t(1, 1, 1; n) = |\{(x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = 8n + 3\}| = 24H(-(8n + 3)),$$

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where $H(-D)$ is the Hurwitz class number with discriminant $-D$. For the further results in this direction, see [1], [2], [5] and [8].

Recently, Sun proved in [6] various relations between $t(a_1,\dots,a_k;n)$ and the numbers of representations of integers by some diagonal quadratic forms. He also gave various conjectures on the relations between $t(a,b,c;n)$ and the numbers of representations by some ternary diagonal quadratic forms.

In this article, we consider the number $t(a,b,c;n)$ of representations by a ternary sum of triangular numbers. We show that for any positive integers a,b,c such that $(a,b,c)=1$, $t(a,b,c;n)$ is equal to the number of representations of a subform of the ternary diagonal quadratic form $ax^2+by^2+cz^2$, if $a+b+c$ is not divisible by 8, or a difference of the numbers of representations of two ternary quadratic forms otherwise.

In Section 3, we prove all conjectures in [6] on ternary sums of triangular numbers, which are Conjectures 6.1∼6.4 and 6.7. In fact, we generalize Conjectures 6.1 and 6.2 in [6], and prove these generalized conjectures. Note that Conjectures 6.5 and 6.6 in [6] are on quaternary sums of triangular numbers, which we have a plan to treat in another paper.

An integral quadratic form $f(x_1,x_2,\dots,x_k)$ of rank k is a degree 2 homogeneous polynomial

$$f(x_1,x_2,\dots,x_k)=\sum_{1\leqslant i,j\leqslant k} a_{ij}x_ix_j \quad (a_{ij}=a_{ji}\in\mathbb{Z}).$$

We always assume that f is positive definite, that is, the corresponding symmetric matrix $(a_{ij})\in M_{k\times k}(\mathbb{Z})$ is positive definite. If $a_{ij}=0$ for any $i\neq j$, then we simply write $f=\langle a_{11},\dots,a_{kk}\rangle$. For an integer n, if the Diophantine equation $f(x_1,x_2,\dots,x_k)=n$ has an integer solution, then we say n is *represented by* f. We define

$$R(f,n)=\{(x_1,\dots,x_k)\in\mathbb{Z}^k : f(x_1,\dots,x_k)=n\},$$

and $r(f,n)=|R(f,n)|$. Since we are assuming that f is positive definite, the above set is always finite. The genus of f, denoted by $\mathrm{gen}(f)$, is the set of all quadratic forms that are locally isometric to f. The number of isometry classes in $\mathrm{gen}(f)$ is called the class number of f.

Any unexplained notations and terminologies on integral quadratic forms can be found in [3] or [4].

2. Representations of ternary sums of triangular numbers

Let a,b and c be positive integers such that $(a,b,c)=1$. Throughout this section, we assume, without loss of generality, that a is odd. We show that the number $t(a,b,c;n)$ is equal to the number of representations of a subform of the ternary diagonal quadratic form $ax^2+by^2+cz^2$, if $a+b+c$ is not divisible by 8, or a difference of the numbers of representations of two ternary quadratic forms otherwise.

Let $f(x,y,z)=ax^2+by^2+cz^2$ be a ternary diagonal quadratic form. Recall that

$$t(a,b,c;n)=|\{(x,y,z)\in\mathbb{Z}^3 : f(x,y,z)=8n+a+b+c,\ xyz\equiv 1 \pmod 2\}|.$$

Lemma 2.1. *Assume that $a+b+c$ is odd. For any positive integer n, we have*

$$t(a,b,c;n)=r(f(x,x-2y,x-2z),8n+a+b+c).$$

In particular, if $a\equiv b\equiv c \pmod 4$, then we have

$$t(a,b,c;n)=r(f(x,y,z),8n+a+b+c).$$

Proof. Let $g(x,y,z)=f(x,x-2y,x-2z)$. Define a map $\phi: T(a,b,c;n)\to R(g,n)$ by $\phi(x,y,z)=(x,\frac{x-y}{2},\frac{x-z}{2})$. Then one may easily show that it is a bijective map.

Now, assume that $a\equiv b\equiv c \pmod 4$. If $ax^2+by^2+cz^2=8n+a+b+c$ for some integers x,y and z, then one may easily show that x,y and z are all odd. The lemma follows directly from this. □

Lemma 2.2. *Assume that $S=a+b+c$, both a and b are odd and c is even. Then, for any positive integer n, we have*

$$t(a,b,c;n) = \begin{cases} r(f(x,y,z),8n+S) & \text{if } S\equiv 2 \pmod 4 \text{ and } c\equiv 4 \pmod 8,\\ r(f(x,y,y-2z),8n+S) & \text{if } S\equiv 2 \pmod 4 \text{ and } c\not\equiv 4 \pmod 8,\\ 2r(f(x,x-4y,z),8n+S) & \text{if } S\equiv 4 \pmod 8 \text{ and } c\equiv 2 \pmod 4,\\ 2r(f(x,x-4y,x-2z),8n+S) & \text{if } S\equiv 4 \pmod 8 \text{ and } c\equiv 0 \pmod 4,\end{cases}$$

and if $S\equiv 0 \pmod 8$, then

$$t(a,b,c;n)=r(f(x,x-2y,x-2z),8n+S)-r\left(f(x,y,z),2n+\frac{S}{4}\right).$$

Proof. Since the proof is quite similar to each other, we only provide the proof of the fourth case, that is, the case when $S\equiv 4 \pmod 8$ and $c\equiv 0 \pmod 4$. Let $g(x,y,z)=f(x,x-4y,x-2z)$. We define a map

$$\psi: \{(x,y,z)\in(\mathbb{Z}_o)^3 : f(x,y,z)=8n+S,\ x\equiv y \pmod 4\}$$
$$\to \{(x,y,z)\in\mathbb{Z}^3 : g(x,y,z)=8n+S\} \ \text{ by } \ \psi(x,y,z)=\left(x,\tfrac{x-y}{4},\tfrac{x-z}{2}\right).$$

From the assumption, it is well defined. Conversely, assume that $g(x,y,z)=8n+S$ for some $(x,y,z)\in\mathbb{Z}^3$. Since

$$\begin{aligned} f(x,x-4y,x-2z)=ax^2+b(x-4y)^2+c(x-2z)^2 &\equiv ax^2+bx^2+cx^2\\ &\equiv Sx^2\equiv S \pmod 8\end{aligned}$$

and $S\equiv 4 \pmod 8$, the integer x is odd. Therefore, the map $(x,y,z)\to(x,x-4y,x-2z)$ is an inverse map of ψ. The lemma follows from this and the fact that

$$t(a,b,c;n)=2|\{(x,y,z)\in(\mathbb{Z}_o)^3 : f(x,y,z)=8n+S,\ x\equiv y \pmod 4\}|.$$

This completes the proof. $\square$

3. Sums of triangular numbers and diagonal quadratic forms

In this section, we generalize some conjectures given by Sun in [6] on the relations between $t(a,b,c;n)$ and the numbers of representations of integers by some ternary quadratic forms, and prove these generalized conjectures.

Let $f(x_1,x_2,\dots,x_k)$ be an integral quadratic form of rank k and let n be an integer. For a vector $\mathbf{d}=(d_1,\dots,d_k)\in(\mathbb{Z}/2\mathbb{Z})^k$, we define

$$R_{\mathbf{d}}(f,n)=\{(x_1,\dots,x_k)\in R(f,n):(x_1,\dots,x_k)\equiv(d_1,\dots,d_k)\ (\mathrm{mod}\ 2)\}.$$

The cardinality of the above set will be denoted by $r_{\mathbf{d}}(f,n)$. Note that

$$t(a,b,c;n)=r_{(1,1,1)}(ax^2+by^2+cz^2,8n+a+b+c).$$

We also define

$$\widetilde{R}_{(1,1)}(ax^2+by^2,N)=\{(x,y)\in R_{(1,1)}(ax^2+by^2,N):x\not\equiv y\ (\mathrm{mod}\ 4)\}.$$

Note that if we define the cardinality of $\widetilde{R}_{(1,1)}(ax^2+by^2,N)$ by $\widetilde{r}_{(1,1)}(ax^2+by^2,N)$, then we have

$$r_{(1,1)}(ax^2+by^2,N)=2\cdot\widetilde{r}_{(1,1)}(ax^2+by^2,N).$$

Lemma 3.1. *Let m be a positive integer.*

(i) *If $m\equiv1\ (\mathrm{mod}\ 4)$, then we have*

$$2r_{(1,0)}(x^2+3y^2,m)=r_{(1,1)}(x^2+3y^2,4m).$$

(ii) *If $m\equiv3\ (\mathrm{mod}\ 4)$, then we have*

$$2r_{(0,1)}(x^2+3y^2,m)=r_{(1,1)}(x^2+3y^2,4m).$$

(iii) *If $m\equiv4\ (\mathrm{mod}\ 8)$, then we have*

$$2r_{(0,0)}(x^2+3y^2,m)=r_{(1,1)}(x^2+3y^2,m).$$

Proof. (i) Note that the map

$$\psi_1:R_{(1,0)}(x^2+3y^2,m)\to\widetilde{R}_{(1,1)}(x^2+3y^2,4m),$$

defined by $\psi_1(x,y)=(x+3y,-x+y)$, is a bijective map.

(ii) If we define a map

$$\psi_2:R_{(0,1)}(x^2+3y^2,m)\to\widetilde{R}_{(1,1)}(x^2+3y^2,4m)\ \text{ by }\ \psi_2(x,y)=(x+3y,-x+y),$$

then one may easily check that it is a bijective map.

(iii) One may easily show that if we define a map

$$\psi_3:R_{(0,0)}(x^2+3y^2,m)\to\widetilde{R}_{(1,1)}(x^2+3y^2,m)\text{ by }\psi_3(x,y)=\left(\tfrac{x+3y}{2},\tfrac{-x+y}{2}\right),$$

then it is a bijective map. $\square$

Lemma 3.2. *Let a, b $(a < b)$ be positive odd integers such that* $\gcd(a, b) = 1$ *and* $a + b \equiv 0 \pmod 8$. *Then*

$$r_{(1,1)}(ax^2 + by^2, m) = r_{(1,1)}(ax^2 + by^2, 4m) \tag{3.1}$$

for any integer m divisible by 8 if and only if $(a, b) \in \{(3, 5), (1, 7), (1, 15)\}$.

Proof. Assume that Equation (3.1) holds for any integer m divisible by 8. Let $a + b = 2^u k$ for some integer $u \geqslant 3$ and an odd integer k. Note that $1 \leqslant a < 2^{u-1}k$.

First, we assume $u \geqslant 5$. Since

$$a \cdot 1^2 + (2^u k - a) \cdot 1^2 = 4 \cdot 2^{u-2}k \quad \text{and} \quad 2^{u-2}k \equiv 0 \pmod 8,$$

there exist odd integers x and y satisfying $ax^2 + (2^u k - a)y^2 = 2^{u-2}k$, which is a contradiction.

Next, assume that $u = 4$. Since

$$a \cdot 7^2 + (16k - a) \cdot 1^2 = 4(4k + 12a) \quad \text{and} \quad 4k + 12a \equiv 0 \pmod 8,$$

there exist two odd integers x_1, y_1 such that $ax_1^2 + (16k - a)y_1^2 = 4k + 12a$. Thus, $4k + 12a \geqslant 16k$ and hence $k \leqslant a$. Now, since $a \cdot 1^2 + (16k - a) \cdot 1^2 = 16k$, there are two positive odd integers x_2, y_2 with $ax_2^2 + (16k - a)y_2^2 = 64k$. Since $16k - a > 8k$ by assumption, we have $y_2^2 = 1$. Furthermore, since $ax_2^2 = a + 48k \leqslant 49a$, $(x_2, a) = (3, 6k), (5, 2k)$ or $(7, k)$. Since a is odd, we have $(a, b) = (1, 15)$ in this case.

Finally, we assume that $u = 3$. Since $a \cdot 1^2 + (8k - a) \cdot 1^2 = 8k$, there are positive odd integers x_3, y_3 such that $ax_3^2 + (8k - a)y_3^2 = 32k$. Hence we have

$$y_3^2 = 1 \quad \text{and} \quad ax_3^2 = a + 24k. \tag{3.2}$$

Note that if $x_3 = 3$, then $(a, b) = (3, 5)$ and if $x_3 = 5$, then $(a, b) = (1, 7)$. Assume that $x_3 \geqslant 7$, that is, $2a \leqslant k$. Since $a \cdot 3^2 + (8k - a) \cdot 1^2 = 8k + 8a$, there are two odd integers x_4, y_4 such that $ax_4^2 + (8k - a)y_4^2 = 32k + 32a$. If $y_4^2 \geqslant 9$, then $a + 72k - 9a \leqslant 32k + 32a$, which is a contradiction to the assumption that $2a \leqslant k$. Hence we have

$$y_4^2 = 1 \quad \text{and} \quad ax_4^2 = 33a + 24k. \tag{3.3}$$

Now, by Equations (3.2) and (3.3), we have $x_4^2 - x_3^2 = 32$. Therefore, $x_3^2 = 49$, $x_4^2 = 81$, and $k = 2a$. which is a contradiction to the assumption that k is odd.

To prove the converse, we define three maps

$$\chi_1 : \widetilde{R}_{(1,1)}(3x^2 + 5y^2, m) \to \widetilde{R}_{(1,1)}(3x^2 + 5y^2, 4m) \text{ by } \chi_1(x, y) = \left(\tfrac{x-5y}{2}, \tfrac{3x+y}{2}\right),$$

$$\chi_2 : \widetilde{R}_{(1,1)}(x^2 + 7y^2, m) \to \widetilde{R}_{(1,1)}(x^2 + 7y^2, 4m) \text{ by } \chi_2(x, y) = \left(\tfrac{3x-7y}{2}, \tfrac{x+3y}{2}\right),$$

and

$$\chi_3 : \widetilde{R}_{(1,1)}(x^2 + 15y^2, m) \to \widetilde{R}_{(1,1)}(x^2 + 15y^2, 4m) \text{ by } \chi_3(x, y) = \left(\tfrac{x+15y}{2}, \tfrac{-x+y}{2}\right).$$

One may easily show that the above three maps are all bijective. $\square$

Theorem 3.3. *Let a, b, c be positive integers such that $(a, b, c) \neq (1, 1, 1)$ and $\gcd(a, b, c) = 1$. Assume that two of three fractions $\frac{b}{a}, \frac{c}{b}, \frac{c}{a}$ are contained in $\left\{1, \frac{5}{3}, 7, 15\right\}$. Then, for any positive integer n, we have*

$$2t(a, b, c; n) = r(ax^2+by^2+cz^2, 4(8n+a+b+c))-r(ax^2+by^2+cz^2, 8n+a+b+c).$$

Proof. Note that all of a, b and c are odd. Furthermore, from the assumption, one may easily show that

$$-a \equiv b \equiv c \pmod 8, \quad a \equiv -b \equiv c \pmod 8 \quad \text{or} \quad a \equiv b \equiv -c \pmod 8.$$

By switching the roles of a, b and c if necessary, we may assume $a \equiv b \equiv -c \pmod 8$. Then we have

$$\left(\tfrac{a}{(a,c)}, \tfrac{c}{(a,c)}\right), \left(\tfrac{b}{(b,c)}, \tfrac{c}{(b,c)}\right) \in \{(3,5), (5,3), (1,7), (7,1), (1,15), (15,1)\}.$$

Let

$$f = f(x, y, z) = ax^2 + by^2 + cz^2 \quad \text{and} \quad N = 8n + a + b + c.$$

One may easily show that if $f(x, y, z) = 4N$, then

$$\left(ax^2, by^2, cz^2\right) \equiv (0,0,4), (0,4,0), (a,4,c), (4,0,0), (4,b,c), \text{ or } (4,4,4) \pmod 8.$$

Let

$$\begin{aligned} A &= \{(x, y, z) \in R(f, 4N) : y \equiv 2 \pmod 4,\ xz \equiv 1 \pmod 2\}, \\ B &= \{(x, y, z) \in R(f, 4N) : x \equiv 2 \pmod 4,\ yz \equiv 1 \pmod 2\}. \end{aligned}$$

Note that

$$r(f, 4N) - r(f, N) = |A| + |B|.$$

Thus it is sufficient to show $t(a, b, c; n) = |A|$ and $t(a, b, c; n) = |B|$. To show the first equality, we apply Lemma 3.2 to show that

$$\begin{aligned} r_{(1,1,1)}(f, N) &= \sum_{y \in \mathbb{Z}} r_{(1,1)}(ax^2 + cz^2, N - by^2) \\ &= \sum_{y \in \mathbb{Z}} r_{(1,1)}(ax^2 + cz^2, 4(N - by^2)) = |A|. \end{aligned}$$

The proof of $t(a, b, c; n) = |B|$ is quite similar to this. This completes the proof. ☐

Remark 3.4. All triples (a, b, c) satisfying the assumption of Theorem 3.3 are listed in Table 1 below. The triples marked with asterisks are exactly those that are listed in Conjecture 6.1 of [6].

Theorem 3.5. *Let a, b be relatively prime positive odd integers such that one of four fractions $\frac{b}{a}, \frac{a}{b}, \frac{3a}{b}, \frac{b}{3a}$ is contained in $\{\frac{5}{3}, 7, 15\}$. Then, for any positive integer n, we have*

$$2t(a, 3a, b; n) = 3r(\langle a, 3a, b\rangle, 8n + 4a + b) - r(\langle a, 3a, b\rangle, 4(8n + 4a + b)).$$

Table 1

$(1,1,7)^*$, $(1,1,15)^*$, $(3,3,5)$, $(1,7,7)^*$, $(3,5,5)$, $(1,7,15)^*$, $(1,9,15)^*$
$(1,15,15)^*$, $(3,5,21)$, $(1,7,49)$, $(1,15,25)^*$, $(3,5,35)$, $(3,5,45)$, $(1,7,105)$
$(3,5,75)$, $(1,15,105)$, $(3,21,35)$, $(1,15,225)$, $(9,15,25)$, $(5,21,35)$, $(7,15,105)$

Proof. Since all the other cases can be treated in a similar manner, we only consider the case when $\frac{b}{3a} = \frac{5}{3}$, that is, $(a, 3a, b) = (1, 3, 5)$. One may easily show that if $x^2 + 3y^2 + 5z^2 = 4(8n + 9)$, then

$$\left(x^2, 3y^2, 5z^2\right) \equiv (0,0,4), (1,3,0), (4,0,0), (4,3,5), \quad \text{or} \quad (4,4,4) \pmod 8.$$

Let

$$f = f(x, y, z) = x^2 + 3y^2 + 5z^2 \quad \text{and} \quad N = 8n + 9.$$

From the above observation, we have

$$\begin{aligned} 3r(f, N) - r(f, 4N) &= 3r_{(0,0,0)}(f, 4N) - r(f, 4N) \\ &= 2r_{(0,0,0)}(f, 4N) - r_{(1,1,0)}(f, 4N) - r_{(0,1,1)}(f, 4N). \end{aligned}$$

Therefore, it suffices to show that

$$2r_{(1,1,1)}(f, N) = 2r_{(0,0,0)}(f, 4N) - r_{(1,1,0)}(f, 4N) - r_{(0,1,1)}(f, 4N).$$

Since $r_{(0,0,0)}(f, 4N) = r(f, N)$ and

$$r(f, N) = r_{(1,1,1)}(f, N) + r_{(1,0,0)}(f, N) + r_{(0,0,1)}(f, N),$$

it is enough to show that

$$r_{(1,0,0)}(f, N) = \frac{1}{2} r_{(1,1,0)}(f, 4N) \quad \text{and} \quad r_{(0,0,1)}(f, N) = \frac{1}{2} r_{(0,1,1)}(f, 4N).$$

To prove the first assertion, we apply (i) of Lemma 3.1 to show that

$$\begin{aligned} r_{(1,0,0)}(f, N) &= \sum_{z \in \mathbb{Z}} r_{(1,0)}(x^2 + 3y^2, N - 5z^2) \\ &= \frac{1}{2} \sum_{z \in \mathbb{Z}} r_{(1,1)}(x^2 + 3y^2, 4(N - 5z^2)) \\ &= \frac{1}{2} r_{(1,1,0)}(f, 4N). \end{aligned}$$

For the second assertion, we apply (iii) of Lemma 3.1 and Lemma 3.2 to show that

$$\begin{aligned} r_{(0,0,1)}(f, N) &= \sum_{z \in \mathbb{Z}} r_{(0,0)}(x^2 + 3y^2, N - 5z^2) \\ &= \frac{1}{2} \sum_{z \in \mathbb{Z}} r_{(1,1)}(x^2 + 3y^2, N - 5z^2) \\ &= \frac{1}{2} r_{(1,1,1)}(x^2 + 3y^2 + 5z^2, N) \end{aligned}$$

$$= \frac{1}{2}\sum_{x\in\mathbb{Z}} r_{(1,1)}(3y^2+5z^2, N-x^2)$$
$$= \frac{1}{2}\sum_{x\in\mathbb{Z}} r_{(1,1)}(3y^2+5z^2, 4(N-x^2))$$
$$= \frac{1}{2} r_{(0,1,1)}(f, 4N).$$

This completes the proof. □

Remark 3.6. All triples $(a, 3a, b)$ satisfying the assumption of Theorem 3.5 are listed in Table 2 below. Those triples marked with asterisks are exactly those that are listed in Conjecture 6.2 of [6].

Table 2

$(1,3,5)^*$	$(1,3,7)^*$	$(1,3,15)^*$	$(1,3,21)^*$	$(1,5,15)^*$	$(1,3,45)$
$(3,5,9)^*$	$(1,7,21)^*$	$(3,5,15)^*$	$(3,7,21)^*$	$(1,15,45)$	$(5,9,15)$

Theorem 3.7. *Let* $(a, b, c) \in \{(1, 2, 15), (1, 15, 18), (1, 15, 30)\}$. *For any positive even integer* n, *we have*

(3.4) $2t(a, b, c; n) = r(\langle a, b, c\rangle, 4(8n + a + b + c)) - r(\langle a, b, c\rangle, 8n + a + b + c)$.

Proof. First, assume that $(a, b, c) = (1, 2, 15)$. Let

$$f = f(x, y, z) = x^2 + 2y^2 + 15z^2 \quad \text{and} \quad N = 8n + 18.$$

One may easily show that if $f(x, y, z) = 4N$, then

$$(x^2, 2y^2, 15z^2) \equiv (0, 0, 0), (1, 0, 7), \quad \text{or} \quad (4, 0, 4) \pmod 8.$$

Hence the right-hand side of Equation (3.4) is

$$r(f, 4N) - r(f, N) = r_{(1,0,1)}(f, 4N).$$

Note that

$$\begin{aligned} r_{(1,1,1)}(f, N) &= \sum_{y\in\mathbb{Z}} r_{(1,1)}(x^2 + 15z^2, (N - 2y^2)) \\ &= \sum_{y\in\mathbb{Z}} r_{(1,1)}(x^2 + 15z^2, 4(N - 2y^2)) \\ &= r_{(1,1,1)}(x^2 + 8y^2 + 15z^2, 4N) \\ &= |\{(x, y, z) \in R(f, 4N) : xz \equiv 1 (\text{mod } 2),\ y \equiv 2 (\text{mod } 4)\}| \end{aligned}$$

by Lemma 3.2. Since

$$\begin{aligned} &|\{(x, y, z) \in R(f, 4N) : xz \equiv 1 (\text{mod } 2),\ y \equiv 0 (\text{mod } 4)\}| \\ &= r(x^2 + 32y^2 + 15z^2, 4N), \end{aligned}$$

it suffices to show that

$$r_{(1,1,1)}(f,N) = r(x^2+32y^2+15z^2,4N). \tag{3.5}$$

It is well known that

$$\mathrm{gen}(f_1 = 4x^2+4y^2+8z^2+2xy) = \{f_1,f_2,f_3\},$$

where $f_2 = 4x^2+6y^2+6z^2+4yz+2xz+2xy$, $f_3 = 2x^2+6y^2+12z^2+6yz+2xz$, and

$$\mathrm{gen}(g_1 = 4x^2+8y^2+18z^2+8yz+4xz) = \{g_1, g_2 = 2x^2+10y^2+24z^2\}.$$

Note that

$$r_{(1,1,1)}(f,N) = r(x^2+2(x-2y)^2+15(x-2z)^2,N) = r(g_1,N).$$

On the other hand, the right-hand side of Equation (3.5) is

$$\begin{aligned}
&r\left(x^2+15y^2+32z^2,4N\right)\\
&= r\left((3x+y)^2+15(x+y)^2+32z^2,4N\right)\\
&= r\left(12x^2+8y^2+16z^2+18xy,2N\right)\\
&= r\left(48x^2+8y^2+16z^2+36xy,2N\right)+r\left(12x^2+32y^2+16z^2+36xy,2N\right)\\
&= 2r\left(f_1,N\right).
\end{aligned}$$

Therefore, it suffices to show that for any positive even integer $n=2m$,

$$2r(f_1,16m+18) = r(g_1,16m+18). \tag{3.6}$$

By the Minkowski-Siegel formula, we have

$$\begin{aligned}
&r(f_1,16m+18)+2r(f_2,16m+18)+r(f_3,16m+18)\\
&= r(g_1,16m+18)+r(g_2,16m+18).
\end{aligned}$$

If $f_1(x,y,z) = 16m+18$, then one may easily check that $x+3y-4z \equiv 0 \pmod 8$, and if $f_2(x,y,z) = 16m+18$, then $x-6y+2z \equiv 0 \pmod 8$. If we define a map

$$\begin{aligned}
\phi_1 :\ &\{(x,y,z) \in R(f_1,16m+18) : x+3y-4z \equiv 0 \pmod{16}\}\\
\to\ &\{(x,y,z) \in R(f_2,16m+18) : x-6y+2z \equiv 0 \pmod{16}\}
\end{aligned}$$

by $\phi_1(x,y,z) = \left(\frac{12x+4y+16z}{16}, \frac{-11x-y+12z}{16}, \frac{x-13y-4z}{16}\right)$, then it is a bijective map. Furthermore, the map

$$\begin{aligned}
\phi_2 :\ &\{(x,y,z) \in R(f_1,16m+18) : x+3y-4z \equiv 8 \pmod{16}\}\\
\to\ &\{(x,y,z) \in R(f_2,16m+18) : x-6y+2z \equiv 8 \pmod{16}\}
\end{aligned}$$

defined by $\phi_2(x,y,z) = \left(\frac{4x+12y-16z}{16}, \frac{-13x+y+4z}{16}, \frac{-x-11y-12z}{16}\right)$ is also bijective. Therefore, we have

$$r(f_1,16m+18) = r(f_2,16m+18). \tag{3.7}$$

Note that the above equation does not hold, in general, if n is odd. If we define two maps

$\phi_3 : R(\langle 8,10,24\rangle, 16m+18) \to R(f_1, 16m+18)$ by $\phi_3(x,y,z) = (y+2z, y-2z, x)$

and

$\phi_4 : R(\langle 2,24,40\rangle, 16m+18) \to R(f_3, 16m+18)$ by $\phi_4(x,y,z) = (x+z, 2y+z, -2z)$,

then one may easily check that both of them are bijective. Hence we have

$$\begin{aligned} r(g_2, 16m+18) &= r\left(\langle 8,10,24\rangle, 16m+18\right) + r\left(\langle 2,24,40\rangle, 16m+18\right) \\ &= r(f_1, 16m+18) + r(f_3, 16m+18) \end{aligned}$$

for any non negative integer m. Therefore, from the Minkowski-Siegel formula given above, we have $2r(f_2, 16m+18) = r(g_1, 16m+18)$ for any non negative integer m. Equation (3.6) follows directly from this and Equation (3.7).

For the other two cases, one may easily show Equation (3.4) by replacing N, f_i, g_i and ϕ_i with the following data:

(1) $(a,b,c) = (1,15,18)$**.** In this case, we let $N = 8n+34$ and

$$\begin{aligned} f_1 &= 4x^2 + 4y^2 + 72z^2 + 2xy, \\ f_2 &= 4x^2 + 16y^2 + 22z^2 + 14yz - 2xz + 4xy, \\ f_3 &= 6x^2 + 16y^2 + 16z^2 - 8yz + 6xz + 6xy, \end{aligned}$$

and

$$g_1 = 4x^2 + 34y^2 + 34z^2 + 8yz + 4xz + 4xy, \quad g_2 = 10x^2 + 18y^2 + 24z^2.$$

Define

$$\begin{aligned} \phi_1 :\ &\{(x,y,z) \in R(f_1, 16m+34) : 3x + y + 4z \equiv 0 \pmod{16}\} \\ \to\ &\{(x,y,z) \in R(f_2, 16m+34) : 3x - y + 2z \equiv 0 \pmod{16}\} \end{aligned}$$

by $\phi_1(x,y,z) = \left(\frac{x-5y-68z}{16}, \frac{-5x-7y+20z}{16}, \frac{-4x+4y-16z}{16}\right)$,

$$\begin{aligned} \phi_2 :\ &\{(x,y,z) \in R(f_1, 16m+34) : 3x + y + 4z \equiv 8 \pmod{16}\} \\ \to\ &\{(x,y,z) \in R(f_2, 16m+34) : 3x - y + 2z \equiv 8 \pmod{16}\} \end{aligned}$$

by $\phi_2(x,y,z) = \left(\frac{9x-5y-52z}{16}, \frac{3x+9y+4z}{16}, \frac{4x-4y+16z}{16}\right)$, and

$$\phi_3 : R(10x^2 + 24y^2 + 72z^2, 16m+34) \to R(f_1, 16m+34)$$

by $\phi_3(x,y,z) = (x-2y, x+2y, z)$,

$$\phi_4 : R(18x^2 + 24y^2 + 40z^2, 16m+34) \to R(f_3, 16m+34)$$

by $\phi_4(x,y,z) = (x+2y, -x+z, -x-z)$.

(2) $(a,b,c) = (1,15,30)$**.** In this case, we let $N = 8n+46$ and

$$\begin{aligned} f_1 &= 4x^2 + 4y^2 + 120z^2 + 2xy, \\ f_2 &= 4x^2 + 16y^2 + 34z^2 + 14yz - 2xz + 4xy, \\ f_3 &= 10x^2 + 16y^2 + 16z^2 + 8yz + 10xz + 10xy, \end{aligned}$$

and

$$g_1 = 4x^2 + 46y^2 + 46z^2 + 32yz + 4xz + 4xy, \quad g_2 = 6x^2 + 30y^2 + 40z^2.$$

Define

$$\begin{aligned} \phi_1 : \ &\{(x,y,z) \in R(f_1, 16m+46) : 3x - y - 4z \equiv 0 \pmod{16}\} \\ \to \ &\{(x,y,z) \in R(f_2, 16m+46) : 3x - y + 2z \equiv 8 \pmod{16}\} \end{aligned}$$

by $\phi_1(x,y,z) = \left(\frac{7x-13y-4z}{16}, \frac{-3x+y-44z}{16}, \frac{-4x-4y+16z}{16}\right)$,

$$\begin{aligned} \phi_2 : \ &\{(x,y,z) \in R(f_1, 16m+46) : 3x - y - 4z \equiv 8 \pmod{16}\} \\ \to \ &\{(x,y,z) \in R(f_2, 16m+46) : 3x - y + 2z \equiv 0 \pmod{16}\} \end{aligned}$$

by $\phi_2(x,y,z) = \left(\frac{9x-11y+20z}{16}, \frac{3x+7y+28z}{16}, \frac{-4x-4y+16z}{16}\right)$, and

$$\phi_3 : R(6x^2 + 40y^2 + 120z^2, 16m+46) \to R(f_1, 16m+46)$$

by $\phi_3(x,y,z) = (x+2y, -x+2y, z)$,

$$\phi_4 : R(24x^2 + 30y^2 + 40z^2, 16m+46) \to R(f_3, 16m+46)$$

by $\phi_4(x,y,z) = (-y-2z, x+y, -x+y)$.

This completes the proof. □

Theorem 3.8. *For any positive integer n such that $n \not\equiv 1 \pmod 3$, we have*

(3.8) $2t(1,1,27;n) = r(x^2+y^2+27z^2, 4(8n+29)) - r(x^2+y^2+27z^2, 8n+29)$.

Proof. Let $N = 8n + 29$ and

$$\begin{aligned} f &= f(x,y,z) = x^2 + y^2 + 27z^2, \\ g &= g(x,y,z) = 8x^2 + 20y^2 + 29z^2 + 4yz + 8xz + 8xy, \\ h &= h(x,y,z) = 2x^2 + 5y^2 + 27z^2 + 2xy. \end{aligned}$$

For any positive integer $m \not\equiv 1 \pmod 3$, we let

$$\delta_m = \begin{cases} 1 & \text{if } m \equiv 0 \pmod 3, \\ 2 & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Note that

$$r(f,m) = \delta_m \left|\{(x,y,z) \in R(f,m) : x \equiv y \pmod 3\}\right|. \tag{3.9}$$

Since

$$r(f,4N) = \delta_N \cdot r(x^2 + (x-3y)^2 + 27z^2, 4N) = \delta_N \cdot r(h, 4N)$$

and

$$\begin{aligned} &\left|\{(x,y,z) \in R(f,4N) : y \equiv 0 \pmod 2\}\right| \\ = \ &\delta_N \cdot r(x^2 + 4(x-3y)^2 + 27z^2, 4N) = \delta_N \cdot r(8x^2 + 5y^2 + 27z^2 + 4xy, 4N) \\ = \ &\delta_N \left|\{(x,y,z) \in R(h,4N) : x \equiv 0 \pmod 2\}\right|, \end{aligned}$$

we have

$$(3.10)\quad |\{(x,y,z)\in R(f,4N): y \text{ is odd}\}| = \delta_N |\{(x,y,z)\in R(h,4N): x \text{ is odd}\}|.$$

One may easily show that if $(x, y, z) \in R(f, 4N)$, then

$$\left(x^2, y^2, 27z^2\right) \equiv (0,0,4),(0,1,3),(0,4,0),(1,0,3),(4,0,0),(4,4,4) \pmod 8.$$

From this and Equation (3.10), the right hand side of Equation (3.8) becomes

$$r(f, 4N) - r(f, N) = 2\delta_N \left|\{(x,y,z) \in R(h,4N) : x \equiv 1 \pmod 2\}\right|.$$

On the other hand, by Equation (3.9),

$$\begin{aligned} t(1,1,27;n) &= r_{(1,1,1)}(f, N) \\ &= \delta_N \left|\{(x,y,z) \in R(f,N) : x \equiv y \pmod 3,\ x \equiv y \equiv z \pmod 2\}\right| \\ &= \delta_N \cdot r(x^2 + (x-6y)^2 + 27(x-2z)^2, N) = \delta_N \cdot r(g, N). \end{aligned}$$

Therefore, it is enough to show that

$$r(g, N) = \left|\{(x,y,z) \in R(h,4N) :\ x \equiv 1 \pmod 2\}\right|.$$

Now, we let

$$\begin{aligned} A &= \{(x,y,z) \in R(g,N) : x \equiv 0 \pmod 2\}, \\ B &= \{(x,y,z) \in R(h,4N) : x \equiv 1 \pmod 2,\ x + z \equiv 0 \pmod 8\}. \end{aligned}$$

Note that $x + z \equiv 8 \pmod{16}$ if $(x, y, z) \in B$. Define a map $\phi : A \to B$ by

$$\phi(x, y, z) = (x - 7z,\ -x - 4y + z,\ -x - z).$$

Then, one may easily show that ϕ is a bijection. Since $g(x+z, y, -z) = g(x, y, z)$ and z_0 is odd for any $(x_0, y_0, z_0) \in R(g, N)$, we have

$$|\{(x,y,z) \in R(g,N) : x \equiv 0 \pmod 2\}| = |\{(x,y,z) \in R(g,N) : x \equiv 1 \pmod 2\}|$$

and thus

$$r(g, N) = 2\left|\{(x,y,z) \in R(g,N) : x \equiv 0 \pmod 2\}\right|.$$

Now, we are ready to prove the assertion. Note that if $(x, y, z) \in R(h, 4N)$ and $x \equiv 1 \pmod 2$, then $z \equiv \pm x \pmod 8$. Therefore, we have

$$\begin{aligned} &\left|\{(x,y,z) \in R(h,4N) : x \equiv 1 \pmod 2\}\right| \\ &= 2\left|\{(x,y,z) \in R(h,4N) : x \equiv 1 \pmod 2,\ x+z \equiv 0 \pmod 8\}\right| \\ &= 2|B| = 2|A| = r(g, N). \end{aligned}$$

This completes the proof. □

Finally, we prove Conjecture 6.7 in [6].

Theorem 3.9. *For a positive integer n, the Diophantine equation*

$$\mathcal{T}_{(1,1,6)}(x, y, z) = \frac{x(x-1)}{2} + \frac{y(y-1)}{2} + 6\frac{z(z-1)}{2} = n$$

has an integer solution if and only if $n \not\equiv 2\cdot 3^{2r-1} - 1 \pmod{3^{2r}}$ for any positive integer r.

Proof. Note that $\mathcal{T}_{(1,1,6)}(x,y,z) = n$ has an integer solution if and only if $f(x,y,z) = x^2 + y^2 + 6z^2 = 8n + 8$ has an integer solution x, y, z such that $xyz \equiv 1 \pmod 2$. Since the ternary quadratic form $f(x,y,z)$ has class number one, it represents every integer that is locally represented (see 102.5 of [4]). Therefore, one may easily check that $f(x,y,z) = 8n+8$ has an integer solution if and only if $n \not\equiv 2 \cdot 3^{2r-1} - 1 \pmod{3^{2r}}$ for any positive integer r.

Now, assume that n is a positive integer such that $n \not\equiv 2 \cdot 3^{2r-1} - 1 \pmod{3^{2r}}$ for any positive integer r. Note that $f(x,y,z) = 8n+8$ has an integer solution x, y, z such that $xyz \equiv 1 \pmod 2$ if and only if $r(f, 8n+8) - r(f, 2n+2) > 0$. By the Minkowski-Siegel formula, we have

$$\frac{r(f, 8n+8)}{r(f, 2n+2)} = 2\frac{\alpha_2(f, 8n+8)}{\alpha_2(f, 2n+2)},$$

where α_2 is the local density over $\mathbb{Z}_2$ (for details, see, for example, [3]). For a positive integer s and a positive odd integer t, one may easily compute by using the result of [7] that

$$\alpha_2(f, 2^s t) = \begin{cases} 2 - 3 \cdot 2^{-s/2} & \text{if} \quad s \equiv 0 \pmod 2, \\ 2 - 2^{(1-s)/2} & \text{if} \quad s \equiv 1 \pmod 2,\ t \equiv 1 \pmod 8, \\ 2 & \text{if} \quad s \equiv 1 \pmod 2,\ t \equiv 5 \pmod 8, \\ 2 - 3 \cdot 2^{(-s-1)/2} & \text{if} \quad s \equiv 1 \pmod 2,\ t \equiv 3 \text{ or } 7 \pmod 8. \end{cases}$$

Therefore, we have $2\alpha_2(f, 8n+8) > \alpha_2(f, 2n+2)$ for any positive integer n. This completes the proof. □

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