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CONTINUOUS SHADOWING AND STABILITY FOR GROUP ACTIONS

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Abstract. Recently, Chung and Lee [2] introduced the notion of topological stability for a finitely generated group action, and proved a group action version of the Walters's stability theorem. In this paper, we introduce the concepts of continuous shadowing and continuous inverse shadowing of a finitely generated group action on a compact metric space X with respect to various classes of admissible pseudo orbits and study the relationships between topological stability and continuous shadowing and continuous inverse shadowing property of group actions. Moreover, we introduce the notion of structural stability for a finitely generated group action, and we prove that an expansive action on a compact manifold is structurally stable if and only if it is continuous inverse shadowing.

1. Introduction

The shadowing property was first established for systems generated by hyperbolic diffeomorphisms and later for those generated by hyperbolic homeomorphisms. Shadowing, or the pseudo orbit tracing property, is one of the interesting concepts in the qualitative theory of smooth dynamical systems. It says that any δ-pseudo orbit can be uniformly approximated by a true orbit with a given accuracy if δ is sufficiently small. The concept of inverse shadowing was established by Corless and Pilyugin [3] and Kloden and Ombach [4] redefined this property using the δ-method. Generally speaking, inverse shadowing means that given a class of approximating methods, one can trace any real orbit with an arbitrary accuracy by an orbit generated with a precise enough method. Moreover the notions of continuous shadowing and continuous inverse shadowing was introduced and discussed in [5].

Furthermore, Walters [11] introduced the notion of topological stability, a kind of stability for homeomorphisms for which continuous pertubations are allowed, and proved that every expansive homeomorphism with the shadowing property on a compact metric space is topologically stable. In [5] Lee

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proved that every expansive homeomorphism with the shadowing property is $\mathfrak{T}_\alpha$-shadowing. In this paper we will obtain a group action version of result. Expansive action with shadowing property motivated by a classical dynamical system that can be considered as an action of the group $\mathbb{Z}$. Very recently, Chung and Lee [2] introduced the notion of topological stability for finitely generated group actions, and Pilyugin et al. [8,9] introduced the notions of shadowing and inverse shadowing for finitely generated group actions which are generalizations of those of topological stability, shadowing and inverse shadowing, respectively, for homeomorphisms on compact metric spaces.

In this paper we introduce the concepts of continuous shadowing and continuous inverse shadowing of a finitely generated group action on a compact metric space X with respect to various classes of admissible pseudo orbits and study the relationships between topological stability and continuous shadowing and continuous inverse shadowing property of group actions. Furthermore we introduce the concept of structural stability for finitely generated group actions on compact smooth manifolds and show that an expansive action is structurally stable if and only if it is continuous inverse shadowing.

2. Preliminaries

Let G be a finitely generated group with the discrete topology and X be a compact metric space with a metric d. Put $Homeo(X)$ the space of all homeomorphisms of X. We denote by $Act(G, X)$ the set of all continuous actions T of G on X; *i.e.,* $T : G \times X \to X$ is a continuous map such that $T(g, \cdot)$ is continuous, $T(e, x) = x$ and $T(g, T(h, x)) = T(gh, x)$ for $x \in X$ and $g, h \in G$, where e is the identity element of G. For simplicity, $T(g, x)$ will be denoted by $T_g(x)$. Let $Homeo(X)^G = \prod_G Homeo(X)$ be the set of homeomorphisms from G to $Homeo(X)$ with the product topology. Then $Act(G, X)$ can be considered as a subset of $Homeo(X)^G$. Let A be a symmetric finitely generating set of G, i.e., for any $a \in A$, $a^{-1} \in A$. If A is a finitely generating set of G, then there always exists a symmetric finitely generating set containing A. Throughout the paper, a finitely generating set A of G means a symmetric finitely generating set. We define a metric d_A on $Act(G, X)$ by

$$d_A(T, S) = \sup\{d(T_a(x), S_a(x)) \mid x \in X, a \in A\}$$

for $T, S \in Act(G, X)$. Then the topology on $Act(G, X)$ induced by d_A coincides with the product topology on $Act(G, X)$ inherited from $Homeo(X)^G$. Hence the space $Act(G, X)$ is a separable complete metrizable topological space, and so a Baire space.

Recently, Chung and Lee [2] introduced the notion of topological stability of a finitely generated group action on a compact metric space. We say that an action $T \in Act(G, X)$ is *topologically stable* with respect to a finitely generating set A of G if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if S is another continuous action of G on X with $d_A(T, S) < \delta$, then there exists a continuous map $f : X \to X$ with $T_g f = f S_g$ for every $g \in G$ and $d(f, Id_X) \le \varepsilon$. An action

T is said to be *topologically stable* if it is topologically stable with respect to A for some finitely generating set A of G.

If T and S are two continuous actions of G on X with $d_A(T,S) < \delta$, then the S-orbit $\{S_g(x)\}_{g\in G}$ of $x \in X$ is nearly a T-orbit in the sense that $d(T_a(S_g(x)), S_{ag}(x)) < \delta$ for all $a \in A$ and $g \in G$. This observation motivates the following definition. Let A be a finitely generating set of G and $\delta > 0$. A *δ-pseudo orbit* of $T \in Act(G,X)$ with respect to A is a sequence $\{x_g\}_{g\in G}$ in X such that $d(T_a(x_g), x_{ag}) < \delta$ for all $a \in A$ and $g \in G$. An action $T \in Act(G,X)$ is said to have the *shadowing property* with respect to A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that any δ-pseudo orbit $\{x_g\}_{g\in G}$ for T with respect to A is ε-traced by some point x of X, that is, $d(T_g(x), x_g) < \varepsilon$ for all $g \in G$.

Let X^G be the compact metric space of all sequences $\xi = \{x_g : g \in G\}$ with elements $x_g \in X$, endowed with the product topology. Suppose that $G = \{g_i | i \in \mathbb{N}\}$ is countable. Then we define a metric D on X^G by

$$D((x_{g_i})_{i\in\mathbb{N}}, (y_{g_i})_{i\in\mathbb{N}}) = \sup_{i\in\mathbb{N}}\{\frac{\bar{d}(x_{g_i}, y_{g_i})}{2^i}\}$$

for any $(x_{g_i})_{i\in\mathbb{N}}, (y_{g_i})_{i\in\mathbb{N}} \in X^G$, where $\bar{d}(x_{g_i}, y_{g_i}) = \min\{d(x_{g_i}, y_{g_i}), 1\}$. Let A be a finitely generating set of G. For $\delta > 0$, let $\Phi_T(\delta, A)$ denote the set of all δ-pseudo orbits of T with respect to A. A mapping $\varphi_A : X \to \Phi_T(\delta, A) \subset X^G$ satisfying $(\varphi_A(x))_e = x$, $(x \in X)$ is said to be a *δ-method* for T with respect to A, where $\varphi_A(x)$ will be denoted by $\{\varphi_A(x)_g\}_{g\in G}$. We say that φ_A is a *continuous δ-method* for T with respect to A if φ_A is continuous. The set of all δ-methods [resp. continuous δ-methods] for T with respect to A will be denoted by $\mathfrak{T}_0(T,\delta,A)$ [resp. $\mathfrak{T}_c(T,\delta,A)$]. Every $S \in Act(G,X)$ with $d_A(T,S) < \delta$ induces a continuous δ-method $\varphi_A : X \to X^G$ for T with respect to A by defining $\varphi_A(x) = \{S_g(x) : g \in G\}$. Let $\mathfrak{T}_h(T,\delta,A)$ denote the set of all continuous δ-method φ_A for T with respect to A which are induced by $S \in Act(G,X)$ with $d_A(T,S) < \delta$. To introduce the notions of continuous shadowing with respect to various classes of admissible pseudo orbits, we define $\mathcal{P}_\alpha(T,\delta,A)$ by

$$\mathcal{P}_\alpha(T,\delta,A) = \bigcup_{\varphi_A\in\mathfrak{T}_\alpha(T,\delta,A)} \varphi_A(X),$$

where $\alpha = 0, c, h$. Clearly we have

$$\mathcal{P}_h(T,\delta,A) \subset \mathcal{P}_c(T,\delta,A) \subset \mathcal{P}_0(T,\delta,A) = \Phi_T(\delta,A).$$

We denote $\mathfrak{T}_\alpha(T,A) = \bigcup_{\delta>0}\mathfrak{T}_\alpha(T,\delta,A)$, where $\alpha = 0, c, h$.

Definition. Let A be a finitely generating set of G. An action $T \in Act(G,X)$ is said to be $\mathfrak{T}_\alpha$-shadowing with respect to A for $\alpha = 0, c, h$ if for every $\epsilon > 0$, there exists $\delta > 0$ and a map $r : \mathcal{P}_\alpha(T,\delta,A) \to X$ such that $d(T_g(r(\mathbf{x})), x_g) < \varepsilon$ for any $\mathbf{x} = \{x_g\}_{g\in G} \in \mathcal{P}_\alpha(T,\delta,A)$ and all $g \in G$. We say that T is $\mathfrak{T}_\alpha$-continuous shadowing with respect to A if the map r is continuous.

Note that the $\mathfrak{T}_0$-shadowing property is equal to the shadowing property in [2]. We observe that the definition of shadowing property (resp. continuous shadowing property) of an action $T \in Act(G, X)$ is independent of the choice of generating sets.

Lemma 2.1. *Let A and B be symmetric finitely generating sets of G. An action $T \in Act(G, X)$ is $\mathfrak{T}_\alpha$-shadowing (resp. $\mathfrak{T}_\alpha$-continuous shadowing) with respect to A if and only if it is $\mathfrak{T}_\alpha$-shadowing (resp. $\mathfrak{T}_\alpha$-continuous shadowing) with respect to B, where $\alpha = 0, c, h$.*

Proof. Assume that $T \in Act(G, X)$ is $\mathfrak{T}_\alpha$-(continuous) shadowing with respect to A. For any $\varepsilon > 0$, there are $\delta_A > 0$ and a (continuous) map $r : \mathcal{P}_\alpha(T, \delta_A, A) \to X$ such that $d(T_g(r(\mathbf{x})), x_g) < \varepsilon$ for any $\mathbf{x} = \{x_g\}_{g\in G} \in \mathcal{P}_\alpha(T, \delta_A, A)$ and $g \in G$. We claim that there exists a $\delta_B > 0$ such that $\mathcal{P}_\alpha(T, \delta_B, B) \subset \mathcal{P}_\alpha(T, \delta_A, A)$. Put $m = \max_{b\in B} l_B(a)$, where l_B is the word length metric on G induced by B. Choose $\delta_1 > 0$ such that $m\delta_1 < \delta_A$. Since X is compact and B is finite, there exists $0 < \delta_B < \delta_1$ such that $d(T_h(x), T_h(y)) < \delta_1$ for $x, y \in X$ with $d(x, y) < \delta_B$ and for $h \in G$ with $l_B(h) \leq m$. For any $a \in A$, we write a as $b_1 \cdots b_{l(a)}$ where $l(a) = l_B(a), b_i \in B$, $i = 1, \ldots, l(a)$. Then for any $\{x_g\}_{g\in G} \in \mathcal{P}_\alpha(T, \delta_B, B)$, we have

$$\begin{aligned}
d(T_a x_g, x_{ag}) &= d(T_{b_1\cdots b_{l(a)}} x_g, x_{b_1\cdots b_{l(a)}g}) \\
&\leq d(T_{b_1\cdots b_{l(a)}} x_g, T_{b_1\cdots b_{l(a)-1}} x_{b_{l(a)}g}) \\
&\quad + d(T_{b_1\cdots b_{l(a)-1}} x_{b_{l(a)}g}, T_{b_1 b_2\cdots b_{l(a)-2}} x_{b_{l(a)-1}b_{l(a)}g}) \\
&\quad + \cdots + d(T_{b_1 b_2} x_{b_3\cdots b_{l(a)}g}, T_{b_1} x_{b_2\cdots b_{l(a)}g}) \\
&\quad + d(T_{b_1} x_{b_2\cdots b_{l(a)}g}, x_{b_1\cdots b_{l(a)}g}) \\
&< (m-1)\delta_1 + \delta_B < (m-1)\delta_1 + \delta_1 = m\delta_1 < \delta_A.
\end{aligned}$$

This means that $\{x_g\}_{g\in G} \in \mathcal{P}_\alpha(T, \delta_A, A)$. Thus $\mathcal{P}_\alpha(T, \delta_B, B) \subset \mathcal{P}_\alpha(T, \delta_A, A)$. Let $\bar{r}$ is restriction of r from $\mathcal{P}_\alpha(T, \delta_B, B)$ to X. Then the (continuous) map $\bar{r} : \mathcal{P}_\alpha(T, \delta_B, B) \to X$ satisfies $d(T_g(r(\mathbf{x})), x_g) < \varepsilon$ for all $\mathbf{x} = \{x_g\}_{g\in G} \in \mathcal{P}_\alpha(T, \delta_B, B)$ and $g \in G$. Thus T is $\mathfrak{T}_\alpha$-(continuous) shadowing with respect to B. □

Definition. An action $T \in Act(G, X)$ is said to be $\mathfrak{T}_\alpha$-shadowing (resp. $\mathfrak{T}_\alpha$-continuous shadowing) if $T \in Act(G, X)$ is $\mathfrak{T}_\alpha$-shadowing (resp. $\mathfrak{T}_\alpha$-continuous shadowing) with respect A for some finitely generating set A of G where $\alpha = 0, c, h$.

Now we introduce the notions of continuous inverse shadowing with respect to various classes of admissible pseudo orbits of a finitely generated group action. First, we recall the concepts of inverse shadowing for group actions which was introduced in [9]. An action $T \in Act(T, A)$ is said to be $\mathfrak{T}_\alpha$-*inverse shadowing property* with respect to A if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $p \in X$, any φ is δ-method for T with respect to A, there is $x \in X$

such that $d(T_g(p), \varphi(x)_g) < \varepsilon$ for all $g \in G$. When discussing inverse shadowing for homeomorphisms, an appropriate choice of the class of admissible pseudo orbits is crucial [3, 4].

Definition. An action $T \in Act(G, X)$ is said to be $\mathfrak{T}_\alpha$-inverse shadowing with respect to A for $\alpha = 0, c, h$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-method $\varphi_A \in \mathfrak{T}_\alpha(T, \delta, A)$ there is a map $s : X \to X$ satisfying $d(T_g(x), \varphi_A(s(x))_g) < \varepsilon$ for $x \in X$ and all $g \in G$. We say that T is $\mathfrak{T}_\alpha$-continuous inverse shadowing with respect to A if the map s is continuous.

Lemma 2.2. *Let A and B be symmetric finitely generating sets of G. An action $T \in Act(G, X)$ is $\mathfrak{T}_\alpha$-inverse shadowing (resp. $\mathfrak{T}_\alpha$-continuous inverse shadowing) with respect to A if and only if it is $\mathfrak{T}_\alpha$-inverse shadowing (resp. $\mathfrak{T}_\alpha$-continuous inverse shadowing) with respect to B, where $\alpha = 0, c, h$.*

Proof. Assume that $T \in Act(G, X)$ is $\mathfrak{T}_\alpha$-(continuous) inverse shadowing property with respect to A. For $\varepsilon > 0$, let δ_A correspond to ε by the $\mathfrak{T}_\alpha$-inverse shadowing property with respect to A. For $\delta_A > 0$, there exists $\delta_B > 0$ such that $\mathfrak{T}_\alpha(T, \delta_B, B) \subset \mathfrak{T}_\alpha(T, \delta_A, A)$ by the same proof of Lemma 2.1. For $\varphi_B \in \mathfrak{T}_\alpha(T, \delta_B, B)$, there is $\varphi_A \in \mathfrak{T}_\alpha(T, \delta_A, A)$ such that $\varphi_A(x) = \varphi_B(x)$ for all $x \in X$. Since T is $\mathfrak{T}_\alpha$-(continuous) inverse shadowing property with respect to A, there is a (continuous) map $s : X \to X$ satisfying

$$d(T_g(x), \varphi_A(s(x))_g) < \varepsilon$$

for all $x \in X$ and $g \in G$. Then T is $\mathfrak{T}_\alpha$-(continuous) inverse shadowing with respect to B. □

Definition. An action $T \in Act(G, X)$ is said to be $\mathfrak{T}_\alpha$-inverse shadowing (resp. $\mathfrak{T}_\alpha$-continuous inverse shadowing) if $T \in Act(G, X)$ is $\mathfrak{T}_\alpha$-inverse shadowing (resp. $\mathfrak{T}_\alpha$-continuous inverse shadowing) with respect A for a finitely generating set A of G.

3. Continuous shadowing and continuous inverse shadowing

In this section, we describe the relationship between continuous shadowing, continuous inverse shadowing for finitely generated group actions. Moreover, we study about topological stability for finitely generated group action by using continuous shadowing and continuous inverse shadowing.

First, we observe that there is no relationship between shadowing and inverse shadowing for finitely generated group actions. Indeed, there is a homeomorphism with inverse shadowing property but it is not shadowing property. Lewowicz [7] showed that there is a pseudo Anosov map on a compact surface which is expansive and $\mathfrak{T}_h$-inverse shadowing but it does not have shadowing property. Conversely, the following example shows that there is a finitely generated group action with shadowing property but it does not have inverse shadowing property. We adapt the counterexample in [11]. First, we recall the notion of shift action for finitely generated group which is introduced in [2]. Let A be a finitely generating set of G. For any $k \in \mathbb{N}$, we put $B(k) = \{g \in G : l_A(g) \le k\}$ where $l_A(g)$ is the minimal word length metric on G induced by A. Let F be a finite set. We denote by F^G the product space $\prod_G F$ endowed with the product topology. We consider the action of G on F^G by left shift, *i.e.*, $(gx)_h = x_{gh}$ for every $g, h \in G$ and $x \in F^G$. We define a metric D on F^G by

$$D(x, y) = 2^{-k},\ k = \sup\{j \in \mathbb{N} : x_g = y_g \text{ for all } g \in B(j)\}$$

for $x, y \in F^G$, where $B(j) = \{g \in G : l_A(g) \le j\}$.

Example 3.1. Let $X = F^G$ and T be a left shift action on X where A is a symmetric finitely generating set. It is easy to show that T is expansive with shadowing property (topologically stable). We show that T does not have $\mathfrak{T}_h$ inverse shadowing with respect to A. Let $\varepsilon = \frac{1}{2}$. For any $\delta > 0$, pick $m \in \mathbb{N}$ with $\frac{1}{2^m} < \delta$. Define $S : G \times X \to X$ by

$$(g, (x_h)_{h\in G})_h = \begin{cases} x_h, & l_A(h) \ge m + n, \\ x_{gh}, & l_A(h) \ge l_A(gh),\ l_A(h) \le m + n - 1, \\ x_{gh}, & l_A(h) < l_A(gh),\ l_A(h) \le m - 1, \\ x_{h^{-1}}, & l_A(h) < l_A(gh),\ l_A(h) = m, \dots, m + n - 1 \end{cases}$$

with $d_A(T, S) = \frac{1}{2^m}$ where $l_A(g) = n$. It is clear that $S \in Act(G, X)$. Choose $\varphi_A \in \mathfrak{T}_h(T, \delta, A)$ such that $\varphi_A(x) = \{S_g(x)\}_{g\in G}$ for given $S \in Act(G, X)$. If T has the inverse shadowing property with respect to $\mathfrak{T}_h(T, A)$, then for any $x \in X$, there exist $y \in X$ such that

$$D(T_g(x), S_g(y)) < \frac{1}{2} = \varepsilon \ \text{ for all } \ g \in G, \tag{1}$$

where $x = \{x_h\}_{h\in G}$, $y = \{y_h\}_{h\in G}$.

Choose $x \in X$ satisfying $x_e \ne x_g$ for any $g \in G \setminus \{e\}$. If $x_e \ne y_e$, then $D(T_e(x), S_e(y)) = 1 > \frac{1}{2} = \varepsilon$. If $x_e = y_e$, then for $a \in A$, $D(T_{a^N}(x), S_{a^N}(y)) = 1 > \frac{1}{2} = \varepsilon$ for $N = 2m + 1$. It is a contradiction to the inequality (1). Thus T is not $\mathfrak{T}_h$-inverse shadowing property with respect to A.

Consequently, we observe that there is no relationship among shadowing and continuous inverse shadowing; inverse shadowing property and continuous shadowing. However, if $T \in Act(G, M)$ is a continuous group action on compact manifold M, then the continuous shadowing (resp. continuous inverse shadowing) implies inverse shadowing (resp. shadowing).

Theorem 3.2. *Let $T \in Act(G, M)$ be an action on compact manifold M. Then*

(i) *if T is $\mathfrak{T}_\alpha$-continuous shadowing, then it is $\mathfrak{T}_\alpha$-inverse shadowing where $\alpha = c, h$,*
(ii) *if T is $\mathfrak{T}_\alpha$-continuous inverse shadowing, then it is $\mathfrak{T}_\alpha$-shadowing, $\alpha = 0, c, h$.*

Proof. (i) Suppose that $T \in Act(G, X)$ is $\mathfrak{T}_c$-continuous shadowing with respect to a finitely generating A of G. For $\varepsilon > 0$, there are $\delta > 0$ and a continuous map $r : \mathcal{P}_c(T, \delta, A) \to M$ such that $d(T_g r(\mathbf{x}), x_g) < \varepsilon$ for all $\mathbf{x} = \{x_g\}_{g\in G} \in \mathcal{P}_c(T, \delta, A)$ and $g \in G$. Let $\varphi_A \in \mathcal{P}_c(T, \delta, A)$ and $x \in M$. Then $d(T_g(r(\varphi_A(x))), \varphi_A(x)_g) < \varepsilon$ for all $g \in G$. Since $r \circ \varphi_A : M \to M$ is continuous and $d(r(\varphi_A(x)), x) < \varepsilon$ for all $x \in M$, the map $r \circ \varphi_A$ is surjective for sufficiently small $\varepsilon > 0$. Choose $y_x \in M$ such that $r(\varphi_A(y_x)) = x$ and define a map $s : M \to M$ by $s(x) = y_x$ for all $x \in M$. Then we have

$$d(T_g(x), \varphi_A(s(x))_g) = d(T_g(r(\varphi_A(y_x))), \varphi_A(y_x)_g) < \varepsilon$$

for all $g \in G$. Therefore T is $\mathfrak{T}_c$-inverse shadowing. Similarly we can show that if T is $\mathfrak{T}_h$-continuous shadowing, then it is $\mathfrak{T}_h$-inverse shadowing.

(ii) Suppose T is $\mathfrak{T}_\alpha$-continuous inverse shadowing property with respect to a finitely generating set A of G. Let $\varepsilon > 0$ be arbitrary. Then we can choose $\delta > 0$ such that for any δ-method $\varphi_A \in \mathfrak{T}_\alpha(T, \delta, A)$, there exists a continuous map $s : M \to M$ satisfying $d(T_g(x), \varphi_A(s(x))_g) < \varepsilon$ for all $x \in M$ and all $g \in G$. If $g = e$, then $d(x, s(x)) < \varepsilon$ for all $x \in M$. Since the map s is continuous, it is surjective for sufficiently small $\varepsilon > 0$. To show that T is $\mathfrak{T}_\alpha$-shadowing, we define a map $r : \mathcal{P}_\alpha(T, \delta, A) \to M$ as follows. For any $\mathbf{y} = \{y_g\}_{g\in G} \in \mathcal{P}_\alpha(T, \delta, A)$, there exist $y \in M$ and $\varphi \in \mathfrak{T}_\alpha(T, \delta, A)$ satisfying $\mathbf{y} = \varphi(y)$. Note that the point y is unique since $y_e = \varphi(y)_e = y$. Since s is surjective, we can choose $x \in M$ with $s(x) = y$, and define $r(\mathbf{y}) = x$. Then r is a desired map. In fact, we have

$$d(T_g(r(\mathbf{y})), y_g) = d(T_g(x), \varphi(s(x))_g) < \epsilon$$

for all $\mathbf{y} = \{y_g\}_{g\in G}$ and $g \in G$. Thus T is $\mathfrak{T}_\alpha$-shadowing with respect to A. $\square$

Here, we observe that $\mathfrak{T}_0$-continuous shadowing does not imply inverse shadowing in general. In fact, let f be an Anosov diffeomorphism satisfying Axiom A on a compact smooth manifold. Then f satisfies strong transversality condition, and so f has the shadowing property (See [10]). By [6], since f is expansive, f has the continuous shadowing property. Moreover, by [3], since f satisfies strong transversality condition, f does not have the $\mathfrak{T}_0$-inverse shadowing property.

Clearly, we know that continuous shadowing implies shadowing but the converse is not true. In fact, Yano [12] constructed an example of a homeomorphism f which is shadowing but it is not topologically stable, and so it is not $\mathfrak{T}_h$-continuous shadowing by Theorem 3.4 below. However, if an action T is expansive, then T is $\mathfrak{T}_\alpha$-shadowing if and only if T is $\mathfrak{T}_\alpha$-continuous shadowing, where $\alpha = 0, c, h$. We recall that an action $T \in Act(G, X)$ is expansive if there exists a constant $\eta > 0$ such that for every $x \neq y$, one has $\sup_{g\in G} d(T_g(x), T_g(y)) > \eta$. The constant η is called an expansive constant of T.

In [6], Lee and Sakai showed that every expansive homeomorphism with shadowing property have the continuous shadowing property. In this section, we obtain a similar result for finitely generated group actions.

Theorem 3.3. *Let $T \in Act(G,X)$ be an expansive group action. Then T is the $\mathfrak{T}_\alpha$-shadowing if and only if T is the $\mathfrak{T}_\alpha$-continuous shadowing, where $\alpha = 0, c, h$*

Proof. We only need to prove if part. Let $\eta > 0$ be an expansive constant of T and A be a finitely generating set of G. For $0 < \varepsilon < \frac{\eta}{3}$, take $\delta > 0$ corresponding to ε by the shadowing property of T with respect to A. Then for any δ-pseudo orbit $\{x_g\}_{g\in G}$ of T with respect to A there exists a unique $y \in X$ satisfying $d(T_g(y), x_g) < \varepsilon$ for all $g \in G$. Hence we can define a map $r : \mathcal{P}_\alpha(T,\delta,A) \to X$ by $r(\{x_g\}_{g\in G})$ is the ε-shadowing point of $\{x_g\}_{g\in G} \in \mathcal{P}_\alpha(T,\delta,A)$. We claim that r is continuous. Since T is expansive, for any $\varepsilon' > 0$ there exists $B(M)$ is finite subset of G such that $\sup\limits_{g\in B(M)} d(T_g(x), T_g(y)) \le \eta$, then $d(x,y) < \varepsilon'$ where $B(M) = \{g \in G \mid l_A(g) \le M\}$ by Lemma 2.10 in [2]. Pick $\delta' > 0$ with $2^M\delta' < \frac{\eta}{3}$. Let $\mathbf{x} = \{x_g\}_{g\in G}$, $\mathbf{x}' = \{x'_g\}_{g\in G} \in \mathcal{P}_\alpha(T,\delta,A)$ be given two δ-pseudo orbits of T with respect to A, and let $r(\mathbf{x}) = y$ and $r(\mathbf{x}') = y'$. If $D(\{x_g\}_{g\in G}, \{x'_g\}_{g\in G}) < \delta'$, then $d(x_g, x'_g) < \frac{\eta}{3}$ for all $g \in B(M)$. We have

$$d(T_g(y), T_g(y')) \le d(T_g(y), x_g) + d(x_g, x'_g) + d(x'_g, T_g(y')) < \varepsilon + \tfrac{\eta}{3} + \varepsilon < \eta$$

for all $g \in B(M)$. By Lemma 2.10 in [2], we have $d(y,y') < \varepsilon'$. Therefore r is continuous. □

Finally, we will study topological stability for group actions.

Theorem 3.4. *If $T \in Act(G,X)$ is $\mathfrak{T}_h$-continuous shadowing, then it is topologically stable.*

Proof. Suppose $T \in Act(G,X)$ is $\mathfrak{T}_h$-continuous shadowing with respect to a finitely generating set A of G. For $\varepsilon > 0$, there are $\delta > 0$ and continuous map $r : \mathcal{P}_h(T,\delta,A) \to X$ such that $d(T_g(r(\mathbf{x})), x_g) < \varepsilon$ for any $\mathbf{x} = \{x_g\}_{g\in G} \in \mathcal{P}_h(T,\delta,A)$ and all $g \in G$. Let $S \in Act(G,X)$ be such that $d_A(T,S) < \delta$ and $Orb(S) = \{O_S(x) : x \in X\}$ where $O_S(x) = \{S_g(x)\}_{g\in G}$. Let $\alpha : Orb(S) \subset \mathcal{P}_h(T,\delta,A) \to X$ be a continuous choice function such that $\alpha(O_S(x)) \in O_S(x)$ and $\alpha(O_S(x)) = \alpha(O_S(y))$ if $y \in O_S(x)$ for each $x \in X$. Define a map $H : X \to X$ by

$$H(x) = (T_g)^{-1} \circ r|_{Orb(S)} \circ O \circ \alpha \circ O(x),\ x \in X,$$

where $g \in G$ satisfying $S_g(x) = \alpha(O_S(x))$ and $O : M \to Orb(S)$ is the orbit map given by $O(x) = \{S_g(x) : g \in G\}$ for $x \in X$. Then H is a continuous map satisfying

$$d(H(x), x) < \varepsilon \text{ for all } x \in X \text{ and } H \circ S_g = T_g \circ H \text{ for any } g \in G.$$

In fact, for any $x \in X$ there exists $h \in G$ such that $S_h(x) = \alpha(O_S(x)) = \overline{x}$. Let $\overline{x} = S_h(x)$, we get

$$d(H(x), x) = d(T_{h^{-1}}(r(O_S(\overline{x}))), S_{h^{-1}}(\overline{x})) < \varepsilon.$$

Let $\overline{y} = r(O_S(\overline{x}))$. We have

$$H(S_a(x)) = T_a \circ T_{h^{-1}}(\overline{y}) = T_a(H(x))$$

for all $a \in A$. This completes the proof. □

Corollary 3.5. *If $T \in Act(G, M)$ is expansive and $\mathfrak{T}_h$-continuous inverse shadowing, then it is topologically stable.*

Proof. Since T is $\mathfrak{T}_h$-continuous inverse shadowing, then T is $\mathfrak{T}_h$-continuous shadowing by Theorem 3.3 and Theorem 3.2. Then it is topologically stable by Theorem 3.4. □

4. Structural stability

In this section, we introduce the notion of structural stability for group actions and show that an expansive group action T is $\mathfrak{T}_d$-continuous inverse shadowing if and only if it is structurally stable. Moreover we prove that if there is $g \in G$ such that homeomorphism T_g is expansive and continuous inverse shadowing, then action T is structurally stable of virtually nilpotent groups. Let M be a compact smooth manifold without boundary. The following theorem is the main result in this section.

Theorem 4.1. *Let G be a finitely generated virtually nilpotent group and T be a continuous action of G on compact manifold M. If there exists an element $g \in G$ such that T_g is expansive and $\mathfrak{T}_d$-continuous inverse shadowing, then T is structurally stable.*

Put $Diff(M)$ be the space of all C^1-diffeomorphisms on M. We denote by $Act(G, M)$ the set of all continuous actions T of G on M such that $T_g \in Diff(M)$ for all $g \in G$. Let $Diff(M)^G = \prod_G Diff(M)$ be the set of C^1-diffeomorphisms from G to $Diff(M)$ with the product topology. Let A be a finitely generating set of G. We define a metric d^1_A on $Act(G, M)$ by

$$d^1_A(T, S) = \sup\{\ d(T_a(x), S_a(x)),\ \|DT_a(x) - DS_a(x)\| \mid x \in X, a \in A\}$$

for $T, S \in Act(G, X)$.

Definition. We say that $T \in Act(G, M)$ is structurally stable with respect to a finitely generating set A if for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $S \in Act(G, M)$ with $d^1_A(T, S) < \delta$, there exists a homeomorphism $h : M \to M$ such that

(i) $T_g \circ h(x) = h \circ S_g(x)$,
(ii) $d(x, h(x)) < \varepsilon$

for all $x \in M$ and $g \in G$.

The following lemma shows that structural stability is independent of the choice of generating sets. The proof is similar to Lemma 2.2 in [2], so we omit it.

Lemma 4.2. *Let A and B be symmetric finitely generating sets of G. An action $T \in Act(G, X)$ is structurally stable with respect to A if and only if it is structurally stable with respect to B.*

Definition. We say that $T \in Act(G, M)$ is structurally stable if T is structurally stable with respect to A for a finitely generating set A of G.

For any $\delta > 0$ and $T \in Act(G, M)$, every $S \in Act(G, M)$ with $d_A^1(T, S) < \delta$ induces a continuous δ-method $\varphi_A : M \to M^G$ for T with respect to A by defining $\varphi_A(x) = \{S_g(x) : g \in G\}$ for each $x \in M$. Let $\mathfrak{T}_d(T, \delta, A)$ denote the set of all continuous δ-method φ_A for T with respect to A which are induced by $S \in Act(G, M)$ with $d_A^1(T, S) < \delta$. $\mathcal{P}_d(T, \delta, A) = \bigcup_{\varphi_A \in \mathfrak{T}_d(T,\delta,A)} \varphi_A(M)$.

Definition. An action $T \in Act(G, M)$ is said to be $\mathfrak{T}_d$-inverse shadowing with respect to A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-method $\varphi_A \in \mathfrak{T}_d(T, \delta, A)$ there is a map $s : M \to M$ satisfying $d(T_g(x), \varphi_A(s(x))_g) < \varepsilon$ for $x \in M$ and all $g \in G$.

We say that T is $\mathfrak{T}_d$-continuous inverse shadowing with respect to A if the map s is continuous.

Similarly to Lemma 2.2, it is not hard to show that $\mathfrak{T}_d$-(continuous)inverse shadowing is independent of the choice of generating sets.

Definition. An action $T \in Act(G, X)$ is said to be $\mathfrak{T}_d$-inverse shadowing [resp. $\mathfrak{T}_d$-continuous inverse shadowing] if $T \in Act(G, X)$ is $\mathfrak{T}_d$-inverse shadowing [resp. $\mathfrak{T}_d$-continuous inverse shadowing] with respect A for a finitely generating set A of G.

Next we will provide a class of structurally stable group actions. Let us recall the definition of a virtually nilpotent group. A subgroup H of G is said to be normal if $gH = Hg$ for all $g \in G$. Let G be a countable group. The lower central series of G is the sequence $\{G_i\}_{i \geq 0}$ of subgroup of G defined by $G_0 = G$ and $G_{i+1} = [G_i, G]$, where $[G_i, G]$ is the subgroup of G generated by all commutators $[a, b] = aba^{-1}b^{-1}$, $a \in G_i, b \in G$. We say that G is nilpotent if there exists $n \geq 0$ such that $G_n = \{e\}$. The such smallest n is called the nilpotent degree of G. A finitely generated group G is called virtually nilpotent if there exists a normal nilpotent subgroup G' of G having finite index (this means that the factor group G/G' is finite).

Lemma 4.3. *If $T \in Act(G, M)$ is structurally stable, then it is $\mathfrak{T}_d$-continuous inverse shadowing.*

Proof. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for every $S \in Act(G, M)$ with $d_A^1(T, S) < \delta$ there is a homeomorphism $f : M \to M$ satisfies $d(f, Id) < \varepsilon$ and $f \circ T_g = S_g \circ f$ for all $g \in G$. Let $\varphi_A \in \mathfrak{T}_d(T, \delta, A)$ and choose $S \in Act(G, M)$ with $d_A^1(T, S) < \delta$ which implies $\varphi_A(x) = \{S_g(x)\}_{g \in G}$. Take a homeomorphism $f : M \to M$ by the structural stability of T. Define a map $s : M \to M$ by

$s(x) = f(x)$ for all $x \in M$.

$$\begin{aligned} d(T_g(x), \varphi_A(s(x))_g) &= d(T_g(x), S_g(s(x))) = d(T_g(x), S_g(f(x))) \\ &= d(T_g(x), f(T_g(x))) < \varepsilon. \end{aligned}$$

Therefore T is $\mathfrak{T}_d$-continuous inverse shadowing. □

Theorem 4.4. *Suppose that $T \in Act(G, M)$ is expansive. Then T is structurally stable if and only if T is $\mathfrak{T}_d$-continuous inverse shadowing.*

Proof. Suppose T is structurally stable, by Lemma 4.3, it is $\mathfrak{T}_d$-continuous inverse shadowing. Conversely, suppose that $T \in Act(G, M)$ is expansive and $\mathfrak{T}_d$-continuous inverse shadowing with respect to finitely generating set A. We claim that T is structurally stable. Let $\eta > 0$ be an expansive constant of T and let $0 < \epsilon < \frac{\eta}{2}$. Then we can find $\delta > 0$ corresponding to ε by $\mathfrak{T}_d$-continuous inverse shadowing with respect to A. Let $S \in Act(G, M)$ with $d^1_A(T, S) < \delta$, and δ-method $\varphi_A \in \mathfrak{T}_d(T, \delta, A)$ such that $\varphi_A(x) = \{\varphi_A(x)_g\}_{g\in G} = \{S_g(x)\}_{g\in G}$ is a δ-pseudo orbit of T with respect to A for all $x \in M$. Then there exists a continuous map $k : M \to M$ such that

$$d(T_g(x), S_g(k(x))) < \varepsilon \text{ for all } x \in M \text{ and } g \in G.$$

Let $\alpha : O(T) \to M$ be a continuous choice function such that $\alpha(O_T(x)) \in O_T(x)$ and $\alpha(O_T(x)) = \alpha(O_T(y))$ if $y \in O_T(x)$ where $O(T) = \{O_T(x) \mid x \in M\}$. Define a map $f : M \to M$ by

$$f(x) = S_g \circ k \circ T_{g^{-1}}(x),$$

where $g \in G$ satisfies $T_{g^{-1}}(x) = \alpha(O_T(x))$. Then f is a conjugacy between T and S. In fact, for any $x \in M$ denote $\alpha(O_T(x))$ by $\overline{x}$. Choose $h \in G$ such that $x = T_h\overline{x}$. Then

$$f(T_a(x)) = S_{h^{-1}} \circ k \circ T_{h^{-1}}(T_a(x)) = S_a \circ S_{a^{-1}h} \circ k \circ T_{h^{-1}a}(x) = S_a(f(x))$$

for all $a \in A$. Moreover we have

$$d(f(x), x) = d(f(T_h(\overline{x})), T_h(\overline{x})) = d(S_h(k(\overline{x})), T_h(\overline{x})) < \varepsilon.$$

Since the map f is continuous, it is surjective for sufficiently small ε. To show that f is injective, we suppose $f(x) = f(y)$ for $x, y \in M$. Since $f \circ T_g = S_g \circ f$, we obtain

$$\begin{aligned} d(T_g(x), T_g(y)) &\le d(T_g(x), f(T_g(x))) + d(f(T_g(x)), f(T_g(y))) \\ &\quad + d(f(T_g(y)), T_g(y)) \\ &= d(T_g(x), f(T_g(x))) + d(S_g(f(x)), S_g(f(y))) \\ &\quad + d(f(T_g(y)), T_g(y)) \\ &< \varepsilon + \varepsilon < \eta \end{aligned}$$

for all $g \in G$. Since T is expansive, $x = y$. Thus f is injective. Therefore f is conjugacy. □

To prove the above theorem, we need following lemmas. This proof is similar to reductive inverse shadowing theorem in [9].

Lemma 4.5. *Let G be a finitely generated group and H be a finitely generated normal subgroup of G. Let $T \in Act(G, M)$ be a continuous action of G on X. If the restriction action $T|_H : H \times M \to M$ is $\mathfrak{T}_d$-continuous inverse shadowing, then T is $\mathfrak{T}_d$-continuous inverse shadowing.*

Proof. Let A be a symmetric finitely generating set of H. We extend A to a symmetric finitely generating set B of G. Let $\varepsilon > 0$. Since M is compact and B is finite, there exists $0 < \delta_B < \varepsilon$ such that $d(T_b(x), T_b(y)) < \frac{\varepsilon}{2}$ for every $b \in B$ and every $x, y \in M$ with $d(x, y) < \delta_B$. We can choose $\delta < \frac{\varepsilon}{2}$ for this δ_B from the $\mathfrak{T}_d$-continuous inverse shadowing for $T|_H$. For any $\varphi_B \in \mathfrak{T}_d(T, \delta, A)$, there exist $S \in Act(G, M)$ with $d^1_B(T, S) < \delta$ such that $\varphi_B(x) = \{S_g(x)\}_{g\in G}$. Then $d(T_b(S_g x), S_{bg}(x)) < \delta$ for all $b \in B$ and $g \in G$. Fix $g \in G$ and $x \in M$. Since $A \subset B$ and H is normal subgroup of G then $d(T_a(S_{hg}(x)), S_{ahg}(x)) < \delta$. So $S|_{Hg}(x) = \{S_{hg}(x)\}_{h\in H}$ is a δ-pseudo orbit of $T|_H$ with respect to A. By assumption, we can choose a continuous map $f : M \to M$ satisfying $d(T_{hg}(x), S_{hg}(f(x))) < \delta_B$ for all $h \in H$.

For any $b \in B$ and $h \in H$, since H is a normal subgroup of G, there is an element $h' \in H$ such that $bh' = hb$. Then

$$\begin{aligned} d(T_{hbg}(x), S_{hbg}(f(x))) &= d(T_{bh'g}(x), S_{bh'g}(f(x))) \\ &\le d(S_{bh'g}(f(x)), T_b(S_{h'g}(f(x)))) \\ &\quad + d(T_b(S_{h'g}(f(x))), T_b(T_{h'g}(x))) \\ &\le \delta + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This means that $d(T_{hg}(x), S_{hg}(f(x))) < \varepsilon$ for all $g, h \in G$ and $x \in X$. By applying $h = e$, we have $d(T_g(x), S_g(f(x))) < \epsilon$ for all $g \in G$. Thus T is $\mathfrak{T}_d$-continuous inverse shadowing. □

Lemma 4.6 ([9])**.** *Let $T \in Act(G, M)$ be a continuous action of a nilpotent group G of class n. If there exists $g \in G$ such that T_g is $\mathfrak{T}_d$-continuous inverse shadowing, then so is T.*

Proof. See Lemma 4.2 in [9]. □

Proof of Theorem 4.1. Let H be a nilpotent normal subgroup of G with finite index. Then H is finitely generated by Proposition 6.6.2 in [1]. Since H has finite index in G, there exists $n \in \mathbb{N}$ such that $g^n \in H$. Since T_g is $\mathfrak{T}_d$-continuous inverse shadowing, then T_{g^n} is also by Lemma 3.5 in [5]. By Lemma 4.6, $T|_H$ is $\mathfrak{T}_d$-continuous inverse shadowing. Applying Theorem 4.4 and Lemma 4.5, we complete the proof. □

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