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WEAKLY ALMOST PERIODIC POINTS AND CHAOTIC DYNAMICS OF DISCRETE AMENABLE GROUP ACTIONS

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Abstract. The aim of this paper is to introduce the notions of (quasi) weakly almost periodic point, measure center and minimal center of attraction of amenable group actions, explore the connections of levels of the orbit's topological structure of (quasi) weakly almost periodic points and study chaotic dynamics of transitive systems with full measure centers. Actually, we showed that weakly almost periodic points and quasi-weakly almost periodic points have distinct orbit's topological structure and proved that there exists at least countable Li-Yorke pairs if the system contains a proper (quasi) weakly almost periodic point and that a transitive but not minimal system with a full measure center is strongly ergodically chaotic.

1. Introduction and preliminaries

For a discrete dynamical system, the most important dynamics are concentrating on a full measure subset from the view of ergodic theory. In order to describe such a full measure set, Zhou [12] introduced the notions of weakly almost periodic point and measure center, and proved that the closure of the set of weakly almost periodic points is actually the measure center. Zhou and He [13] raised the concept of quasi-weakly almost periodic point and showed that weakly almost periodic points and quasi-weakly almost periodic points have completely distinct ergodic properties. Concretely, the support of each invariant measure generated by the orbit of a given weakly almost periodic point is its minimal center of attraction, and a point is quasi-weakly almost periodic if and only if it belongs to its minimal center of attraction. In 2012, Huang and Zhou [7] introduced the conceptions of weakly almost periodic point, quasi-weakly almost periodic point and measure center for continuous semi-flows and obtained some similar results of [12, 13]. One can see [3–6, 8, 9] for the recent results about group actions. In this paper, we considered the version of

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group actions of [7, 12, 13]. Indeed, we introduced the conceptions of weakly almost periodic point, quasi-weakly almost periodic point and measure center of amenable group actions, showed that weakly almost periodic points and quasi-weakly almost periodic points have distinct orbit's topological structure and proved that there exists at least countable Li-Yorke pairs if the system contains a proper (quasi) weakly almost periodic point and that a transitive but not minimal amenable group action with a full measure center is strongly ergodically chaotic.

1.1. Sets in discrete amenable groups

Throughout let $(G,\cdot)$ be a discrete infinite countable *amenable* topological group and (G,X) be a topologically dynamical system or G*-system* for short, on a compact metric space (X,d). A sequence $\{F_i\}_{i=1}^{\infty}$ of finite subsets of G is called a *Følner sequence* if

$$\lim_{i\to\infty}\frac{|gF_i\Delta F_i|}{|F_i|}=0, \forall g\in G,$$

where $|\cdot|$ is the counting measure on G. Since $(G,\cdot)$ is assumed to be amenable, it always has a Følner sequence (cf., e.g. [1]). Denote by $\mathcal{F}_G$ the collection of all Følner sequences of G.

Fix a Følner sequence $\mathcal{F}=\{F_n\}_{n=1}^{\infty}$ of G, we define the density of a set $S\subset G$ with respect to $\mathcal{F}$ by

$$d_{\mathcal{F}}(S)=\lim_{i\to\infty}\frac{|S\cap F_i|}{|F_i|}$$

if the above limit exists. Otherwise we can define respectively the upper density and lower density of S with respect to $\mathcal{F}$ by

$$\overline{d}_{\mathcal{F}}(S)=\limsup_{i\to\infty}\frac{|S\cap F_i|}{|F_i|} \text{ and } \underline{d}_{\mathcal{F}}(S)=\liminf_{i\to\infty}\frac{|S\cap F_i|}{|F_i|}.$$

$S\subset G$ is called a syndetic set if there is a compact subset K of G such that $G=KS$, i.e., $G=\bigcup_{s\in S}Ks$. One can see [1] for more details of amenable groups.

1.2. Related concepts of dynamical systems

Let (X,G) be a G-system. The orbit of a point $x\in X$ is denoted by $G(x)=\{gx : g\in G\}$ and $\mathcal{C}\ell(A)$ denotes the closure of $A\subset X$ in X. Given $x\in X$, denote by $\omega(x,G)$ the ω*-limit set* of x under G, that is $\omega(x,G)=\{y\in X :$ there exists a sequence $\{g_i\}_{i=1}^{\infty}\subset G$ such that $g_ix\to y\}$. $x\in X$ is an *almost periodic point* if for each $\varepsilon>0$, the recurrent time set of x entering its ε-neighborhood $B(x,\varepsilon)$, denoted by $N(x,B(x,\varepsilon))$, is a syndetic set of G; a recurrent point if for each $\varepsilon>0$ there is a non-identity element $g\in G$ such that $gx\in B(x,\varepsilon)$; a non-wandering point if for any neighborhood U of x there exists a non-identity element $g\in G$ such that $gU\cap U\neq\emptyset$. Denoted by $A(G)$, $R(G)$ and $\Omega(G)$ the sets of all almost periodic points, all recurrent points and all non-wandering points of a given G-system (X,G). Obviously, $A(G) \subset R(G) \subset \Omega(G)$.

The meeting time set of nonempty open subsets U and V of X is defined by

$$N(U,V) = \{g \in G : gU \cap V \neq \emptyset\}.$$

$K \subset X$ is said to be invariant if $GK = K$. (X,G) is said to be *transitive* if $N(U,V) \neq \emptyset$ for any pair of nonempty open subsets U and V of X. $x \in X$ is called a *transitive point* if the orbit of x is dense in X. We say that (X,G) is *point-transitive* if there is a transitive point in X. (X,G) is said to be

(1) *topological ergodic* if for any pair of nonempty open subsets U and V of X, there exists a Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$of G such that $N(U,V)$ has positive upper density with respect to $\mathcal{F}$;

(2) *strongly ergodic* if $N(U,V)$ is a syndetic set for any pair of nonempty open subsets U and V of X;

(3) *weakly mixing* if $(X \times X, G)$, the self-product system of (X,G), is transitive;

(4) *strongly mixing* if $N(U,V)$ is cofinite, that is the cardinality of $G - N(U,V)$ is finite for any pair of nonempty open subsets U and V of X, where $A - B$ stands for the difference set of A and B. Similarly hereinafter.

$x \in X$ is said to be an *equi-continuous point* if for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $y \in X$ with $d(x,y) < \delta$, we have $d(gx, gy) < \varepsilon$ for each $g \in G$. (X,G) is said to be *equi-continuous* if for each $\varepsilon > 0$ there is $\delta > 0$ such that $d(gx, gy) < \varepsilon$ for every $g \in G$ whenever $x, y \in X$ with $d(x,y) < \delta$.

Let $V \subset X$ and $\delta > 0$. We write

$$S_G(V,\delta) = \{g \in G : \exists\, x, y \in V \text{ such that } d(gx, gy) > \delta\}.$$

(X,G) is *sensitive* if there exists $\delta > 0$ such that $S_G(V,\delta) \neq \emptyset$ for each nonempty open subset V of X; *ergodically sensitive* if there exists $\delta > 0$ such that for each nonempty open subset V of X, $S_G(V,\delta)$ has positive upper density with respect to some Følner sequence of G; *syndetic sensitive* if there exists $\delta > 0$ such that $S_G(V,\delta)$ is syndetic for each nonempty open subset V of X.

1.3. Preparations of ergodic theory

Let (X,G) be a G-system. Denote by $\mathcal{B}(X)$ the Borel σ-algebra of X and respectively by $M(X)$, $M(X,G)$ and $E(X,G)$ the collections of all probability measures, all G-invariant measures and all G-ergodic measures of (X,G). Then

$$\emptyset \neq E(X,G) \subseteq M(X,G) \subseteq M(X)$$

and $M(X)$ is a compact metrizable convex space with weak*-topology. Given $x \in X$, then x determines an element δ_x in $M(X)$ as follows: for each $A \in \mathcal{B}(X)$,

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

A point $x \in X$ is said to be a *support point* of an invariant measure μ if $\mu(U) > 0$ for each neighborhood U of x. The set of all support points of μ is called the *support* of μ, denoted by S_μ. For each Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^\infty$ of G, it is easy to see that $\frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx} \in M(X)$ and the set of all limit points of $\{\frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx}\}_{i=1}^\infty$, denoted by $M_{\mathcal{F},x}$, is contained in $M(X,G)$. For convenience, we write $M_x = \bigcup_{\mathcal{F}\in\mathcal{F}_G} M_{\mathcal{F},x}$ for any given $x \in X$ and $M_{X_0} = \bigcup_{x\in X_0} M_x$ for each nonempty set $X_0 \subset X$.

2. Weakly almost periodic points and minimal centers of attraction of amenable group actions

In the sequel, we always assume that (X,G) is a G-system. Firstly, we introduce the notions of weakly almost periodic points and quasi-weakly almost periodic points of amenable group actions.

Definition 2.1. $x \in X$ is said to be a weakly almost periodic point of (X,G) if for each $\varepsilon > 0$ and every Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^\infty$ of G,

$$\underline{d}_{\mathcal{F}}(N(x,B(x,\varepsilon))) = \liminf_{i\to\infty}\frac{|\{g\in F_i : gx \in B(x,\varepsilon)\}|}{|F_i|} > 0.$$

Denote by $W(G)$ the set of all weakly almost periodic points of (X,G).

Definition 2.2. $x \in X$ is said to be a quasi-weakly almost periodic point of (X,G) if for each $\varepsilon > 0$, there exists a Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^\infty$ of G such that

$$\overline{d}_{\mathcal{F}}(N(x,B(x,\varepsilon))) = \limsup_{i\to\infty}\frac{|\{g\in F_i : gx \in B(x,\varepsilon)\}|}{|F_i|} > 0.$$

Denote by $QW(G)$ the set of all quasi-weakly almost periodic points of (X,G).

Clearly $W(G) \subset QW(G)$. Moreover, $W(G)$ and $QW(G)$ are invariant if G is abelian, i.e., for all $a,b\in G$, $ab = ba$.

Proposition 2.1. *$W(G)$ and $QW(G)$ are invariant under G if G is abelian.*

Proof. Suppose $x \in W(G)$. It suffices to prove $tx \in W(G)$ for each $t \in G$. Given $t \in G$ with $t \neq e$ and let $y = tx$. For any $\varepsilon > 0$, write $T_\varepsilon = \{g \in G : gy \in B(y,\varepsilon)\}$. As t is continuous, there exists $\delta > 0$ such that $tB(x,\delta) \subset B(y,\varepsilon)$. Denote $S_\delta = \{g \in G : gx \in B(x,\delta)\}$, then $S_\delta t \subset T_\varepsilon$. Next we prove that T_ε has positive lower density with respect to any Følner sequence $\{F_i\}_{i=1}^\infty$ of G. In fact, since $\{F_i\}_{i=1}^\infty$ is a Følner sequence of G, so does $\{F_it^{-1}\}_{i=1}^\infty$. Note $|gF_i\Delta F_i| = |(gF_i\Delta F_i)t^{-1}| = |gF_it^{-1}\Delta F_it^{-1}|$, $|F_it^{-1}| = |F_i|$ and $x \in W(G)$, we have

$$\frac{|T_\varepsilon \cap F_i|}{|F_i|} \geq \frac{|S_\delta t \cap F_i|}{|F_i|} = \frac{|S_\delta \cap F_it^{-1}|}{|F_it^{-1}|} > 0.$$

Thus $tx \in W(G)$.

Similarly, we can prove $tx \in QW(G)$ for any $x \in QW(G)$ and $t \in G$. $\square$

In the following, we introduce the concept of minimal center of attraction of amenable group actions.

Definition 2.3. Suppose $X_0 \subset X$ is nonempty. A subset E of X is said to be a center of attraction of X_0 if E is closed and invariant and

$$d_{\mathcal{F}}(N(x, B(E,\varepsilon))) = \lim_{n\to\infty} \frac{|\{g \in F_n : gx \in B(E,\varepsilon)\}|}{|F_n|} = 1$$

for any $\varepsilon > 0$, $x \in X_0$ and each Følner sequence $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ of G. Here $B(E,\varepsilon)$ denotes the ε-neighborhood of E, i.e., $B(E,\varepsilon) = \{x \in X : d(x,E) < \varepsilon\}$.

E is called the minimal center of attraction of X_0, denoted by C_{X_0}, if E is a center of attraction of X_0 and there is no proper subsets of E satisfying the above conditions. When X_0 is a singleton, say $X_0 = \{x\}$, we say that $C_{\{x\}}$ is the minimal center of attraction of x, denoted simply by C_x.

Similar to [12], we introduce the conception of measure center of amenable group actions as follows.

Definition 2.4. A subset F of X is called the measure center of (X,G) with respect to X_0 if F is closed, invariant and $m(F) = 1$ for all $m \in M_{X_0}$ and there is no proper subsets of F satisfying these properties. The measure center of (X,G) with respect to X_0 is denoted by $MC(X_0)$. When $X_0 = X$, we denote it by $MC(G)$.

Next we need to present some properties of weakly almost periodic points and measure centers. Before doing that, we give two lemmas as follows.

Lemma 2.1 ([10])**.** *Suppose $m_i, m \in M(X)$, $i = 1,2,\dots$ and $m_i \to m$ under weak*-topology. Then $m(E) \le \liminf_{i\to\infty} m_i(E)$ for each open subset E of X and $\limsup_{i\to\infty} m_i(F) \le m(F)$ for every closed subset F of X.*

Lemma 2.2. *Let $X_0 \subset X$. Then $MC(X_0) = \mathcal{C}\ell(\bigcup_{m\in M_{X_0}} S_m)$. In particular. $MC(G) = \mathcal{C}\ell(\bigcup_{m\in M_X} S_m)$.*

Proof. Obviously, $MC(X_0) \subset \mathcal{C}\ell(\bigcup_{m\in M_{X_0}} S_m)$. Since for all $m \in M_{X_0}$,

$$m(\mathcal{C}\ell(\bigcup_{m\in M_{X_0}} S_m)) = 1,$$

if there is $x \in \mathcal{C}\ell(\bigcup_{m\in M_{X_0}} S_m)$ such that $x \notin MC(X_0)$, then there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \cap MC(X_0) = \emptyset$. By the definition of the support of an invariant measure, there exists $\mu \in M_{X_0}$ such that $\mu(B(x,\varepsilon)) > 0$ which implies $\mu(MC(X_0)) < 1$. This is a contradiction. $\square$

Lemma 2.3. *$\underline{d}_{\mathcal{F}}(N(x,U)) + \underline{d}_{\mathcal{F}}(N(x, X-U)) \le 1$ for any $x \in X$, $U \subset \mathcal{B}(X)$ and $\mathcal{F} \in \mathcal{F}_G$.*

Proof. Since the proof is easy, we leave it to the reader. $\square$

Proposition 2.2. *$MC(X_0) = C_{X_0}$ for each nonempty subset X_0 of X. In particular, $C_X = MC(G)$ and $C_x = \mathcal{C}\ell(\bigcup_{m\in M_x} S_m)$ for all $x \in X$.*

Proof. By Lemma 2.2, $MC(X_0) = \mathcal{C}\ell(\bigcup_{m\in M_{X_0}} S_m)$. We prove firstly that $MC(X_0)$ is a center of attraction of X_0. If not, then there exist $\varepsilon_0 > 0$ and $x \in X_0$ such that

$$d_{\mathcal{F}}(N(x, B(MC(X_0), \varepsilon_0))) < 1$$

with respect to some Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$ of G (otherwise, we can take a subsequence of $\mathcal{F}$). Write $S = \{g \in G : gx \in B(MC(X_0), \varepsilon_0)\}$. Choose $\mu \in M_{\mathcal{F},x}$, then there exists a subsequence $\mathcal{F}_0 := \{F_{n_i}\}_{i=1}^{\infty}$ of $\mathcal{F}$ such that

$$\mu_{n_i} = \frac{1}{|F_{n_i}|}\sum_{g\in F_{n_i}} \delta_{gx} \longrightarrow \mu \in M_{\mathcal{F},x} \subset M_{X_0}.$$

Then from Lemma 2.1,

$$\begin{aligned}\mu(X - B(MC(X_0), \varepsilon_0)) &\geq \limsup_{i\to\infty} \frac{1}{|F_{n_i}|}\sum_{g\in F_{n_i}} \delta_{gx}(X - B(MC(X_0), \varepsilon_0)) \\ &= \limsup_{i\to\infty} \frac{|(G - S) \cap F_{n_i}|}{|F_{n_i}|}.\end{aligned}$$

Clearly, $\mathcal{F}_0 = \{F_{n_i}\}_{i=1}^{\infty}$ is also a Følner sequence of G and

$$d_{\mathcal{F}_0}(N(x, B(MC(X_0), \varepsilon_0))) < 1.$$

So

$$\mu(X - B(MC(X_0), \varepsilon_0)) \geq \limsup_{i\to\infty} \frac{|(G - S) \cap F_{n_i}|}{|F_{n_i}|} = 1 - \liminf_{i\to\infty} \frac{|S \cap F_{n_i}|}{|F_{n_i}|} > 0.$$

It is in contradiction with $\mu(MC(X_0)) = 1$. So $C_{X_0} \subset MC(X_0)$.

Next we prove that $MC(X_0) = C_{X_0}$. Otherwise, $C_{X_0} \subsetneq MC(X_0)$, take $y \in MC(X_0) - C_{X_0}$, then there are $\varepsilon > 0$ and $\delta > 0$ such that $B(y, \delta) \cap B(C_{X_0}, \varepsilon) = \emptyset$, that is $B(y, \delta) \subset X - B(C_{X_0}, \varepsilon)$. Clearly there is $m \in M_{X_0}$ such that $S_m \cap B(y, \delta) \neq \emptyset$, so $m(B(y, \delta)) > 0$. Without loss of generality, assume that $\frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx} \to m$ for some Følner sequence $\{F_i\}_{i=1}^{\infty}$ of G and $x \in X_0$, then by Lemma 2.1 and Lemma 2.3,

$$\begin{aligned}m(B(C_{X_0}, \varepsilon)) &\leq \liminf_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx}(B(C_{X_0}, \varepsilon)) \\ &\leq 1 - \liminf_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx}(X - B(C_{X_0}, \varepsilon)) \\ &\leq 1 - \liminf_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx}(B(y, \delta)) \\ &\leq 1 - m(B(y, \delta)) < 1,\end{aligned}$$

which is in contradiction with C_{X_0} being the minimal center of attraction of X_0. Thus $C_{X_0} = MC(X_0)$. ∎

We show in the following several propositions the relationships between the omega-limit set of a given weakly almost periodic point and the support of each invariant measure generalized by such a point.

Proposition 2.3. *Given $x \in X$. Then $x \in W(G) \Leftrightarrow \omega(x,G) = S_m$ for each Følner sequence $\mathcal{F}$ of G and all $m \in M_{\mathcal{F},x}$.*

Proof. Suppose that $x \in W(G)$, $\mathcal{F} = \{F_k\}_{k=1}^{\infty}$ is a Følner sequence of G and $m \in M_{\mathcal{F},x}$. Then there exists a sequence $\{m_i\}_{i=1}^{\infty}$ of $\{\frac{1}{|F_k|}\sum_{g\in F_k}\delta_{gx}\}_{k=1}^{\infty}$ such that $m_i \to m$ under weak*-topology. Without loss of generality, we assume that $m_i = \frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx}$ for every $i \geq 1$. For each $\varepsilon > 0$, take $0 < \delta < \varepsilon$. By Lemma 2.1, we have

$$\begin{aligned} m(B(x,\varepsilon)) \geq m(\mathcal{C}\ell(B(x,\delta))) &\geq \limsup_{i\to\infty} m_i(\mathcal{C}\ell(B(x,\delta))) \\ &\geq \limsup_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx}(B(x,\delta)) > 0. \end{aligned}$$

Given $g \in G$. For each neighborhood U of gx, $g^{-1}U$ is a neighborhood of x and $m(U) = m(g^{-1}U) > 0$. Since $G(x)$ is dense in $\omega(x,G)$, for any $y \in \omega(x,G)$ and any neighborhood U of y, $U \cap G(x) \neq \emptyset$. So there is $g_1 \in G$ such that $g_1x \in U$ which means that U is also a neighborhood of g_1x. It follows from the previous conclusion that $m(U) > 0$. Thus every point of $\omega(x,G)$ is a support point of m.

Conversely, if $x \notin W(G)$, from the definition of weakly almost points, there exist a Følner sequence $\mathcal{F}_0 = \{F_i\}_{i=1}^{\infty}$ of G and $\varepsilon_0 > 0$ such that $\underline{d}_{\mathcal{F}_0}(N(x,B(x,\varepsilon_0))) = 0$, that is

$$\liminf_{i\to\infty} \frac{|\{g \in G : gx \in B(x,\varepsilon_0)\} \cap F_i|}{|F_i|} = 0.$$

Without loss of generality, assume that $m_i = \frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx}$ converges to some invariant measure μ under weak*-topology. By Lemma 2.1

$$\mu(B(x,\varepsilon_0)) \leq \liminf_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx}(B(x,\varepsilon_0)) = 0,$$

which implies that x is not a support point of μ, so $\omega(x,G) \neq S_\mu$. □

Proposition 2.4. $x \in C_x \Rightarrow \omega(x,G) = \mathcal{C}\ell(\bigcup_{m\in M_x} S_m)$.

Proof. Suppose $x \in C_x$. Note that C_x is invariant and closed, so $\mathcal{C}\ell(G(x)) \subset C_x$. By the definition of minimal centers of attraction, we have, by Proposition 2.2, $C_x = \omega(x,G) = \mathcal{C}\ell(\bigcup_{m\in M_x} S_m)$. □

Proposition 2.5. *Given $x \in X$. Then $x \in W(G) \Leftrightarrow x \in C_x = S_m$ for every Følner sequence $\mathcal{F}$ of G and $m \in M_{\mathcal{F},x}$.*

Proof. From Proposition 2.3, $x \in W(G) \Leftrightarrow \omega(x,G) = S_\mu$ for each Følner sequence $\mathcal{F}$ of G and $\mu \in M_{\mathcal{F},x}$. Since $S_m = \mathcal{C}\ell(\bigcup_{\mu \in M_x} S_\mu) = C_x$ for any $m \in M_x$, $x \in W(G) \Leftrightarrow x \in S_m = C_x, \forall m \in M_x$. □

Proposition 2.6. $x \in QW(G) \Rightarrow x \in C_x$.

Proof. If $x \in QW(G)$, then for each $\varepsilon > 0$ there is a Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^\infty$ of G such that

$$\overline{d}_{\mathcal{F}}(N(x, B(x, \frac{\varepsilon}{2}))) = \limsup_{i\to\infty} \frac{|\{g \in F_i : gx \in B(x, \frac{\varepsilon}{2})\}|}{|F_i|} > 0.$$

Without loss of generality, we assume that $\mu_i = \frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx} \to \mu \in M_{\mathcal{F},x}$ (otherwise, we take a subsequence of $\mathcal{F}$). Noting $x \in QW(G)$, we have by Lemma 2.1,

$$\mu(B(x,\varepsilon)) \geq \mu(\mathcal{C}\ell(B(x, \frac{\varepsilon}{2}))) \geq \limsup_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx}(B(x, \frac{\varepsilon}{2})) > 0.$$

From Proposition 2.2, $x \in C_x$. □

Proposition 2.7. *Let $x \in R(G)$. Then $C_x = \omega(x,G) \Rightarrow x \in QW(G)$.*

Proof. Since $x \in R(G)$, $x \in \omega(x,G) = C_x$. If $x \notin QW(G)$, by the definition of quasi-weakly almost periodic points, there exists $\varepsilon_0 > 0$ such that $N(x, B(x,\varepsilon_0))$ has upper density zero with respect to each Følner sequence of G. So for any $m \in M_x$ there exists a Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^\infty$ of G such that $m \in M_{\mathcal{F},x}$. Without loss of generality, we suppose $\lim_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx} = m$. Then by the previous conclusion and Lemma 2.1, we have

$$\begin{aligned} m(B(x,\varepsilon_0)) &\leq \liminf_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx}(B(x,\varepsilon_0)) \\ &\leq \limsup_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx}(B(x,\varepsilon_0)) \\ &= \overline{d}_{\mathcal{F}}(N(x, B(x,\varepsilon_0))) = 0 \end{aligned}$$

which is a contradiction with Proposition 2.2. The proof is ended. □

From the above propositions, it is not hard to see

$$x \in QW(G) \Leftrightarrow x \in C_x \Leftrightarrow \omega(x,G) = \mathcal{C}\ell(\bigcup_{m\in M_x} S_m). \tag{2.1}$$

3. Weakly almost periodic points and measure centers of amenable group actions

Let (X,G) be a G-system. A given $x \in X$ is a support point of (X,G) if for each $\varepsilon > 0$ there exists $m \in M(X,G)$ such that $m(B(x,\varepsilon)) > 0$. Denote by $S(G)$ the set of all support points of (X,G).

About the relationships between (quasi) weakly-almost periodic points and support points of a G-system, we have the following Proposition 3.1.

Proposition 3.1. $W(G) \subset QW(G) \subset S(G)$.

Proof. Since the proof is similar to that of Proposition 2.6, we omit it. □

Proposition 3.2. *If X has a countable basis $\{U_i\}_{i=1}^{\infty}$ satisfying that for any $i \geq 1$, there exist $x_i \in X$ and a Følner sequence $\mathcal{F}$ of G such that $\overline{d}_{\mathcal{F}}(N(x_i, U_i)) > 0$, then there exists an invariant measure m with $S_m = X$.*

Proof. By assumptions, for each $i \geq 1$ there exist U_i and $x_i \in X$ and a Følner sequence $\mathcal{F}_i = \{F_j^i\}_{j=1}^{\infty}$ of G such that $\overline{d}_{\mathcal{F}_i}(N(x_i, U_i)) > 0$. Choose $\mu_i \in M_{\mathcal{F}_i, x_i}$ and put $m = \sum_{i=1}^{\infty} \frac{1}{2^i}\mu_i$. Next we will prove $S_m = X$. For any nonempty open subset U of X, there is a nonempty open set V such that $\mathcal{C}\ell(V) \subset U$. Since $\{U_i\}_{i=1}^{\infty}$ is a basis of X, there exists $p > 0$ such that $U_p \subset V$. Then we have

$$\begin{aligned}
\mu_p(U) \geq \mu_p(\mathcal{C}\ell(V)) &\geq \limsup_{j\to\infty} \frac{1}{|F_j^p|} \sum_{g \in F_j^p} \delta_{gx_p}(\mathcal{C}\ell(V)) \\
&\geq \limsup_{j\to\infty} \frac{1}{|F_j^p|} \sum_{g \in F_j^p} \delta_{gx_p}(V) \\
&\geq \limsup_{j\to\infty} \frac{1}{|F_j^p|} \sum_{g \in F_j^p} \delta_{gx_p}(U_p) \\
&= \limsup_{j\to\infty} \frac{|\{g \in F_j^p : gx_p \in U_p\}|}{|F_j^p|} > 0.
\end{aligned}$$

Thus $m(U) > \frac{1}{2^p}\mu_p(U) > 0$ which yields $S_m = X$. □

4. The chaotic dynamics of weakly almost periodic points of amenable group actions

In this section, we mainly discuss some chaotic properties of a given G-system with proper weakly almost periodic points. Firstly we review some needed notions. Given a G-system, a pair $(x, y) \in X \times X$ is said to be *proximal* if there exists $\{t_i\}_{i=1}^{\infty} \subset G$ such that $\lim_{i\to\infty} t_i x = z$ and $\lim_{i\to\infty} t_i y = z$; *distal* if $\inf\{d(tx, ty) : t \in G\} > 0$; a *Li-Yorke pair* if (x, y) is proximal and $\sup\{d(tx, ty) : t \in G\} > 0$.

Theorem 4.1. *Let (X, G) be a G-system and $x \in QW(G) - W(G)$. Then there exists $\mu \in M_x$ such that $S_\mu = C_x$ if and only if there is a Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$ of G such that for any neighborhood U of x,*

$$\liminf_{i\to\infty} \frac{|\{g \in F_i : gx \in U\}|}{|F_i|} > 0. \tag{4.1}$$

In this case, μ is a limit point of $\{\frac{1}{|F_i|}\sum_{g\in F_i} \delta_{gx}\}_{i=1}^{\infty}$.

Proof. Assume that $x \in QW(G) - W(G)$, by (2.1), $C_x = \omega(x,G)$. Suppose that (4.1) holds and μ is a limit point of $\{\frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx}\}_{i=1}^{\infty}$. For any open set U with $U \cap \omega(x,G) \neq \emptyset$, since $G(x)$ is dense in $\omega(x,G)$, there exists $t \in G$ such that $tx \in U$. By the continuity of t, there exists $\varepsilon > 0$ such that $tB(x,\varepsilon) \subset U$, then $\mu(U) \geq \mu(tB(x,\varepsilon))$. As μ is invariant, $\mu(tB(x,\varepsilon)) = \mu(B(x,\varepsilon))$. Take $0 < \delta < \varepsilon$ such that $\mathcal{C}\ell(B(x,\delta)) \subseteq B(x,\varepsilon)$, then by Lemma 2.1,

$$\begin{aligned}\liminf_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx}(B(x,\delta)) &\leq \limsup_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx}(B(x,\delta))\\ &\leq \limsup_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx}(\mathcal{C}\ell(B(x,\delta)))\\ &\leq \mu(\mathcal{C}\ell(B(x,\delta))) \leq \mu(B(x,\varepsilon)).\end{aligned}$$

By (4.1), we have $\mu(U) \geq \mu(B(x,\varepsilon)) > 0$, so $\omega(x,G) \subset S_\mu$. From Proposition 2.2, the proof process of Proposition 2.6 and noting $C_x = \omega(x,G)$, we have $S_\mu = C_x$.

On the other hand, if for any Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$ of G there exists a neighborhood U of x such that

$$\liminf_{i\to\infty} \frac{|\{g \in F_i : gx \in U\}|}{|F_i|} = 0. \tag{4.2}$$

For each $\mu \in M_x$, there is a Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$ of G such that $\lim_{i\to\infty} \frac{1}{|F_i|}\sum_{g\in F_i}\delta_{gx} = \mu$. For the Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$, there exists an open set V satisfying (4.2). So by Lemma 2.1,

$$\mu(V) \leq \liminf_{i\to\infty} \frac{1}{|F_i|}\delta_{gx}(V)) = \liminf_{i\to\infty} \frac{|\{g \in F_i : gx \in V\}|}{|F_i|} = 0$$

which implies that $x \notin S_\mu$. But by (2.1), $x \in C_x$, so $S_\mu \neq C_x$ for any $\mu \in M_x$. □

Next we present some chaotic properties of the systems with proper (quasi) weakly almost periodic points. To do that, the following lemma is necessary.

Lemma 4.1 ([2]). *Let (X,G) be a G-system. Then for any $x \in X$, there exists $y \in A(G)$ such that x and y are proximal.*

Theorem 4.2. *Let (X,G) be a G-system. If there exists a proper weakly almost periodic point $x \in X$, i.e., $x \in W(G) - A(G)$, then (X,G) contains at least countable Li-Yorke pairs.*

Proof. Based on Lemma 4.1, there exists $y \in A(G)$ such that x and y are proximal. Denote $A = \mathcal{C}\ell(G(y))$, then there is $\varepsilon > 0$ such that $a := d(B(x,\varepsilon), A) > 0$. Set $S = \{g \in G : gx \in B(x,\varepsilon)\}$, then S has positive lower density with respect to any Fløner sequence of G. Note that for all $g \in S$, $gy \in A$ and $gx \in B(x,\varepsilon)$, so $d(gx,gy) > d(B(x,\varepsilon),A) > 0$ which implies $\sup_{t\in G} d(tx,ty) > 0$, so (x,y) is a Li-Yorke pair.

Clearly for each $t \in G$, (tx, ty) is also a Li-Yorke pair. Since G is infinite countable, (X, G) contains countable Li-Yorke pairs. $\square$

Remark 4.1. Similarly, we can prove that if there exists a proper quasi-weakly almost periodic point $x \in X$, i.e., $x \in QW(G) - W(G)$, then (X, G) contains countable Li-Yorke pairs.

In [11], the authors introduced the notions of ergodic chaos and strongly ergodic chaos. In the following, we introduce such two chaotic concepts of amenable group actions.

Definition 4.1. A G-system (X, G) is said to be ergodically chaotic, EC for short, if (X, G) is topologically ergodic and ergodically sensitive.

Definition 4.2. A G-system (X, G) is said to be strongly ergodically chaotic, SEC for short, if (X, G) is strongly ergodic and syndetic sensitive.

Obviously SEC implies EC, and there is an example in [11] showing that EC and SEC are two distinct notions.

Proposition 4.1. *Let (X, G) be a G-system and $x \in R(G)$ with $\mathcal{C}\ell(G(x)) = X$. Then (X, G) is topologically ergodic if and only if for any $\varepsilon > 0$, there is a Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$ of G such that $N(B(x, \varepsilon), B(x, \varepsilon))$ has positive upper density with respect to $\mathcal{F}$.*

Proof. The necessity is obvious, so we prove only the sufficiency.

Suppose that U and V are two nonempty open subsets of X. Since $\mathcal{C}\ell(G(x)) = X$, for each $\varepsilon > 0$, there exist g_1 and $g_2 \in G$ such that $g_1(B(x, \varepsilon)) \subset U$ and $g_2(B(x, \varepsilon)) \subset V$. From the assumptions, there exists a Følner sequence $\mathcal{F} = \{F_i\}_{i=1}^{\infty}$ of G such that $N(B(x, \varepsilon), B(x, \varepsilon))$ has positive upper density with respect to $\mathcal{F}$. For any $g \in N(B(x, \varepsilon), B(x, \varepsilon))$, $B(x, \varepsilon) \cap g^{-1}(B(x, \varepsilon)) \neq \emptyset$. So

$$\begin{aligned} g_1(B(x, \varepsilon) \cap g^{-1}(B(x, \varepsilon))) &\subset g_1(B(x, \varepsilon)) \cap g_1 g^{-1}(B(x, \varepsilon)) \\ &= g_1(B(x, \varepsilon)) \cap g_1 g^{-1} g_2^{-1}(g_2(B(x, \varepsilon))) \subset U \cap g_1 g^{-1} g_2^{-1}(V) \end{aligned}$$

which implies that $g_2 g g_1^{-1} \in N(U, V)$. Thus $g_2 N(B(x, \varepsilon), B(x, \varepsilon)) g_1^{-1} \subset N(U, V)$. Next we only need to prove that $g_2 N(B(x, \varepsilon), B(x, \varepsilon)) g_1^{-1}$ has positive upper density with respect to the Følner sequence $\{g_2 F_i g_1^{-1}\}_{i=1}^{\infty}$. Actually we have

$$\frac{|N(U, V) \cap g_2 F_i g_1^{-1}|}{|g_2 F_i g_1^{-1}|} \geq \frac{|N(B(x, \varepsilon), B(x, \varepsilon)) \cap F_i|}{|F_i|}.$$

Since $\{F_i\}_{i=1}^{\infty}$ is a Følner sequence of G, so does $\{g_2 F_i g_1^{-1}\}_{i=1}^{\infty}$. In fact, for every $g \in G$,

$$|g g_2 F_i g_1^{-1} \Delta g_2 F_i g_1^{-1}| = |g g_2 F_i \Delta g_2 F_i| \leq |g g_2 F_i \Delta F_i| + |g_2 F_i \Delta F_i|$$

and note that

$$\lim_{i \to \infty} \frac{|g g_2 F_i \Delta F_i|}{|F_i|} = 0, \quad \lim_{i \to \infty} \frac{|g_2 F_i \Delta F_i|}{|F_i|} = 0, \quad |g_2 F_i g_1^{-1}| = |F_i|,$$

thus

$$\lim_{i\to\infty} \frac{|gg_2F_ig_1^{-1}\Delta g_2F_ig_1^{-1}|}{|g_2F_ig_1^{-1}|} = 0$$

and so $\{g_2F_ig_1^{-1}\}_{i=1}^{\infty}$ is a Følner sequence of G.

Because $N(B(x,\varepsilon), B(x,\varepsilon))$ has positive upper density with respect to $\mathcal{F}$, that is

$$\limsup_{i\to\infty} \frac{|N(B(x,\varepsilon), B(x,\varepsilon)) \cap F_i|}{|F_i|} > 0$$

which implies that

$$\limsup_{i\to\infty} \frac{|N(U,V) \cap g_2F_ig_1^{-1}|}{|g_2F_ig_1^{-1}|} > 0.$$

Since $\{g_2F_ig_1^{-1}\}_{i=1}^{\infty}$ is also a Følner sequence of G, (X,G) is topologically ergodic. The proof is completed. □

Proposition 4.2. *Let (X,G) be a G-system and $x \in R(G)$ with $\mathcal{C}\ell(G(x)) = X$. Then (X,G) is ergodicly sensitive if and only if there exists $\delta > 0$ such that for any $\varepsilon > 0$, $S_G(B(x,\varepsilon),\delta)$ has positive upper density with respect to some Følner sequence of G.*

Proof. The necessity is obvious, we only need to prove the sufficiency. Take a nonempty open subset $V \subset X$, since $\mathcal{C}\ell(G(x)) = X$ there exist $g_0 \in G$ and $\varepsilon > 0$ such that $g_0(B(x,\varepsilon)) \subset V$. By the sufficient assumption, there exists $\delta > 0$ such that $S_G(B(x,\varepsilon),\delta)$ has positive upper density with respect to some Følner sequence of G. Take arbitrarily $g \in S_G(B(x,\varepsilon),\delta)$, then there are $y, z \in B(x,\varepsilon)$ such that $d(gy, gz) > \delta$. Clearly, $g_0y, g_0z \in V$, thus

$$d(gy, gz) = d(gg_0^{-1}(g_0y), gg_0^{-1}(g_0z)) > \delta.$$

So $gg_0^{-1} \in S_G(V,\delta)$ which implies $S_G(B(x,\varepsilon),\delta)g_0^{-1} \subset S_G(V,\delta)$. Similar to the proof of Proposition 4.1, we can prove easily that $S_G(B(x,\varepsilon),\delta)g_0^{-1}$ has positive upper density with respect to some Følner sequence of G. The proof is ended. □

Theorem 4.3. *A transitive but not minimal G-system with a full measure center is strongly ergodically chaotic.*

Proof. For nonempty open subsets U and V of X, let $B = \bigcup_{g\in G} gU$ and $C = B \cap V$. Let μ be such an invariant measure satisfying $\mu(B) > 0$ and $\mu(C) > 0$. Then there exists a compact set $A \subseteq G$ such that

$$\mu\left(\bigcup_{g\in A} gU\right) > \mu(B) - \frac{\mu(C)}{2} > 0.$$

Thus for each $g_1 \in G$, $\mu(g_1(\bigcup_{g\in A} gU)) > 0$ and

$$g_1\left(\bigcup_{g\in G} gU\right)\bigcap V \neq \emptyset.$$

So for each $g_1 \in G$ there exists $g \in A$ such that $g_1 g(U) \cap V \neq \emptyset$. Write

$$S = \{gg_1 : \text{for each } g_1 \in G, \text{ there exists } g \in A \text{ such that } g_1 g(U) \cap V \neq \emptyset\}.$$

For convenience, let $A_0 = A \cup A^{-1}$. Clearly A_0 is compact and $SA_0 = G$. Then S is a syndetic subset of G which means that (X, G) is syndetic transitive.

Next we prove that (X, G) is syndetic sensitive.

Since (X, G) is not minimal, there exists $x \in X$ such that $\mathcal{C}\ell(G(x)) \neq X$. Choose $y \in X - \mathcal{C}\ell(G(x))$ and let $4\delta := d(y, \mathcal{C}\ell(G(x))$. Write $V = B(y, \delta)$, then for any nonempty open set U of X, by the previous conclusion, $N(U, V) = \{g \in G : gU \cap V \neq \emptyset\}$ is syndetic. Thus there is a compact set $M \subseteq G$ such that $MN(U, V) = G$. Let $P = M \cup M^{-1}$, then P is also compact and $PN(U, V) = G$.

Choose $\varepsilon > 0$ such that for all $z \in B(x, \varepsilon)$ and $p \in P$, $d(pz, px) < \delta$. Since $N(U, B(x, \varepsilon)) = \{g \in G : gU \cap B(x, \varepsilon) \neq \emptyset\}$ is syndetic, there exists a compact subset M_1 of G such that $M_1 N(U, B(x, \varepsilon)) = G$. Let $Q = M_1 \cup M_1^{-1}$. We also have $QN(U, B(x, \varepsilon)) = G$.

Take $v \in U$. For any $g \in G$, there exists $q \in Q$ such that $qgv \in B(x, \varepsilon)$. Then for all $p \in P$, we have $pqgv \in B(\mathcal{C}\ell(G(x)), \delta)$. Choose $w \in U$, since $qg \in G$, there exists $p_0 \in P$ such that $p_0 qg(w) \in B(y, \delta)$. Thus

$$d(p_0 qg(v), p_0 qg(w)) > 2\delta.$$

Since g is arbitrary and $p_0 q \in PQ$, let $J = \{g \in G : d(gy, gz) > 2\delta\}$, then $QPJ = G$. Since QP is compact, then J is syndetic. □

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