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FRENET TYPE FORMULAE FOR 2, 3-PLANES IN MINKOWSKI SPACE $\mathbb{L}^6$

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Abstract. We prove the Frenet type formulae for smooth one-parameter family of 2-planes or 3-planes in the Lorentz-Minkowski space $\mathbb{L}^6$. We consider two cases separately: the planes are spacelike or the planes are timelike.

1. Introduction

The 6-dimensional Lorentz-Minkowski space $\mathbb{L}^6$ is $\mathbb{R}^6$ endowed with the Lorentzian metric

$$\begin{aligned} &g(u,v) = \textstyle\sum_{i=1}^{5} u_i v_i - u_6 v_6, \\ &u = (u_1, \ldots, u_6),\ v = (v_1, \ldots, v_6). \end{aligned}$$

A vector $u \in \mathbb{L}^6$ is spacelike if $g(u,u) > 0$, timelike if $g(u,u) < 0$ and null or lightlike if $g(u,u) = 0$ [3]. For a smooth one-parameter family of 2 or 3-planes P_t in $\mathbb{L}^6$, we prove Frenet type formulae for a basis of $\mathbb{L}^6$ which includes the basis of P_t. We consider three cases separately: I) P_t is spacelike, that is, $g|_{P_t}$ is positive definite, II) P_t is timelike, that is, $g|_{P_t}$ is nondegenerate but not positive definite and III) P_t is null, that is, $g|_{P_t}$ is degenerate.

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The Frenet formulae for a smooth regular curve in the 3-dimensional Euclidean space $\mathbb{E}^3$ says that

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $'$ denotes the differention with respect to the arclength, and T, N and B are the frenet frames, and κ is the curvature and τ is the torsion of the curve. We can use the Frenet formula in the study of ruled surfaces in $\mathbb{E}^3$: If we consider T as the direction vector of the lines in the ruled surface, then the Frenet formulae gives a description of the behavior of the lines.

Generalizing the Frenet formulae, Frank and Giering studied the behavior of smooth one-parameter family of k-planes in the Euclidean space $\mathbb{E}^n$ to classify $(k+1)$-dimensional minimal susbmanifolds in $\mathbb{E}^n$ foliated by k-planes with $k < n-1$ [1]: Let P_t be a smooth one-parameter family of k-planes with orthonormal basis $\{f_1(t), f_2(t), \dots, f_k(t)\}$ for $k < n-1$ and $t \in I$. The subspace

$$A(t) = Span\{f_1(t), \dots, f_k(t), f_1'(t), \dots, f_k'(t)\}$$

is called the asymptotic bundle. Then $\dim A(t) = k+m$ with $0 \le m \le k$. Frank and Giering showed that there exists an orthonormal basis of $\mathbb{R}^n$

$$\{e_1(t), \dots, e_k(t), e_{k+1}(t), \dots, e_{k+m}(t), e_{k+m+1}(t), \dots, e_n(t)\}$$

on some subinterval $J \subset I$, for which $Span\{e_1(t), \dots, e_k(t)\} = Span\{f_1(t), \dots, f_k(t)\}$, $A(t) = Span\{e_1(t), \dots, e_k(t), e_{k+1}(t), \dots, e_{k+m}(t)\}$ and the following equations hold (see Satz 5 in [1], [2]):

$$\begin{aligned}
&e_i' = \alpha_i^j e_j + \kappa^i e_{k+i} \\
&e_{m+\rho}' = \alpha_{m+\rho}^l e_l \\
&e_{k+i}' = -\kappa^i e_i + \tau_i^l e_{k+l} + \omega^i e_{k+m+1} + \gamma_i^\lambda e_{k+m+\lambda} \\
&e_{k+m+1}' = -\omega^l e_{k+l} - \beta^\lambda e_{k+m+\lambda} \\
&e_{k+m+\xi}' = -\gamma_l^\xi e_{k+l} + \beta^\xi e_{k+m+1} + \beta_\xi^\lambda e_{k+m+\lambda},
\end{aligned}$$

where

$$\begin{aligned}
&\alpha_j^h = -\alpha_j^h,\ \tau_i^l = -\tau_l^i,\ \beta_\xi^\lambda = -\beta_\lambda^\xi \\
&i, l = 1, 2, \dots, m \\
&j, h = 1, 2, \dots, k \\
&\lambda, \xi = 2, \dots, n-k-m \\
&\rho = 1, 2, \dots, k-m.
\end{aligned}$$

In the case of lines in $\mathbb{R}^3$, the equation is

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa^1 & 0 \\ -\kappa^1 & 0 & \omega^1 \\ 0 & -\omega^1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

We obtain analogous formulae for 2-planes or 3-planes in $\mathbb{L}^6$. The results in this paper can be easily generalized and used in the study of ruled k-dimensional minimal submanifolds in $\mathbb{L}^n$ for $k < n-1$ and ruled minimal submanifolds in the n-dimensional hyperbolic space $\mathbb{H}^n$.

2. The behavior of 2-planes in $\mathbb{L}^6$

In [4], the author gave a detailed proof of the Frenet type formulae for smooth one-parameter family of 2-planes in $\mathbb{R}^4$. We first consider smooth one-parameter family of spacelike 2-planes in $\mathbb{L}^4$. The case of $\mathbb{L}^6$ is a straightforward generalization (cf. Remark 1).

Theorem 1. *Let $\{P_t\}$ be a smooth one-parameter family of spacelike non-parallel planes in $\mathbb{L}^4$ passing through the origin. Locally, there is a one-parameter family of orthonormal frame $\{e_1(t), e_2(t), \ldots, e_4(t)\}$ of $\mathbb{L}^4$ such that $e_1(t)$ and $e_2(t)$ span P_t and one of the following holds with $' = \frac{d}{dt}$.*

I) $A(t)$ is spacelike or timelike with $A(t) = Span\{e_1(t), e_2(t), e_3(t)\}$, and the following equations hold:

$$e_1' = \alpha e_2 + \kappa e_3, \quad e_2' = -\alpha e_1, \quad e_3' = -\kappa e_1 + \eta e_4, \quad e_4' = -\eta e_3,$$

for smooth α and κ, or

II) $\dim A(t) = 4$ and

$$e_1' = \alpha e_2 + \kappa e_3,\ e_2' = -\alpha e_1 + \tau e_4,\ e_3' = -\kappa e_1 + \eta e_4,\ e_4' = -\tau e_2 - \eta e_3,$$

for smooth α, κ, τ and η.

The proof is similar to that of Theorem A in [4]. The case of 2-planes in $\mathbb{L}^6$ is a straightforward generalization.

Proof. Let $\{f_1(t), f_2(t)\}$ be an orthonormal basis of $\{P_t\}$ smooth in t. For $f(t) = \sum_{i=1,2} \gamma_i(t) f_i(t)$ with $\gamma_1(t)$ and $\gamma_2(t)$ smooth and $\gamma_1(t)^2 +$

$\gamma_2(t)^2 = 1$, let

$$\overset{\circ}{f}(t) = f'(t) - \sum_{i=1,2} g\left(f'(t), f_i(t)\right) f_i(t) \tag{1}$$

the projection of $f'(t)$ onto $P_t^\perp$. Note that $P_t^\perp$ is timelike. Omitting t for simplicity, we have

$$\overset{\circ}{f}_1 = f_1' - g\left(f_1', f_2\right) f_2, \quad \overset{\circ}{f}_2 = f_2' - g\left(f_2', f_1\right) f_1.$$

Hence

$$\overset{\circ}{f} = f' - \sum_{i=1,2} g\left(f', f_i\right) f_i = \sum_{i=1,2} \gamma_i \left(f_i' - \sum_{j=1,2} g\left(f_i', f_j\right) f_j \right) = \sum_{i=1,2} \gamma_i \overset{\circ}{f}_i .$$

Therefore

$$g\left(\overset{\circ}{f}, \overset{\circ}{f}\right) = \sum_{i,j=1,2} \gamma_i \gamma_j \, g\left(\overset{\circ}{f}_i, \overset{\circ}{f}_j\right).$$

Note that, for fixed t, $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$ is a quadratic form in γ_1 and γ_2. We have three possibilities for all $t \in I$ (if necessary, we replace I with a suitable subinterval): i) $A(t)$ is spacelike and $\dim A(t) = 3$, or ii) $A(t)$ is timelike and $\dim A(t) = 3$, or iii) $\dim A(t) = 4$.

If i) holds, then $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right) \geq 0$. Fot a fixed $t_0 \in I$, we may assume that $g\left(\overset{\circ}{f}(t_0), \overset{\circ}{f}(t_0)\right)$ attains maximum at $(\gamma_1(t_0), \gamma_2(t_0)) = (1, 0)$. Then $g\left(\overset{\circ}{f}_2(t_0), \overset{\circ}{f}_2(t_0)\right) = 0$. Hence $\overset{\circ}{f}_2(t_0) = f_2'(t_0) - g\left(f_2'(t_0), f_1(t_0)\right) f_1(t_0) = 0$.

To find $e_1(t)$ and $e_2(t)$, first let $e_1(t)$ be the unit vector maximizing $g\left(\overset{\circ}{f}(t), \overset{\circ}{f}(t)\right)$ for each $t \in I$. Then $e_1(t)$ is smooth in t and $g\left(\overset{\circ}{e}_1(t), \overset{\circ}{e}_1(t)\right) > 0$. Choose e_2 in such a way that $\{e_1(t), e_2(t)\}$ is an orthonormal basis of P_t smooth in t. Then e_2 is the unit vector minimizing $g\left(\overset{\circ}{f}(t), \overset{\circ}{f}(t)\right)$, whose value is 0. Define e_3 by

$$g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)^{\frac{1}{2}} e_3 := \overset{\circ}{e}_1 = e_1' - g\left(e_1', e_2\right) e_2.$$

Then an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{L}^4$, smooth in t, satisfies

$$\begin{aligned} e_1' &= g\left(e_1', e_2\right) e_2 + g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)^{\frac{1}{2}} e_3, \\ e_2' &= g\left(e_2', e_1\right) e_1 = -g\left(e_1', e_2\right) e_1, \\ e_3' &= g\left(e_3', e_4\right) e_4 - g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)^{\frac{1}{2}} e_1, \\ e_4' &= -g\left(e_3', e_4\right) e_3. \end{aligned} \tag{2}$$

If ii) holds, then then $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right) \leq 0$. For each $t \in I$, let e_1 be the unit vector minimizing $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$, and let $\{e_1, e_2\}$ be an orthonormal basis of P_t smooth in t. Then $g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right) < 0$ and $g\left(\overset{\circ}{e}_2, \overset{\circ}{e}_2\right) = 0$. Define e_3 by

$$\left(-g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)\right)^{\frac{1}{2}} e_3 := \overset{\circ}{e}_1 = e_1' - g\left(e_1', e_2\right) e_2.$$

Choose e_4 so that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathbb{L}^4$ smooth in t. Then e_1, e_2, e_3 and e_4 satisfies (2). This completes the proof of I).

If iii) holds, then $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$ has positive maximum and negative minimum for each fixed t. Let e_1 be the unit vector maximizing $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$, and let e_2 be the unit vector minimizing $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$ for each t. Since $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$ is a quadratic form in γ_1 and γ_2, we have $g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_2\right) = 0$. Let e_3 and e_4 be defined by

$$\begin{aligned} g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)^{\frac{1}{2}} e_3 &: = \overset{\circ}{e}_1 = e_1' - g\left(e_1', e_2\right) e_2 \\ \left(-g\left(\overset{\circ}{e}_2, \overset{\circ}{e}_2\right)\right)^{\frac{1}{2}} e_4 &: = \overset{\circ}{e}_2 = e_2' - g\left(e_2', e_1\right) e_1. \end{aligned}$$

Then the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{L}^4$ satisfies

$$\begin{aligned} e_1' &= g(e_1', e_2)\, e_2 + g\left(\mathring{e}_1, \mathring{e}_1\right)^{\frac{1}{2}} e_3, \\ e_2' &= -g(e_1', e_2)\, e_1 + \left(-g\left(\mathring{e}_2, \mathring{e}_2\right)\right)^{\frac{1}{2}} e_4, \\ e_3' &= -g\left(\mathring{e}_1, \mathring{e}_1\right)^{\frac{1}{2}} e_1 + g(e_3', e_4)\, e_4, \\ e_4' &= -\left(-g\left(\mathring{e}_2, \mathring{e}_2\right)\right)^{\frac{1}{2}} e_2 - g(e_3', e_4)\, e_3. \end{aligned}$$

This completes the proof. □

REMARK 1. The generalization of the above theorem to $\mathbb{L}^6$ is straightforward. For example, in the case of spacelike 2-planes in $\mathbb{L}^6$, first we define f for a given orthonormal basis $\{f_1, f_2\}$ of P_t as above. If $\dim A = 4$ and A is spacelike, then we find e_1, e_2, e_3 and e_4 as above, and choose e_5 and e_6 so that $\{e_1, \ldots, e_6\}$ is a smooth orthonormal basis of $\mathbb{L}^6$. Then we have

$$\begin{aligned} e_1' &= g(e_1', e_2)\, e_2 + g\left(\mathring{e}_1, \mathring{e}_1\right)^{\frac{1}{2}} e_3, \\ e_2' &= -\, g(e_1', e_2)\, e_1 + g\left(\mathring{e}_2, \mathring{e}_2\right)^{\frac{1}{2}} e_4. \end{aligned}$$

Moreover,

$$\begin{aligned} e_3' &= -g\left(\mathring{e}_1, \mathring{e}_1\right)^{\frac{1}{2}} e_1 + g(e_3', e_4)\, e_4 + g(e_3', e_5)\, e_5 + g(e_3', e_6)\, e_6, \\ e_4' &= -g\left(\mathring{e}_2, \mathring{e}_2\right)^{\frac{1}{2}} e_2 + g(e_4', e_3)\, e_3 + g(e_4', e_5)\, e_5 + g(e_4', e_6)\, e_6, \\ e_5' &= g(e_5', e_3)\, e_3 + g(e_5', e_4)\, e_4 + g(e_5', e_6)\, e_6, \\ e_6' &= g(e_6', e_3)\, e_3 + g(e_6', e_4)\, e_4 + g(e_6', e_5)\, e_5. \end{aligned}$$

The remaining cases can be dealt with similarly. The case that P_t are timelike is similar, and we consider the proof only in $\mathbb{L}^4$.

THEOREM 2. *Let $\{P_t\}$ be a smooth one-parameter family of timelike non-parallel planes in $\mathbb{L}^4$ passing through the origin. There is a one-parameter family of orthonormal frame $\{e_1(t), e_2(t), e_3(t), e_4(t)\}$ of $\mathbb{L}^4$ such that $e_1(t)$ and $e_2(t)$ span P_t and the following equations hold:*

$$e_1' = \alpha e_2 + \kappa e_3, \;\; e_2' = -\alpha e_1 + \tau e_4, \;\; e_3' = -\kappa e_1 + \eta e_4, \;\; e_4' = -\tau e_3 + \eta e_3,$$

for smooth α, κ, τ and η. Furthermore, if $\dim A(t) = 3$ *then* $\tau = 0$.

Proof. Let $\{f_1, f_2\}$ be a smooth one-parameter family of orthonormal basis of P_t. Let $f = \sum_{i=1,2} \gamma_i f_i$ for smooth γ_1 and γ_2 satisfying $\gamma_1^2 + \gamma_2^2 = 1$. Then $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right) \geq 0$. Let e_1 be the unit vector maximizing $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$, and let e_2 be unit vector minimizing $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$. Then $\{e_1, e_2\}$ spans P_t.

If $\dim A(t) = 3$, then $g\left(\overset{\circ}{e}_2, \overset{\circ}{e}_2\right) = 0$. Define e_3 by

$$g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)^{\frac{1}{2}} e_3 := \overset{\circ}{e}_1 = e_1' - g\left(e_1', e_2\right) e_2,$$

and let e_4 be a smooth unit vector field perpendicular to e_1, e_2 and e_3.

If $\dim A(t) = 4$, then $g\left(\overset{\circ}{e}_2, \overset{\circ}{e}_2\right) \neq 0$. Define e_3 and e_4 by

$$g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)^{\frac{1}{2}} e_3 := \overset{\circ}{e}_1 = e_1' - g\left(e_1', e_2\right) e_2$$

$$g\left(\overset{\circ}{e}_2, \overset{\circ}{e}_2\right)^{\frac{1}{2}} e_4 := \overset{\circ}{e}_2 = e_2' - g\left(e_2', e_1\right) e_1.$$

Then we have

$$e_1' = g\left(e_1', e_2\right) e_2 + g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)^{\frac{1}{2}} e_3,$$

$$e_2' = -g\left(e_1', e_2\right) e_1 + g\left(\overset{\circ}{e}_2, \overset{\circ}{e}_2\right)^{\frac{1}{2}} e_4,$$

$$e_3' = -g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)^{\frac{1}{2}} e_1 + g\left(e_3', e_4\right) e_4,$$

$$e_4' = -g\left(\overset{\circ}{e}_2, \overset{\circ}{e}_2\right)^{\frac{1}{2}} e_2 - g\left(e_3', e_4\right) e_3.$$

This completes the proof. □

3. The behavior of 3-planes in $\mathbb{L}^6$

We state the result in full generality, that is, $\dim A = 6$. If $\dim A = 4$, then $\kappa_2 = 0$ and $\kappa_3 = 0$, and if $\dim A = 5$, then $\kappa_3 = 0$ in the following theorem.

THEOREM 3. *Let $\{P_t\}$ be a smooth one-parameter family of spacelike or timelike non-parallel 3-planes in $\mathbb{L}^6$ passing through the origin. There is a one-parameter family of orthonormal frame $\{e_1(t), \ldots, e_6(t)\}$ of $\mathbb{L}^6$ such that $e_1(t)$, $e_2(t)$ and e_3 span P_t and the following equations hold:*

$$\begin{aligned} e_1' &= \alpha_1^2 e_2 + \alpha_1^3 e_3 + \kappa_1 e_4, \quad e_2' = -\alpha_1^2 e_1 + \alpha_2^3 e_3 + \kappa_2 e_5, \\ e_3' &= -\alpha_1^3 e_1 - \alpha_2^3 e_2 + \kappa_3 e_6, \quad e_4' = -\kappa_1 e_1 + \eta_4^5 e_5 + \eta_4^6 e_6, \\ e_5' &= -\kappa_2 e_2 - \eta_4^5 e_4 + \eta_5^6 e_6, \quad e_6' = -\kappa_3 e_3 - \eta_4^6 e_4 - \eta_5^6 e_5, \end{aligned}$$

where α_i^j, κ_i and η_{3+i}^{3+j}, for $i, j = 1, 2, 3$, are smooth.

The proof is a straightforward generalization of the proof of Theorem 1.

Proof. We give the proof only for the case that P_t is spacelike. The proof for the case that P_t is timelike is similar. Let $\{f_1(t), f_2, f_3(t)\}$ be an orthonormal basis of P_t smooth in $t \in I$. Let $f = \sum_{i=1}^3 \gamma_i f_i$ for smooth γ_i satisfying $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$. Let

$$\overset{\circ}{f} = f' - \sum_{i=1}^{3} g\left(f', f_i\right) f_i = \sum_{i=1}^{3} \gamma_i \overset{\circ}{f_i} .$$

Then

$$g\left(\overset{\circ}{f}, \overset{\circ}{f}\right) = \sum_{i,j=1}^{3} \gamma_i \gamma_j g\left(\overset{\circ}{f_i}, \overset{\circ}{f_j}\right)$$

is a quadratic from in γ_i, $i = 1, 2, 3$. Since $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{S}^2$, $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$ attains positive maximum and negative minimum for each fixed t. Let e_1 and e_3 be the unit vector maximizing and minimizing $g\left(\overset{\circ}{f}, \overset{\circ}{f}\right)$ respectively. Let e_2 be the remaining eigenvector of the symmetric matrix $g\left(\overset{\circ}{f_i}, \overset{\circ}{f_j}\right)_{ij}$, for $i, j = 1, 2, 3$. Since $P_t^{\perp}$ is timelike, $g\left(\overset{\circ}{e_3}, \overset{\circ}{e_3}\right) < 0$ and

$g\left(\overset{\circ}{e}_2, \overset{\circ}{e}_2\right) > 0$. Define e_4m e_5 and e_6 by

$$g\left(\overset{\circ}{e}_1, \overset{\circ}{e}_1\right)^{\frac{1}{2}} e_4 := \overset{\circ}{e}_1 = e_1' - g\left(e_1', e_2\right) e_2 - g\left(e_1', e_3\right) e_3$$

$$g\left(\overset{\circ}{e}_2, \overset{\circ}{e}_2\right)^{\frac{1}{2}} e_5 := \overset{\circ}{e}_2 = e_2' - g\left(e_2', e_1\right) e_1 - g\left(e_2', e_3\right) e_3$$

$$\left(-g\left(\overset{\circ}{e}_3, \overset{\circ}{e}_3\right)\right)^{\frac{1}{2}} e_6 := \overset{\circ}{e}_3 = e_3' - g\left(e_3', e_1\right) e_1 - g\left(e_3', e_2\right) e_2.$$

Then $\{e_1, \ldots, e_6\}$ is the desired orthonormal basis of $\mathbb{L}^6$. □

References

[1] H. Frank and O. Giering, *Verallegemeinerte Regelflächen.* Math. Z. **150** (1976), 261–271.
[2] H. Hagen, *Die minimalen* $(k+1)$*-Regelflaechen.* Arch. Math. **42**, (1984), 76–84.
[3] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York-London, 1983.
[4] S. Park, *Minimal and Constant mean curvature surfaces in* $\mathbb{S}^3$ *foliated by circles,* preprint.

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