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SOME METRIC ON EINSTEIN LORENTZIAN WARPED PRODUCT MANIFOLDS

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Abstract. In this paper, let $M = B \times_{f^2} F$ be an Einstein Lorentzian warped product manifold with 2−dimensional base. We study the geodesic completeness of some metric with constant curvature. First of all, we discuss the existence of nonconstant warping functions on M. As the results, we have some metric g admits nonconstant warping functions f. Finally, we consider the geodesic completeness on M.

1. Introduction

R.L. Bishop and B. O'Neill introduced singly warped products or simply warped products to construct Riemannian manifolds with negative sectional curvature([5]). Later, we study the existence of some metric on Riemannian warped product manifolds([7], [12], [18]). And we consider the existence and the completeness of some metric on Lorentzian warped product manifolds([2], [3], [4], [8], [11], [14], [15], [16], [17], [19], [25], [26]).

In the present work, we study multiply warped products or multi-warped products. One can also generalize singly warped products to

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multiply warped products. A multiply warped product (M, g) is a product manifold of the form $M = B \times_{f_1} F_1 \times \cdots \times_{f_m} F_m$ with the metric $g = g_B \oplus f_1^2 g_{F_1} \oplus \cdots \oplus f_m^2 g_{F_m}$, where for each $i \in \{1, \cdots, m\}$, $f_i : B \to (0, \infty)$ is smooth and (F_i, g_{F_i}) is a pseudo-Riemannian manifold. In particular, when $B = (a, b)$ with the negative definite metric $g_B = -dt^2$, the corresponding multiply warped product $M = (a, b) \times_{f_1} F_1 \times \cdots \times_{f_m} F_m$ with the metric $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus \cdots \oplus f_m^2 g_{F_m}$ is called a Lorentzian multiply warped product, where for each $i \in \{1, \cdots, m\}$, (F_i, g_{F_i}) is a Riemannian manifold and $-\infty \leq a < b \leq \infty$([27]).

In a recently, we study an Einstein manifold. We obtain some results an Einstein warped product manifold([6], [9], [10], [13], [20], [21], [22], [23]). In [1], the author may also consider for that purpose special case of an Einstein warped product manifold $M = B \times_{f^2} F$ with $2-$dimensional base, $B = (a, b) \times_{f'^2} \mathbb{R}$, where $-\infty \leq a < b \leq \infty$. And we study the existence of nonconstant warping functions on M([24]).

In this paper, we study an Einstein Lorentzian warped product manifold $M = B \times_{f^2} F$ with $2-$dimensional base, $B = (a, b) \times_{f'^2} \mathbb{R}$ when (a, b) with the negative definite metric $-dt^2$, where $-\infty \leq a < b \leq \infty$. First of all, we study the existence of nonconstant warping functions f depends on the signs of λ_0. As a results, we have some metric g admits nonconstant warping functions f. Finally, we consider the geodesic completeness on M.

2. Preliminaries

We denote by Ric_F be the Ricci curvature of (F, g_F) and Ric_B be the Ricci curvature of (B, g_B). We denote by Ric^B and Ric^F the lifts to M of Ricci curvatures of B and F, respectively.

Proposition 2.1. *The Ricci curvature Ric of the warped product manifold $M = B \times_{f^2} F$ satisfies*

(i) $Ric(V, W) = Ric^F(V, W) + g(V, W)[\ (\frac{\Delta f}{f} - (p-1)\frac{||df||^2}{f^2})\ \pi\ ]$,

(ii) $Ric(X, V) = 0$,

(iii) $Ric(X,Y) = Ric^B(X,Y) - \frac{p}{f}H^f(X,Y)$

for any vertical vectors V, W and any horizontal vectors X, Y. We are defined by df is the gradient of f for g_B and H^f is the Hessian of f for g_B. We denote by Δf is the Laplacian of f for g_B and $p = \dim F$ ([1]).

Corollary 2.2. *The warped product $M = B \times_{f^2} F$ is Einstein manifold (with $Ric = \lambda g$) if and only if g_F, g_B and f satisfy*

(i) *(F, g_F) is Einstein (with $Ric_F = \lambda_0 g_F$),*

(ii) $\frac{\Delta f}{f} - (p-1)\frac{\|df\|^2}{f^2} + \frac{\lambda_0}{f^2} = \lambda,$

(iii) $Ric_B - \frac{p}{f}H^f = \lambda g_B.$

Obviously, (ii) and (iii) are two differential equations for f on (B, g_B)([1]).

Remark 2.3. Using Corollary 2.2 (ii) and (iii), we replace the unique equation

(2.1)

$$Ric_B - \frac{p}{f}H^f = \frac{1}{2}[\ s_B + 2p\frac{\Delta f}{f} - p(p-1)\frac{\|df\|^2}{f^2} + p\frac{\lambda_0}{f^2} - (p+q-2)\lambda\ ]g_B,$$

where $q = \dim B$.

Proposition 2.4. *In the special case of a warped product $B \times_{f^2} F$ over $2-$dimensional base, we have $Ric_B = \frac{1}{2}s_B g_B$ and $q = 2$. Hence equation (2.1) implies that*

$$H^f = -\frac{1}{2}[\ 2\Delta f - (p-1)\frac{||df||^2}{f} + \frac{\lambda_0}{f} - \lambda f\ ]g_B. \tag{2.2}$$

Lemma 2.5. *Let $B = (a,b) \times_{f'(t)^2} \mathbb{R}$ be $2-$dimensional manifold for $t \in (a,b)$ and $u \in \mathbb{R}$, where $-\infty \leq a < b \leq \infty$. On (B, g_B) the equation $H^f = -f'' g_B$ admits a nonconstant solution f if and only if, locally at* *points where $df \neq 0$, there exists local coordinates (t, u) such that f is a function of t alone.*

Proof. By a proof similar Lemma 9.117 in [1], then $H^f = -f'' g_B$. □

With the notations of the Lemma 2.5, we have an ordinary differential equation for in the variable t

$$2f''(t) + (p-1)\frac{f'(t)^2}{f(t)} + \frac{\lambda_0}{f(t)} - \lambda f(t) = 0, \tag{2.3}$$

where $\|df\|^2 = -[f'(t)]^2$ and $\Delta f = 2f''(t)$.

The following notation and Remark 2.6 are needed to show the geodesic completeness.

Notation. Let $M = (a, b) \times_{f_1} F_1 \times \cdots \times_{f_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus \cdots \oplus f_m^2 g_{F_m}$, where $-\infty \leq a < b \leq \infty$. If $\mathcal{B} = \{f_1, \cdots, f_m\}$ and for some $k \in \{1, \cdots, m\}$ and for some subset $\{\bar{f}_1, \cdots, \bar{f}_k\}$ of $\mathcal{B}$, then

$$r[\bar{f}_1, \cdots, \bar{f}_k] = \prod_{i=1}^{k} \bar{f}_i \quad \text{and} \quad h[\bar{f}_1, \cdots, \bar{f}_k] = \sum_{i=1}^{k} \bar{f}_i^{\,2}.$$

Also, it is assumed that $h[\bar{f}_1] = 1$ for any $\bar{f}_1$([27]).

Remark 2.6. Let $M = (a, b) \times_{f_1} F_1 \times \cdots \times_{f_m} F_m$ be a Lorentzian multiply warped product with metric $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus \cdots \oplus f_m^2 g_{F_m}$, where $-\infty \leq a < b \leq \infty$. Suppose that $(F_i,\ g_{F_i})$ is a complete Riemannian manifold for any $i \in \{1, \cdots, m\}$ and $\mathcal{B} = \{f_1, \cdots, f_m\}$. Then

every future directed time-like geodesic is future(respectively past) complete if and only if $\lim\limits_{t \to b-} \int_{t_1}^{t} \frac{r[\bar{f}_1, \cdots, \bar{f}_k](s)}{\sqrt{r[\bar{f}_1, \cdots, \bar{f}_k]^2(s) + h[\bar{f}_1, \cdots, \bar{f}_k]}} ds = \infty$

(respectively $\lim\limits_{t \to a+} \int_{t}^{t_1} \frac{r[\bar{f}_1, \cdots, \bar{f}_k](s)}{\sqrt{r[\bar{f}_1, \cdots, \bar{f}_k]^2(s) + h[\bar{f}_1, \cdots, \bar{f}_k]}} ds = \infty$) for some $t_1 \in (a, b)$ and for any $k \in \{1, \cdots, m\}$ and for any subset $\{\bar{f}_1, \cdots, \bar{f}_k\}$ of $\mathcal{B}$(cf. Theorem 4.8 in [27]).

3. The existence of nonconstant warping functions

Let $M = (a, b) \times_{f'(t)^2} \mathbb{R} \times_{f(t)^2} F$ be an Einstein Lorentzian warped product manifold, where $f(t)$ and $f'(t)$ are smooth functions and $-\infty \leq a < b \leq \infty$. Let $\dim F = p > 1$.

First of all, if we denote $f(t) = z(t)^{\frac{2}{p+1}}$, then equation (2.3) can be changed into

$$(3.1) \qquad [z'(t)]^2 = -\frac{(p+1)^2\, \lambda_0}{4(p-1)}\, z(t)^{2-\frac{4}{p+1}} + \frac{(p+1)\lambda}{4}[z(t)]^2,$$

where $z(t)$ is a positive function. Thus we study positive solution $z(t)$ of equation (3.1).

Theorem 3.1. *Suppose that $\lambda_0 = 0$. If λ is a constant, then there exists a nonconstant solution $z(t)$ of equation (3.1).*

(i) *For $\lambda = 0$, there does not exist a nonconstant solution of equation* (3.1).

(ii) *For $\lambda > 0$, we have a solution $z(t) = e^{\pm\sqrt{\frac{(p+1)\lambda}{4}}\, t+c}$, where c is a constant.*

(iii) *For $\lambda < 0$, there does not exist a solution of equation* (3.1).

Proof. For $\lambda_0 = 0$, equation (3.1) implies that

$$(3.2) \qquad [z'(t)]^2 = \frac{(p+1)\lambda}{4}\, [z(t)]^2.$$

(i) For $\lambda = 0$, equation (3.2) implies that $[z'(t)]^2 = 0$ and $z'(t) = 0$. An integration gives $z(t) = c$, where c is a positive constant. Because $z(t) = c$ is not a nonconstant, thus $z(t) = c$ is not our solution.

(ii) For $\lambda > 0$, equation (3.2) implies that we get $z'(t) = \pm\sqrt{\frac{(p+1)\lambda}{4}}\, u(t)$. An integration gives $\ln|z(t)| = \pm\sqrt{\frac{(p+1)\lambda}{4}}\, t + c$, where c is a constant.

Therefore we have $z(t) = e^{\pm\sqrt{\frac{(p+1)\lambda}{4}}\, t+c}$, where c is a constant.

(iii) For $\lambda < 0$, equation (3.2) implies that $[z'(t)]^2 < 0$. Which is a contradiction. Hence there does not exist a solution of equation (3.1). □

Remark 3.2. Let M be an Einstein Lorentzian warped product manifold. From above Theorem 3.1 (ii), for $\lambda_0 = 0$ and $\lambda > 0$, we have that equation (2.3) satisfies a nonconstant warping function $f(t) = e^{\pm\sqrt{\frac{\lambda}{p+1}}\ t+\frac{2c}{p+1}}$ on $(-\infty, \infty)$, where c is a constant.

Theorem 3.3. *Suppose that* $\lambda_0 > 0$. *If* λ *is a constant, then there exists a nonconstant solution* $z(t)$ *of equation (3.1).*

(i) *For* $\lambda \leq 0$, *there does not exist a solution of equation (3.1).*

(ii) *For* $\lambda > 0$, *we have* $z(t) = \left(\sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}}\cosh(\sqrt{\frac{\lambda}{p+1}}\ t + c)\right)^{\frac{p+1}{2}}$, *where* c *is a constant.*

Proof. (i) For $\lambda \leq 0$, equation (3.1) implies that $[z'(t)]^2 < 0$. Which is a contradiction. Therefore there does not exist a solution of equation (3.1).

(ii) For $\lambda > 0$, first of all, equation (3.1) implies that we rewritten as

$$\int \frac{1}{z(t)\sqrt{-\frac{(p+1)^2\ \lambda_0}{4(p-1)}\ z(t)^{-\frac{4}{p+1}} + \frac{(p+1)\lambda}{4}}}\ du = \pm\int\ dt.$$

Putting $\dfrac{(p+1)^2\lambda_0}{4(p-1)} = I > 0$ and $\dfrac{(p+1)\lambda}{4} = J > 0$, then we get the equation

$$\int \frac{1}{z(t)\sqrt{J - Iz(t)^{\frac{-4}{p+1}}}}\ du = \pm\int\ dt.$$

By using trigonometric substitution, $z(t)^{\frac{-2}{p+1}} = \dfrac{\sqrt{J}}{\sqrt{I}}\sin\theta$, then we obtain

$$-\int \csc\theta\ d\theta = \pm\int \frac{2\sqrt{J}}{p+1}\ dt.$$

Upon integration, we become $\ln|\ \csc\theta + \cot\theta\ | = \pm\dfrac{2\sqrt{J}}{p+1}\ t + c$, where c is a constant. Here we have $\ln|\sqrt{J}z(t)^{\frac{2}{p+1}} + \sqrt{Jz(t)^{\frac{4}{p+1}} - I}| = \pm\frac{2\sqrt{J}}{p+1}\ t + c + \ln\sqrt{I}$, where c is a constant.

Therefore we have $z(t) = \left(\sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}}\cosh(\sqrt{\frac{\lambda}{p+1}}t + c)\right)^{\frac{p+1}{2}}$, where c is a constant. □

REMARK 3.4. Let M be an Einstein Lorentzian warped product manifold. From above Theorem 3.3 (ii), for $\lambda_0 > 0$ and $\lambda > 0$, we have that equation (2.3) satisfies a nonconstant warping function $f(t) = \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}}\cosh(\sqrt{\frac{\lambda}{p+1}}\ t + c\ )$ on $\left(-\sqrt{\frac{p+1}{\lambda}}\ c,\ \infty\right)$, where c is a constant.

THEOREM 3.5. *Suppose that $\lambda_0 < 0$. If λ is a constant, then there exist nonconstant solutions $z(t)$ of equation (3.1).*

(i) *For $\lambda = 0$, we have $z(t) = \left(\pm\sqrt{\frac{-\lambda_0}{p-1}}\ t + \frac{2c}{p+1}\right)^{\frac{p+1}{2}}$, where c is a constant.*

(ii) *For $\lambda > 0$, we get $z(t) = \left(\sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}}\sinh(\pm\sqrt{\frac{\lambda}{p+1}}\ t + c\ )\right)^{\frac{p+1}{2}}$, where c is a constant.*

(iii) *For $\lambda < 0$, we become $z(t) = \left(\sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}}\cos(\pm\sqrt{\frac{-\lambda}{p+1}}\ t + c\ )\right)^{\frac{p+1}{2}}$, where c is a constant.*

Proof. (i) For $\lambda = 0$, equation (3.1) implies that we have equation

$$z'(t) = \pm\sqrt{\frac{-(p+1)^2\lambda_0}{4(p-1)}}\ z(t)^{1-\frac{2}{p+1}}.$$

Therefore we have $z(t) = \left(\ \pm\sqrt{\frac{-\lambda_0}{p-1}}\ t + \frac{2c}{p+1}\right)^{\frac{p+1}{2}}$, where c is a constant.

(ii) For $\lambda > 0$. By a proof similar to Theorem 3.3 (ii), equation (3.1) implies that we rewritten as

$$\int \frac{1}{z(t)\sqrt{-\frac{(p+1)^2\ \lambda_0}{4(p-1)}\ z(t)^{-\frac{4}{p+1}} + \frac{(p+1)\lambda}{4}}}\ du = \pm \int\ dt.$$

Putting $-\dfrac{(p+1)^2\lambda_0}{4(p-1)} = I > 0$ and $\dfrac{(p+1)\lambda}{4} = J > 0$, then we have the equation

$$\int \frac{1}{z(t)\sqrt{\ Iz(t)^{\frac{-4}{p+1}}\ + J}}\ du = \pm \int\ dt.$$

By using trigonometric substitution, $z(t)^{\frac{-2}{p+1}} = \dfrac{\sqrt{J}}{\sqrt{I}} \tan\theta$, then we obtain

$$-\int \csc\theta\ d\theta = \pm \int \frac{2\sqrt{J}}{p+1} dt.$$

Therefore we have $z(t) = \left(\ \sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}} \sinh(\pm\sqrt{\frac{\lambda}{p+1}}\ t + c\ )\right)^{\frac{p+1}{2}}$, where c is a constant.

(iii) For $\lambda < 0$. By a proof similar to Theorem 3.3 (ii) and Theorem 3.5 (ii), putting $-\dfrac{(p+1)^2\lambda_0}{4(p-1)} = I > 0$ and $-\dfrac{(p+1)\lambda}{4} = J > 0$, then we have

$$\int \frac{1}{z(t)\sqrt{Iz(t)^{\frac{-4}{p+1}} - J}}\ du = \pm \int dt.$$

By using trigonometric substitution, $z(t)^{\frac{-2}{p+1}} = \dfrac{\sqrt{J}}{\sqrt{I}} \sec\theta$, then we get

$$\int d\theta = \pm \int \frac{2\sqrt{J}}{p+1}\ dt.$$

Therefore we have $z(t) = \left(\sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}}\ \cos(\pm\sqrt{\frac{-\lambda}{p+1}}\ t +\ c)\right)^{\frac{p+1}{2}}$, where c is a constant. □

Remark 3.6. Let M be an Einstein Lorentzian warped product manifold. From above Theorem 3.5, we consider that equation (2.3) satisfies nonconstant warping functions $f(t)$.

(i) For $\lambda_0 < 0$ and $\lambda = 0$, we become $f(t) = \sqrt{\dfrac{-\lambda_0}{p-1}}\, t + \dfrac{2c}{p+1}$ on $\left(-\dfrac{2c}{p+1}\sqrt{\dfrac{p-1}{-\lambda_0}},\ \infty\right)$, where c is a constant.

(ii) For $\lambda_0 < 0$ and $\lambda > 0$, we get $f(t) = \sqrt{\dfrac{-(p+1)\lambda_0}{(p-1)\lambda}}\sinh(\sqrt{\dfrac{\lambda}{p+1}}\, t + c)$ on $\left(-\sqrt{\dfrac{p+1}{\lambda}}\, c,\ \infty\right)$, where c is a constant.

(iii) For $\lambda_0 < 0$ and $\lambda < 0$, we have $f(t) = \sqrt{\dfrac{(p+1)\lambda_0}{(p-1)\lambda}}\cos(\sqrt{\dfrac{-\lambda}{p+1}}\, t + c\ )$ on $\left(\ (2n\pi - \dfrac{\pi}{4} - c)\sqrt{\dfrac{p+1}{-\lambda}},\ (2n\pi + \dfrac{3\pi}{4} - c)\sqrt{\dfrac{p+1}{-\lambda}}\ \right)$, where c is a constant and n is a integer.

Remark 3.7. The behaviour of the nonconstant warping functions depends on the signs of λ_0 and λ. Then we reduced to the following sets of solutions besides the constant case when $c = 0$ and $p > 1$ is an integer.

λ_0	0	$p-1$	$-(p-1)$	$-(p-1)$	$-(p-1)$
λ	$p+1$	$p+1$	0	$p+1$	$-(p+1)$
$f(t)$	$e^{\pm t}$	$\cosh(t)$	t	$\sinh(t)$	$\cos(t)$

4. The existence and the completeness of some metric

Let $M = (a, b) \times_{f'(t)^2} \mathbb{R} \times_{f(t)^2} F$ be an Einstein Lorentzian warped product manifold, where $f(t)$ and $f'(t)$ are smooth functions and $-\infty \leq a < b \leq \infty$. Let $\dim F = p > 1$.

Remark 4.1. From the Remark 3.2, we have positive smooth functions $f(t)$ and $f'(t)$. Then we have the metric

$$g = -dt^2 + \frac{\lambda}{p+1} e^{\sqrt{\frac{\lambda}{p+1}}\ 2t + \frac{4c}{p+1}}\ du^2 + e^{\sqrt{\frac{\lambda}{p+1}}\ 2t + \frac{4c}{p+1}}\ g_F,$$

where c is a constant.

THEOREM 4.2. *Let M be an Einstein Lorentzian warped product manifold. Suppose that $(\mathbb{R}, du^2)$ and (F, g_F) are complete. If $\lambda_0 = 0$ and $\lambda > 0$, then on M the resulting metric is that every future directed time-like geodesics is future (or past) complete.*

Proof. For $\lambda_0 = 0$ and $\lambda > 0$, the metric of Remark 4.1 simplifies to

$$g = -dt^2 + \alpha^2 e^{2\alpha t}\ du^2 + e^{2\alpha t}\ g_F$$

on $(-\infty, \infty)$, where α is a positive constant.

Let $\mathcal{B} = \{\alpha e^{\alpha t},\ e^{\alpha t}\}$, where α is a positive constant. For some $t_1 \in (-\infty, \infty)$, then

(i) $\displaystyle\int_{t_1}^{\infty} \frac{\alpha e^{\alpha t}}{\sqrt{\alpha^2 e^{2\alpha t} + 1}}\ dt \geq \int_{t_1}^{\infty} \frac{1}{\sqrt{2}}\ dt = +\infty,$

(ii) $\displaystyle\int_{t_1}^{\infty} \frac{e^{\alpha t}}{\sqrt{e^{2\alpha t} + 1}}\ dt \geq \int_{t_1}^{\infty} \frac{1}{\sqrt{2}}\ dt = +\infty,$

(iii) $\displaystyle\int_{t_1}^{\infty} \frac{\alpha e^{\alpha t} e^{\alpha t}}{\sqrt{\alpha^2 e^{2\alpha t} + e^{2\alpha t} + \alpha^2 e^{2\alpha t} e^{2\alpha t}}}\ dt \geq \int_{t_1}^{\infty} \frac{1}{\sqrt{3}}\ dt = +\infty,$ where α is a positive constant.

Therefore from the Remark 2.6, on M every future directed time-like geodesic is future complete. On the other hand, by similar methods, on M every future directed time-like geodesic is past incomplete. □

REMARK 4.3. From the Remark 3.4, we have positive smooth functions $f(t)$ and $f'(t)$. Then we have the metric

$$g = -dt^2 + \frac{\lambda_0}{p-1} \sinh^2(\sqrt{\frac{\lambda}{p+1}}\ t + c)\ du^2 + \frac{(p+1)\lambda_0}{(p-1)\lambda} \cosh^2(\sqrt{\frac{\lambda}{p+1}}\ t + c)\ g_F,$$

where c is a constant.

THEOREM 4.4. *Let M be an Einstein Lorentzian warped product manifold. Suppose that $(\mathbb{R}, du^2)$ and (F, g_F) are complete. If $\lambda_0 > 0$ and $\lambda > 0$, then on M the resulting metric is that every future directed time-like geodesics is future (or past) complete.*

Proof. For $\lambda_0 > 0$ and $\lambda > 0$, the metric of Remark 4.3 simplifies to

$$g = -dt^2 + \alpha^2 \sinh^2(\alpha t)\ du^2 + \cosh^2(\alpha t)\ g_F$$

on $(0, \infty)$, where α is a positive constant.

Let $\mathcal{B} = \{\alpha \sinh(\alpha t),\ \cosh(\alpha t)\}$, where α is a positive constant. For some $t_1 \in (0, \infty)$, then

(i) $\displaystyle\int_{t_1}^{\infty} \frac{\alpha \sinh(\alpha t)}{\sqrt{\alpha^2 \sinh^2(\alpha t) + 1}}\ dt \geq \int_{t_1}^{\infty} \frac{1}{\sqrt{2}}\ dt = +\infty,$

(ii) $\displaystyle\int_{t_1}^{\infty} \frac{\cosh(\alpha t)}{\sqrt{\cosh^2(\alpha t) + 1}} dt \geq \int_{t_1}^{\infty} \frac{1}{\sqrt{2}}\ dt = +\infty,$

(iii) $\displaystyle\int_{t_1}^{\infty} \frac{\alpha \sinh(\alpha t) \cosh(\alpha t)}{\sqrt{\alpha^2 \sinh^2(\alpha t) + \cosh^2(\alpha t) + \alpha^2 \sinh^2(\alpha t) \cosh^2(\alpha t)}}\ dt \geq \int_{t_1}^{\infty} \frac{1}{\sqrt{3}} dt$

$= +\infty$, where α is a positive constant.

Therefore from the Remark 2.6, on M every future directed time-like geodesic is future complete but past incomplete. □

REMARK 4.5. From the Remark 3.6, we have positive smooth functions $f(t)$ and $f'(t)$. Then we have the metrics.

(i) For $\lambda_0 < 0$ and $\lambda = 0$, we have

$$g = -dt^2 + \frac{-\lambda_0}{p-1}\ du^2 + \left(\sqrt{\frac{-\lambda_0}{p-1}}\ t + \frac{2c}{p+1}\right)^2\ g_F,$$

where c is a constant.

(ii) For $\lambda_0 < 0$ and $\lambda > 0$, we become

$$g = -dt^2 + \frac{-\lambda_0}{p-1}\cosh^2(\sqrt{\frac{\lambda}{p+1}}\,t + c)\,du^2 + \frac{-(p+1)\lambda_0}{(p-1)\lambda}\sinh^2(\sqrt{\frac{\lambda}{p+1}}\,t + c)\,g_F,$$

where c is a constant.

(iii) For $\lambda_0 < 0$ and $\lambda < 0$, we get

$$g = -dt^2 + \frac{-\lambda_0}{p-1}\sin^2(\sqrt{\frac{-\lambda}{p+1}}\,t + c\,)du^2 + \frac{(p+1)\lambda_0}{(p-1)\lambda}\cos^2(\sqrt{\frac{-\lambda}{p+1}}\,t + c\,)\,g_F,$$

where c is a constant and n is an integer.

Theorem 4.6. *Let M be an Einstein Lorentzian warped product manifold. Suppose that $(\mathbb{R}, du^2)$ and (F, g_F) are complete. If $\lambda_0 < 0$ and $\lambda = 0$, then on M the resulting metric is that every future directed time-like geodesic is future (or past) complete.*

Proof. For $\lambda_0 < 0$ and $\lambda = 0$, from the Remark 4.5 (i), the warping function $f(t)$ is a linear function and $f'(t)$ is a constant function. Because $f'(t)$ is not a nonconstant, thus we can not discuss geodesic complete. □

Theorem 4.7. *Let M be an Einstein Lorentzian warped product manifold. Suppose that $(\mathbb{R}, du^2)$ and (F, g_F) are complete. If $\lambda_0 < 0$ and $\lambda > 0$, then on M the resulting metric is that every future directed time-like geodesic is future (or past) complete.*

Proof. For $\lambda_0 < 0$ and $\lambda > 0$, the metric of Remark 4.5 (ii) simplifies to

$$g = -dt^2 + \alpha^2\cosh^2(\alpha t)\,du^2 + \sinh^2(\alpha t)\,g_F,$$

on $(0, \infty)$, where α is a positive constant.

Let $\mathcal{B} = \{\alpha\cosh(\alpha t),\ \sinh(\alpha t)\}$, where α is a positive constant. By a proof similar to Theorem 4.4, for some $t_1 \in (0, \infty)$, from the Remark 2.6 implies that on M every future directed time-like geodesic is future complete but past incomplete. □

Theorem 4.8. *Let M be an Einstein Lorentzian warped product manifold. Suppose that $(\mathbb{R}, du^2)$ and (F, g_F) are complete. If $\lambda_0 < 0$ and $\lambda < 0$, then on M the resulting metric is that every future directed time-like geodesic is not future (or past) complete.*

Proof. For $\lambda_0 < 0$ and $\lambda < 0$, from the Remark 4.5 (iii), we have $f(t) = \cos\left(\sqrt{\frac{-\lambda}{p+1}}\, t + c\right)$ and $f'(t) = \sqrt{\frac{-\lambda}{p+1}} \sin\left(\sqrt{\frac{-\lambda}{p+1}}\, t + c\right)$, where c is a constant. Because we can consider the existence of a nonconstant warping function on only a finite interval, thus we can not discuss the completeness. □

Remark 4.9. Let $M = (a,\ b) \times_{f'(t)^2} \mathbb{R} \times_{f(t)^2} F$ be an Einstein Lorentzian warped product manifold. The behaviour of the metrics depends on the signs of λ_0 and λ. Then we reduced to the following sets of metrics besides the constant case when $c = 0$ and $p > 1$ is an integer.

	λ_0	λ	metric
(i)	0	$p+1$	$g = -dt^2 + e^{2t} du^2 + e^{2t} g_F$
(ii)	$p-1$	$p+1$	$g = -dt^2 + \sinh^2 t\ du^2 + \cosh^2 t\ g_F$
(iii)	$-(p-1)$	0	$g = -dt^2 + du^2 + t^2 g_F$
(iv)	$-(p-1)$	$p+1$	$g = -dt^2 + \cosh^2 t\ du^2 + \sinh^2 t\ g_F$
(v)	$-(p-1)$	$-(p+1)$	$g = -dt^2 + \sin^2 t\ du^2 + \cos^2 t\ g_F$

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