

## FUZZY CONNECTIONS ON ADJOINT TRIPLES

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ABSTRACT. In this paper, we introduce the notion of residuated and Galois connections on adjoint triples and investigate their properties. Using the properties of residuated and Galois connections, we solve fuzzy relation equations and give their examples.

### 1. Introduction

Ward et al.[19] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool as algebraic structures for many valued logics [2,6-10,18]. Pawlak [11,12] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. For an extension of classical rough sets, many researchers [2,9,10] developed  $L$ -lower and  $L$ -upper approximation operators in complete residuated lattices.

Abdel-Hamid [1] introduced the notion of adjoint triples. By using this concepts, Medina et al.[3-5] developed information systems and decision rules. Sanchez [15] introduced the theory of fuzzy relation equations with various types of composition: max-min, min-max, min- $\alpha$ . Fuzzy

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relation equations with new types of composition( continuous t-norm [16], residuated lattice [13,14]) is developed.

In this paper, we show that there exists the residuated connection between fuzzy relational erosion and fuzzy relational dilation on adjoint triples. Moreover, we study residuated and Galois connections on adjoint triples and investigate their properties. Using the properties of residuated and Galois connections, we solve fuzzy relation equations and give their examples.

## 2. Preliminaries

DEFINITION 2.1. [2,9,17] Let  $X$  and  $Y$  be sets. Let  $(P_1, \leq_1)$  and  $(P_2, \leq_2)$  be posets. Let  $\delta, \gamma : P_1^X \rightarrow P_2^Y$  and  $\epsilon, \rho : P_2^Y \rightarrow P_1^X$ .

(1)  $(P_1^X, \delta, \epsilon, P_2^Y)$  is called a *residuated connection* if  $\delta(f) \leq_2 g$  iff  $f \leq_1 \epsilon(g)$  for all  $f \in P_1^X, g \in P_2^Y$ .

(2)  $(P_1^X, \gamma, \rho, P_2^Y)$  is called a *Galois connection* if  $g \leq_2 \gamma(f)$  iff  $f \leq_1 \rho(g)$  for all  $f \in P_1^X, g \in P_2^Y$ .

DEFINITION 2.2.[2] Let  $X$  be a set and  $(P, \leq)$  be a poset. An operator  $C : P^X \rightarrow P^X$  is called a *fuzzy closure operator* on  $X$  if it satisfies the following conditions:

(C1)  $f \leq C(f)$  and  $C(C(f)) \leq C(f)$ , for all  $f \in P^X$ .

(C2) If  $f \leq g$ , then  $C(f) \leq C(g)$  for all  $f, g \in P^X$ .

An operator  $I : P^X \rightarrow P^X$  is called a *fuzzy interior operator* on  $X$  if it satisfies the conditions

(I1)  $I(f) \leq f$  and  $I(f) \leq I(I(f))$  for all  $f \in P^X$ ,

(I2) If  $f \leq g$ , then  $I(f) \leq I(g)$  for all  $f, g \in P^X$ .

DEFINITION 2.3. [3-5] Let  $(P_1, \leq_1), (P_2, \leq_2), (P_3, \leq_3)$  be posets. We say that the mappings  $\& : P_1 \times P_2 \rightarrow P_3, \rightarrow : P_2 \times P_3 \rightarrow P_1$  and  $\Rightarrow : P_1 \times P_3 \rightarrow P_2$  is called an *adjoint triple* if it satisfies the following conditions:

$x \leq_1 (y \rightarrow z)$  iff  $x \& y \leq_3 z$  iff  $y \leq_2 (x \Rightarrow z)$  for  $x \in P_1, y \in P_2, z \in P_3$ .

EXAMPLE 2.4. Let  $[0, 1]_m$  be a regular partition of  $[0, 1]$  in  $m$  pieces with  $[0, 1]_m = \{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}, 1\}$ . A discretization of a t-norm  $T :$

$[0, 1] \times [0, 1] \rightarrow [0, 1]$  is the operator  $T^0 : [0, 1]_m \times [0, 1]_n \rightarrow [0, 1]_k$  defined as

$$T^0(x, y) = \frac{[kT(x, y)]}{k}$$

where  $[x] = \bigwedge\{n \in Z \mid x \leq n\}$  is the ceiling function. For this operator, the corresponding implication operators  $\rightarrow^0 : [0, 1]_n \times [0, 1]_k \rightarrow [0, 1]_m$  and  $\Rightarrow^0 : [0, 1]_m \times [0, 1]_k \rightarrow [0, 1]_n$  defined as

$$y \rightarrow^0 z = \frac{\langle m(y \rightarrow z) \rangle}{m}, \quad x \Rightarrow^0 z = \frac{\langle n(x \rightarrow z) \rangle}{n}$$

where  $\langle x \rangle = \bigvee\{n \in Z \mid n \leq x\}$  is the floor function.

Let  $x \leq y \rightarrow^0 z = \frac{\langle m(y \rightarrow z) \rangle}{m}$ . Since  $x - 1 \leq \langle x \rangle \leq x$ ,  $x \leq \frac{\langle m(y \rightarrow z) \rangle}{m} \leq \frac{m(y \rightarrow z)}{m} = y \rightarrow z$ . Hence  $T(x, y) \leq z$ . Since  $x \leq [x] < x + 1$ ,

$$T^0(x, y) = \frac{[kT(x, y)]}{k} < \frac{kT(x, y) + 1}{k} \leq \frac{z + 1}{k}.$$

Hence  $T^0(x, y) = \frac{[kT(x, y)]}{k} \leq z$ .

Let  $T^0(x, y) = \frac{[kT(x, y)]}{k} \leq z$ . Since  $kT(x, y) \leq [kT(x, y)]$ ,  $T(x, y) \leq z$  iff  $y \leq x \rightarrow z$ . Hence

$$x \Rightarrow^0 z = \frac{\langle n(x \rightarrow z) \rangle}{n} > \frac{\langle n(x \rightarrow z) \rangle}{n} \geq y.$$

Other cases are similarly proved.

### 3. Fuzzy connections on adjoint triples

LEMMA 3.1. Let  $P_i$  be complete lattices for  $i \in \{1, 2, 3\}$ . Let  $(\&, \rightarrow, \Rightarrow)$  be an adjoint triple with respect to  $(P_1, \leq_1)$ ,  $(P_2, \leq_2)$ ,  $(P_3, \leq_3)$ . Then the following properties hold.

- (1) If  $x_1 \leq_1 x_2$ , then  $x_1 \& y \leq_3 x_2 \& y$ .
- (2) If  $y_1 \leq_2 y_2$ , then  $x \& y_1 \leq_3 x \& y_2$ .
- (3)  $\rightarrow, \Rightarrow$  are order-preserving on the second argument and order-reversing on the first argument.
- (4)  $y \leq_2 (x \Rightarrow (x \& y))$ ,  $x \leq_1 (y \rightarrow (x \& y))$ .
- (5)  $x \& (x \Rightarrow z) \leq_3 z$ ,  $(y \rightarrow z) \& y \leq_3 z$ .
- (6)  $y \leq_2 ((y \rightarrow z) \Rightarrow z)$ ,  $x \leq_1 ((x \Rightarrow z) \rightarrow z)$ .
- (7)  $(\bigvee_i x_i) \& y = \bigvee_i (x_i \& y)$  and  $x \& (\bigvee_i y_i) = \bigvee_i (x \& y_i)$ .
- (8)  $x \Rightarrow (\bigwedge_i z_i) = \bigwedge_i (x \rightarrow z_i)$  and  $(\bigvee_i x_i) \Rightarrow z = \bigwedge_i (x_i \Rightarrow z)$ .

$$(9) \ y \rightarrow (\bigwedge_i z_i) = \bigwedge_i (y \rightarrow z_i) \text{ and } (\bigvee_i y_i) \rightarrow z = \bigwedge_i (y_i \rightarrow z).$$

*Proof.* (1) Since  $x_1 \leq_1 x_2 \leq_1 (y \rightarrow x_2 \& y)$ ,  $x_1 \& y \leq_3 x_2 \& y$ .

(2) Since  $y_1 \leq_2 y_2 \leq_2 (y \Rightarrow x \& y_2)$ ,  $x \& y_1 \leq_3 x \& y_2$ .

(3) Let  $x_1 \leq_1 x_2$ . Since  $x_1 \& (x_2 \Rightarrow z) \leq_3 x_2 \& (x_2 \Rightarrow z) \leq_3 z$ ,  $(x_2 \Rightarrow z) \leq_2 (x_1 \Rightarrow z)$ .

Let  $z_1 \leq_3 z_2$ . Since  $(x \rightarrow z_1) \& x \leq_3 z_1 \leq_3 z_2$ ,  $(x \rightarrow z_1) \leq_1 (x \rightarrow z_2)$ .

Other cases are similarly proved.

(4) It follows  $x \& y \leq_3 x \& y$ .

(5) It follows  $(x \Rightarrow z) \leq_2 (x \Rightarrow z)$  iff  $x \& (x \Rightarrow z) \leq_3 z$ . Moreover,  $(y \rightarrow z) \leq_1 (y \rightarrow z)$  iff  $(y \rightarrow z) \& y \leq_3 z$ .

(6) By (5),  $(y \rightarrow z) \& y \leq_3 z$  iff  $y \leq_2 ((y \rightarrow z) \Rightarrow z)$ . Moreover,  $x \& (x \Rightarrow z) \leq_3 z$  iff  $x \leq_1 ((x \Rightarrow z) \rightarrow z)$ .

(7) By (1),  $(\bigvee_i x_i) \& y \geq_3 \bigvee_i (x_i \& y_i)$ . Since  $x_i \leq_2 (y \rightarrow x_i \& y) \leq_2 (y \rightarrow \bigvee_i (x_i \& y))$ ,  $\bigvee_i x_i \leq_2 (y \rightarrow \bigvee_i (x_i \& y))$  iff  $(\bigvee_i x_i) \& y \leq_3 \bigvee_i (x_i \& y_i)$ .

(8) By (3),  $x \Rightarrow (\bigwedge_i z_i) \leq_2 \bigwedge_i (x \Rightarrow z_i)$ . Since  $x \& (\bigwedge_i (x \Rightarrow z_i)) \leq_3 x \& (x \Rightarrow z_i) \leq_3 z_i$ , then  $x \& (\bigwedge_i (x \Rightarrow z_i)) \leq_3 \bigwedge_i (x \& (x \Rightarrow z_i)) \leq_3 \bigwedge_i z_i$ . Thus  $\bigwedge_i (x \Rightarrow z_i) \leq_2 x \Rightarrow (\bigwedge_i z_i)$ .

Moreover, by (3),  $(\bigvee_i x_i) \Rightarrow z \leq_2 \bigwedge_i (x_i \Rightarrow z)$ . Since  $(\bigvee_i x_i) \& \bigwedge_i (x_i \Rightarrow z) \leq_3 (\bigvee_i x_i) \& (x_i \Rightarrow z) = \bigvee_i (x_i \& (x_i \Rightarrow z)) \leq_3 z$ , then  $\bigwedge_i (x_i \Rightarrow z) \leq_2 (\bigvee_i x_i) \Rightarrow z$ .  $\square$

**DEFINITION 3.2.** [5] Let  $X, Y$  be sets and  $P_i$  be complete lattices. Let  $(\&, \rightarrow, \Rightarrow)$  be an adjoint triple with respect to  $(P_1, \leq_1)$ ,  $(P_2, \leq_2)$ ,  $(P_3, \leq_3)$ .

(1) The *fuzzy relational erosion with respect to*  $R \in P_1^{X \times Y}$ ,  $\epsilon_R : P_3^X \rightarrow P_2^Y$  is defined as

$$\epsilon_R(f)(y) = \bigwedge_{x \in X} (R(x, y) \Rightarrow f(x)).$$

(2) The *fuzzy relational dilation with respect to*  $R$ ,  $\delta_R : P_2^Y \rightarrow P_3^X$  is defined as

$$\delta_R(g)(x) = \bigvee_{y \in Y} (R(x, y) \& g(y)).$$

(3) The *fuzzy relational property-oriented erosion with respect to*  $R$ ,  $\epsilon_{R_p} : P_3^Y \rightarrow P_1^X$  is defined as

$$\epsilon_{R_p}(g)(x) = \bigwedge_{y \in Y} (R(x, y) \rightarrow g(y)).$$

(4) The fuzzy relational property-oriented dilation with respect to  $R$ ,  $\delta_{R_p} : P_1^X \rightarrow P_3^Y$  is defined as

$$\delta_{R_p}(f)(y) = \bigvee_{x \in X} (f(x) \& R(x, y)).$$

**THEOREM 3.3.** *Let  $X, Y$  be sets and  $P_i$  be complete lattices. Let  $(\&, \rightarrow, \Rightarrow)$  be an adjoint triple with respect to  $(P_1, \leq_1), (P_2, \leq_2), (P_3, \leq_3)$ . Then the following properties hold.*

- (1)  $(P_2^Y, \delta_R, \epsilon_R, P_2^Y)$  is a residuated connection.
- (2)  $\delta_R(\epsilon_R(f)) \leq_3 f$  iff  $g \leq_2 \epsilon_R(\delta_R(g))$  for all  $f \in P_3^X, g \in P_2^Y$ .
- (3) If  $f_1 \leq_3 f_2$  and  $g_1 \leq_2 g_2$ , for all  $f_1, f_2 \in P_3^X, g_1, g_2 \in P_2^Y$ ,

$$\epsilon_R(f_1) \leq_2 \epsilon_R(f_2), \delta_R(g_1) \leq_3 \delta_R(g_2).$$

(4)  $\bigvee_{i \in \Gamma} \delta_R(g_i) = \delta_R(\bigvee_{i \in \Gamma} g_i)$  for all  $g_i \in P_2^Y$ .

(5)  $\bigwedge_{i \in \Gamma} \epsilon_R(f_i) = \epsilon_R(\bigwedge_{i \in \Gamma} f_i)$  for all  $f_i \in P_3^X$ .

(6)  $\delta_R(g) = \delta_R(\epsilon_R(\delta_R(g)))$  for all  $g \in P_2^Y$ . If  $g = g_0$  is a solution of  $\delta_R(g) = f$ , then  $g_1 = \epsilon_R(f)$  is a solution of  $\delta_R(g) = f$  such that  $g_0 \leq \epsilon_R(f)$ .

(7)  $\epsilon_R(\delta_R(\epsilon_R(f))) = \epsilon_R(f)$  for all  $f \in P_3^X$ . If  $f = f_1$  is a solution of  $\epsilon_R(f) = g$ , then  $f_2 = \delta_R(g)$  is a solution of  $\epsilon_R(f) = g$  such that  $\delta_R(g) \leq f_1$ .

(8)  $\delta_R \circ \epsilon_R : P_3^X \rightarrow P_3^X$  is a fuzzy interior operator.

(9)  $\epsilon_R \circ \delta_R : P_2^Y \rightarrow P_2^Y$  is a fuzzy closure operator.

*Proof.* (1) We show that  $\delta_R(g) \leq_3 f$  iff  $g \leq_2 \epsilon_R(f)$  for all  $f \in P_3^X, g \in P_2^Y$ . For  $f \in P_3^X, g \in P_2^Y$ ,  $\delta_R(g)(x) \leq_3 f(x)$  iff  $\bigvee_{y \in Y} (R(x, y) \& g(y)) \leq_3 f(x)$  iff  $g(y) \leq_2 \bigwedge_{x \in Y} (R(x, y) \Rightarrow f(x))$  iff  $g(y) \leq_2 \epsilon_R(f)(y)$ .

(2) For  $f \in P_3^X, g \in P_2^Y$ ,  $\delta_R(g)(x) \leq_3 \delta_R(g)(x)$  iff  $g(y) \leq_2 \epsilon_R(\delta_R(g))(y)$  and  $\epsilon_R(f)(y) \leq_2 \epsilon_R(f)(x)$  iff  $\delta_R(\epsilon_R(f))(x) \leq_3 f(x)$ .

(3) Since  $g_1(y) \leq_2 g_2(y) \leq_2 \epsilon_R(\delta_R(g_2))(y)$ , then  $\delta_R(g_1)(x) \leq \delta_R(g_2)(x)$ . Moreover, since  $\delta_R(\epsilon_R(f_1))(x) \leq_3 f_2(x)$ ,  $\epsilon_R(f_1)(y), \epsilon_R(f_2)(y)$ .

(4) From (3),  $\bigvee_{i \in \Gamma} \delta_R(g_i) \leq_3 \delta_R(\bigvee_{i \in \Gamma} g_i)$ .

Since  $g_i \leq_2 \epsilon_R(\delta_R(g_i))$  and  $\bigvee_{i \in \Gamma} g_i \leq_2 \epsilon_R(\bigvee_{i \in \Gamma} \delta_R(g_i))$ ,  $\delta_R(\bigvee_{i \in \Gamma} g_i) \leq_3 \bigvee_{i \in \Gamma} \delta_R(g_i)$ .

(5) From (3),  $\epsilon_R(\bigwedge_{i \in \Gamma} f_i) \leq_2 \bigwedge_{i \in \Gamma} \epsilon_R(f_i)$ .

Since  $\delta_R(\epsilon_R(f_i)) \leq_3 f_i$  and  $\delta_R(\bigwedge_{i \in \Gamma} \epsilon_R(f_i)) \leq_3 \bigwedge_{i \in \Gamma} \delta_R(\epsilon_R(f_i)) \leq_3 \bigwedge_{i \in \Gamma} f_i$ ,  $\bigwedge_{i \in \Gamma} \epsilon_R(f_i) \leq_2 \epsilon_R(\bigwedge_{i \in \Gamma} f_i)$ .

(6) By (2),  $\delta_R(g) = \delta_R(\epsilon_R(\delta_R(g)))$  for all  $g \in P_2^Y$ . If  $\delta_R(g_0) = f$ , Then  $\delta_R(\epsilon_R(\delta_R(g_0))) = \delta_R(\epsilon_R(f)) = \delta_R(g_0) = f$ . Moreover,  $g_0 \leq_2 \epsilon_R(\delta_R(g_0)) = \epsilon_R(f)$ .

(7) It is similarly proved as (6).

(8) For each  $f, f_1, f_2 \in L^X$ ,  $\delta_R \circ \epsilon_R(f) \leq_3 f$  and  $(\delta_R \circ \epsilon_R)(\delta_R \circ \epsilon_R)(f) = \delta_R \circ \epsilon_R(f)$ , if  $f_1 \leq_3 f_2$ ,

$$(\delta_R \circ \epsilon_R)(f_1) \leq_3 (\delta_R \circ \epsilon_R)(f_2)$$

(9) It is similarly proved as (8). □

**COROLLARY 3.4.** *Let  $X, Y$  be sets and  $P_i$  be complete lattices. Let  $(\&, \rightarrow, \Rightarrow)$  be an adjoint triple with respect to  $(P_1, \leq_1), (P_2, \leq_2), (P_3, \leq_3)$ . Then the following properties hold.*

(1)  $(P_1^X, \delta_{R_p}, \epsilon_{R_p}, P_3^Y)$  is a residuated connection.

(2)  $\delta_{R_p}(\epsilon_{R_p}(g) \leq_3 g$  iff  $f \leq_1 \epsilon_{R_p}(\delta_{R_p}(f))$  for all  $f \in P_1^X, g \in P_3^Y$ .

(3) If  $f_1 \leq_3 f_2$  and  $g_1 \leq_2 g_2$ , for all  $f_1, f_2 \in P_1^X, g_1, g_2 \in P_3^Y$ ,

$$\epsilon_{R_p}(g_1) \leq_1 \epsilon_{R_p}(g_2), \delta_{R_p}(f_1) \leq_3 \delta_{R_p}(f_2).$$

(4)  $\bigvee_{i \in \Gamma} \delta_{R_p}(f_i) = \delta_{R_p}(\bigvee_{i \in \Gamma} f_i)$  for all  $f_i \in P_1^X$ .

(5)  $\bigwedge_{i \in \Gamma} \epsilon_{R_p}(g_i) = \epsilon_{R_p}(\bigwedge_{i \in \Gamma} g_i)$  for all  $g_i \in P_3^Y$ .

(6)  $\delta_{R_p}(f) = \delta_{R_p}(\epsilon_{R_p}(\delta_{R_p}(f)))$  for all  $f \in P_1^Y$ . If  $f = f_0$  is a solution of  $\delta_{R_p}(f) = g$ , then  $f_1 = \epsilon_{R_p}(g)$  is a solution of  $\delta_{R_p}(f) = g$  such that  $f_0 \leq_1 \epsilon_{R_p}(g)$ .

(7)  $\epsilon_{R_p}(\delta_{R_p}(\epsilon_{R_p}(g))) = \epsilon_{R_p}(g)$  for all  $g \in P_3^Y$ . If  $g = g_1$  is a solution of  $\epsilon_{R_p}(g) = f$ , then  $g_2 = \delta_{R_p}(f)$  is a solution of  $\epsilon_{R_p}(g) = f$  such that  $\delta_{R_p}(f) \leq_3 g_1$ .

(8)  $\delta_{R_p} \circ \epsilon_{R_p} : P_3^Y \rightarrow P_3^Y$  is a fuzzy interior operator.

(9)  $\epsilon_{R_p} \circ \delta_{R_p} : P_1^X \rightarrow P_1^X$  is a fuzzy closure operator.

**THEOREM 3.5.** *Let  $X, Y$  be sets and  $P_i$  be complete lattices. Let  $(\&, \rightarrow, \Rightarrow)$  be an adjoint triple with respect to  $(P_1, \leq_1), (P_2, \leq_2), (P_3, \leq_3)$ . An operation  $\gamma_R : P_1^X \rightarrow P_2^Y$  is defined as*

$$\gamma_R(f)(y) = \bigwedge_{x \in X} (f(x) \Rightarrow R(x, y)).$$

An operation  $\rho_R : P_2^Y \rightarrow P_1^X$  is defined as

$$\rho_R(g)(x) = \bigwedge_{y \in Y} (g(y) \rightarrow R(x, y)).$$

Then following properties hold.

- (1)  $(P_1^X, \gamma_R, \rho_R, P_2^Y)$  is a Galois connection.
- (2)  $f \leq_1 \rho_R(\gamma_R(f))$  and  $g \leq_2 \gamma_R(\rho_R(g))$  for all  $f \in P_1^X$  and  $g \in P_2^Y$ .
- (3) If  $f_1 \leq_1 f_2$  for all  $f_1, f_2 \in P_1^X$ , then  $\gamma_R(f_2)(y) \leq_2 \gamma_R(f_1)(y)$ .
- (4) If  $g_1 \leq_2 g_2$  for all  $g_1, g_2 \in P_2^Y$ , then  $\rho_R(g_2)(x) \leq_1 \rho_R(g_1)(x)$ .
- (5)  $\bigwedge_{i \in \Gamma} \gamma_R(f_i) = \gamma_R(\bigvee_{i \in \Gamma} f_i)$  for all  $f_i \in P_1^X$ .
- (6)  $\bigwedge_{i \in \Gamma} \rho_R(g_i) = \rho_R(\bigvee_{i \in \Gamma} g_i)$  for all  $g_i \in P_2^Y$ .
- (7)  $\gamma_R(f) = \gamma_R(\rho_R(\gamma_R(f)))$  for all  $f \in P_1^X$ . If  $f = f_1$  is a solution of  $\gamma_R(f) = g$ , then  $f = \rho_R(g)$  is a solution of  $\gamma_R(f) = g$  such that  $f_1 \leq_1 \rho_R(g)$ .
- (8)  $\rho_R(g) = \rho_R(\gamma_R(\rho_R(g)))$  for all  $g \in P_2^Y$ . If  $g = g_1$  is a solution of  $\rho_R(g) = f$ , then  $g = \gamma_R(f)$  is a solution of  $\rho_R(g) = f$  such that  $g_1 \leq_2 \gamma_R(f)$ .
- (9)  $\rho_R \circ \gamma_R : P_1^X \rightarrow P_1^X$  and  $\gamma_R \circ \rho_R : P_2^Y \rightarrow P_2^Y$  are fuzzy closure operators.

*Proof.* (1) We show that  $g \leq_2 \gamma_R(f)$  iff  $f \leq_1 \rho_R(g)$  for all  $f \in P_1^X, g \in P_2^Y$ .

For  $f \in P_1^X, g \in P_2^Y, g(y) \leq_2 \gamma_R(f)(y) = \bigwedge_{x \in X} (f(x) \Rightarrow R(x, y))$  iff  $f(x) \leq_1 \bigwedge_{y \in Y} (g(y) \rightarrow R(x, y))$  iff  $f(x) \leq_1 \rho_R(g)(x)$ .

(2) It follows from  $\rho_R(g) \leq_1 \rho_R(g)$  iff  $g \leq_2 \gamma_R(\rho_R(g))$  and  $\gamma_R(f) \leq_2 \gamma_R(f)$  iff  $f \leq_1 \rho_R(\gamma_R(f))$ .

(3) Since  $f_1 \leq_1 f_2 \leq_1 \rho_R(\gamma_R(f_2)), \gamma_R(f_2) \leq_2 \gamma_R(f_1)$ .

(4) Since  $g_1 \leq_2 g_2 \leq_2 \gamma_R(\rho_R(g_2)), \rho_R(g_2) \leq_1 \rho_R(g_1)$ .

(5) By (3), since  $f_1 \leq_1 f_2$ , then  $\gamma_R(f_2) \leq_2 \gamma_R(f_1)$ . Hence  $\gamma_R(\bigvee_{i \in \Gamma} f_i) \leq_2 \bigwedge_{i \in \Gamma} \gamma_R(f_i)$ .

Since  $f_i \leq_1 \rho_R(\gamma_R(f_i))$  and  $\bigvee_{i \in \Gamma} f_i \leq_1 \bigvee_{i \in \Gamma} \rho_R(\gamma_R(f_i)) \leq_1 \rho_R(\bigwedge_{i \in \Gamma} \gamma_R(f_i))$ , Thus  $\gamma_R(\bigvee_{i \in \Gamma} f_i) \leq_2 \bigwedge_{i \in \Gamma} \gamma_R(f_i)$ ,

(6) It is similarly proved as (5).

(7) By (2),  $\gamma_R(f) = \gamma_R(\rho_R(\gamma_R(f)))$  for all  $f \in P_1^X$ . If  $\gamma_R(f_1) = g$ , then  $\gamma_R(\rho_R(\gamma_R(f_1))) = \gamma_R(\rho_R(g)) = \gamma_R(f_1) = g$ . Moreover,  $f_1 \leq_1 \rho_R(\gamma_R(f_1)) = \rho_R(g)$ .

(8) It is similarly proved as (7).

(9) For each  $g, h \in P_2^Y, g \leq_2 \gamma_R \circ \rho_R(g)$  and  $(\gamma_R \circ \rho_R) \circ (\gamma_R \circ \rho_R)(g) = \gamma_R \circ \rho_R(g)$ . If  $g \leq_2 h$ , then  $\rho_R(h) \leq_1 \rho_R(g)$ . Moreover  $(\gamma_R \circ \rho_R)(g) \leq_2 (\gamma_R \circ \rho_R)(h)$ . Hence  $\gamma_R \circ \rho_R : P_2^Y \rightarrow P_2^Y$  is a fuzzy closure operator. Similarly,  $\rho_R \circ \gamma_R$  is a fuzzy closure operator.

□

EXAMPLE 3.6. Let  $X = \{x, y, z\}$  be a set of cars and  $Y = \{a, b\}$  be a set of attributes. Let  $([0, 1], \odot, \rightarrow, 0, 1)$  be a t-norm (ref.[2,6-8]) as

$$x \odot y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}.$$

Let  $[0, 1]_m$  be a regular partition of  $[0, 1]$  in  $m$  pieces with  $[0, 1]_m = \{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}, 1\}$ .

Let  $\&^0 : [0, 1]_3 \times [0, 1]_4 \rightarrow [0, 1]_2, \Rightarrow^0 : [0, 1]_3 \times [0, 1]_2 \rightarrow [0, 1]_4, \rightarrow^0 : [0, 1]_4 \times [0, 1]_2 \rightarrow [0, 1]_3$  defined as

$$x \&^0 y = \frac{[2(x \odot y)]}{2}, \quad x \Rightarrow^0 y = \frac{\langle 4(x \rightarrow y) \rangle}{4} \quad x \rightarrow^0 y = \frac{\langle 3(x \rightarrow y) \rangle}{3}$$

$\&^0$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\Rightarrow^0$	0	$\frac{1}{2}$	1	$\rightarrow^0$	0	$\frac{1}{2}$	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{3}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1	1	$\frac{1}{4}$	$\frac{2}{3}$	1	1
$\frac{2}{3}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{2}$	$\frac{1}{3}$	1	1
1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	0	$\frac{1}{2}$	1	$\frac{3}{4}$	0	$\frac{2}{3}$	1

where  $[x] = \bigwedge\{n \in Z \mid x \leq n\}, \langle x \rangle = \bigvee\{n \in Z \mid n \leq x\}$ .

(1) Define  $R : X \times Y \rightarrow [0, 1]_3$  as

$$R(x, a) = \frac{1}{3}, R(y, a) = 1, R(z, a) = \frac{2}{3}$$

$$R(x, b) = 0, R(y, b) = \frac{2}{3}, R(z, b) = 1.$$

For  $f_1 \in [0, 1]_2^X$  with  $f_1 = (\frac{1}{2}, 0, \frac{1}{2})$ . Then  $\epsilon_R(f_1) = (0, \frac{1}{4})$ .  $f_1 = (\frac{1}{2}, 0, \frac{1}{2}) \in [0, 1]_2^X$  is a solution of  $\epsilon_R(f_1) = (0, \frac{1}{4})$ . Also,  $\delta_R(0, \frac{1}{4}) = (0, 0, \frac{1}{2})$  is a solution of  $\epsilon_R(f_1) = (0, \frac{1}{4})$  such that  $\delta_R(0, \frac{1}{4}) = (0, 0, \frac{1}{2}) \leq_2 f_1 = (\frac{1}{2}, 0, \frac{1}{2})$ .

For  $g_0 \in [0, 1]_4^X$  with  $g_0 = (\frac{3}{4}, \frac{1}{4})$ . Then  $\delta_R(g_0) = (\frac{1}{2}, 1, \frac{1}{2})$ .  $g_0 = (\frac{3}{4}, \frac{1}{4}) \in [0, 1]_4^X$  is a solution of  $\delta_R(g) = (\frac{1}{2}, 1, \frac{1}{2})$ . Also,  $\delta_R(\frac{1}{2}, 1, \frac{1}{2}) = (\frac{3}{4}, \frac{1}{2})$  is a solution of  $\delta_R(g) = (\frac{1}{2}, 1, \frac{1}{2})$  such that  $g_0 = (\frac{3}{4}, \frac{1}{4}) \leq_2 \delta_R(\frac{1}{2}, 1, \frac{1}{2}) = (\frac{3}{4}, \frac{1}{2})$ .

Then  $([0, 1]_4^Y, \delta_R, \epsilon_R, [0, 1]_2^X)$  is a residuated connection.

(2) Define  $R : X \rightarrow [0, 1]_4$  for  $c \in \{a, b\}$  as

$$R(x, a) = \frac{3}{4}, R(y, a) = \frac{1}{4}, R(z, a) = 1$$

$$R(x, b) = \frac{1}{2}, R(y, b) = 1, R(z, b) = \frac{1}{4}.$$

For  $f_1 \in [0, 1]_3^X$  with  $f_1 = (\frac{1}{3}, \frac{2}{3}, 1)$ . Then  $\delta_{R_p}(f_1) = (\frac{1}{2}, 1)$ .  $f_1 = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}) \in [0, 1]_3^X$  is a solution of  $\delta_{R_p}(f_1) = (\frac{1}{2}, 1)$ . Also,  $\epsilon_{R_p}(\frac{1}{2}, 1) = (\frac{2}{3}, 1, \frac{1}{3})$  is a solution of  $\delta_{R_p}(f) = (\frac{1}{2}, 1)$  such that  $f_1 = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}) \leq_1 \epsilon_{R_p}(\frac{1}{2}, 1) = (\frac{2}{3}, 1, \frac{1}{3})$ .



For  $g_0 \in [0, 1]_2^X$  with  $g_0 = (\frac{1}{2}, \frac{1}{2})$ . Then  $\epsilon_{R_p}(g_0) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ .  $g_0 = (\frac{1}{2}, \frac{1}{2}) \in [0, 1]_2^X$  is a solution of  $\epsilon_{R_p}(g) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ . Also,  $\delta_{R_p}(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) = (\frac{1}{2}, \frac{1}{2})$  is a solution of  $\epsilon_{R_p}(g) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$  such that  $g_0 = (\frac{1}{2}, \frac{1}{2}) = \delta_{R_p}(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) = (\frac{1}{2}, \frac{1}{2})$ .

Then  $([0, 1]_3^Y, \delta_{R_p}, \epsilon_{R_p}, [0, 1]_2^X)$  is a residuated connection.

(3) Define  $R : X \times Y \rightarrow [0, 1]_2$  as

$$\begin{aligned} R(x, a) &= 0, R(y, a) = \frac{1}{2}, R(z, a) = \frac{1}{2} \\ R(x, b) &= \frac{1}{2}, R(y, b) = 1, R(z, b) = 0. \end{aligned}$$

Since  $\gamma_R(f)(a) = \bigwedge_{x \in X} (f(x) \Rightarrow R(x, a))$ ,  $f_1 = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \in [0, 1]_3^X$  is a solution of  $\gamma_R(f) = (\frac{1}{2}, \frac{1}{4})$ . Also,  $\rho_R(\frac{1}{2}, \frac{1}{4}) = (\frac{1}{3}, 1, \frac{2}{3})$  is a solution of  $\gamma_R(f) = (\frac{1}{2}, \frac{1}{4})$  such that  $f_1 = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \leq_1 \rho_R(\frac{1}{2}, \frac{1}{4}) = (\frac{1}{3}, 1, \frac{2}{3})$ .

Since  $\rho_R(g)(x) = \bigwedge_{x \in X} (g(a) \rightarrow R(x, a))$ ,  $g = g_1 = (\frac{1}{4}, \frac{1}{2}) \in [0, 1]_4^Y$  is a solution of  $\rho_R(g) = (\frac{2}{3}, 1, \frac{1}{3})$ . Also,  $g = \gamma_R(\frac{2}{3}, 1, \frac{1}{3}) = (\frac{1}{4}, \frac{1}{2})$  is a solution of  $\rho_R(g) = (\frac{2}{3}, 1, \frac{1}{3})$  such that  $g_1 = (\frac{1}{4}, \frac{1}{2}) = \gamma_R(\frac{2}{3}, 1, \frac{1}{3})$ .

Then  $([0, 1]_3^X, \gamma_R, \rho_R, [0, 1]_4^Y)$  is a Galois connection.

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