

CONVERGENCE OF A CONTINUATION METHOD UNDER MAJORANT CONDITIONS

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ABSTRACT. The paper is devoted to study local convergence of a continuation method under the assumption of majorant conditions. The method is used to approximate a zero of an operator in Banach space and is of third order. It is seen that the famous Kantorovich-type and Smale-type conditions are special cases of our majorant conditions. This infers that our result is a generalized one in comparison to results based on Kantorovich-type and Smale-type conditions. Finally a number of numerical examples have been computed to show applicability of the convergence analysis.

1. Introduction

In the study of present work, we are concerned with the problem of approximating a zero τ of an operator F , where F is a nonlinear operator defined on an open convex subset \mathbb{D} of a Banach space \mathbb{W}_1 with values in a Banach space \mathbb{W}_2 . One of the most important and challenging problem in scientific computing is that of finding efficiently the solutions of nonlinear equations in Banach spaces

$$(1) \quad F(w) = 0$$

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by iterative methods. An iterative process is a mathematical procedure that, starts from one or several initial approximations of a solution τ , a sequence of values $\{w_n\}_{n \in \mathbf{N}}$ is constructed so that each value of the sequence is a better approximation to the subsequent approximation of solution τ and, the real sequence $\{\|w_n - \tau\|\}_{n \in \mathbf{N}}$ is convergent to zero.

One of the most famous methods to solve this problem is Newton's method defined by $w_{n+1} = w_n - F'(w_n)^{-1}F(w_n)$, $n = 0, 1, 2, \dots$ where $w_0 \in \mathbb{D}$ is an initial point. Usually, the study about convergence issue of Newton's method includes local and semilocal convergence analysis. The local convergence issue is, based on the information around a solution, to seek estimates of the radii of convergence balls, while the semilocal one is, based on the information around an initial point, to give criteria ensuring the convergence. An interesting problem is to find the radius of convergence of the solution. The convergence ball of an iterative method is very important because it shows the extent of difficulty for choosing initial guesses for iterative methods. One of the famous result on Newton's method is the well known Kantorovich theorem [7], which guarantees convergence of that method to a solution, using semilocal conditions. It does not require a priori existence of a solution, proving instead the existence of the solution and its uniqueness on some region. Also, Smale's point theory [15] assumes that the nonlinear operator is analytic at the initial point, which is an important result concerning Newton's method.

On the other hand for some requirement higher order methods are most suitable to be considered for finding roots of nonlinear operator equations. The well known third order iterative methods extensively studied by many researchers [2,4–6,12] are Chebyshev, Halley and Super-Halley methods and they have established convergence analysis under different types of continuity conditions.

The continuation method was known as early as in the 1930s. It was used by Kinematician in the 1960s for solving mechanism synthesis problems. It also gives a set of certain answers and a simple iteration process to obtain solutions more exactly. Mathematically, it is a parameter based method giving a continuous connection between two functions $f, g : \mathbb{W}_1 \rightarrow \mathbb{W}_2$ and is defined as a continuous map $h : [0, 1] \times \mathbb{W}_1 \rightarrow \mathbb{W}_2$ such that $h(\alpha, w) = \alpha f(w) + (1 - \alpha)g(w)$, $\alpha \in [0, 1]$ and $h(0, w) = g(w)$, $h(1, w) = f(w)$. For better review one can see [3, 11].

Let $F'(w_0)^{-1} \in \mathcal{BL}(\mathbb{W}_2, \mathbb{W}_1)$ exists at some point $w_0 \in \mathbb{D}$, where $\mathcal{BL}(\mathbb{W}_2, \mathbb{W}_1)$ is the set of bounded linear operators from \mathbb{W}_2 into \mathbb{W}_1 . The third order Chebyshev method used for solving (1) is given as for $n = 0, 1, \dots$

$$(2) \quad w_{n+1} = A_0(w_n) = w_n - \left[I + \frac{1}{2} \mathcal{L}_F(w_n) \right] F'(w_n)^{-1} F(w_n),$$

and the Convex acceleration of Newton's method (super-Halley method) is for $n = 0, 1, \dots$

$$(3) \quad \begin{aligned} w_{n+1} &= A_1(w_n) \\ &= w_n - \left[I + \frac{1}{2} \mathcal{L}_F(w_n) (I - \mathcal{L}_F(w_n))^{-1} \right] F'(w_n)^{-1} F(w_n), \end{aligned}$$

where I is identity operator and $\mathcal{L}_F(w)$ is the linear operator given by

$$\mathcal{L}_F(w) = F'(w)^{-1} F''(w) F'(w)^{-1} F(w), \quad w \in \mathbb{D}.$$

The continuation method between (2) and (3) for $\alpha \in [0, 1]$ can now be defined as

$$(4) \quad w_{n+1} = \alpha A_1(w_n) + (1 - \alpha) A_0(w_n), \quad n = 0, 1, \dots$$

Replacing Eq. (2) and (3) in (4) and rearranging we get

$$(5) \quad w_{n+1} = w_n - \left[I + \frac{1}{2} \mathcal{L}_F(w_n) (I + \alpha \mathcal{L}_F(w_n) H_\alpha(w_n)) \right] F'(w_n)^{-1} F(w_n)$$

where,

$$H_\alpha(w) = (I - \mathcal{L}_F(w))^{-1}.$$

As order of convergence of both the methods are three, so as of the continuation method (5). The convergence of this method and its variants are studied by Prashanth and Gupta [8–10] under Lipschitz, Hölder and ω -continuity conditions on the second derivative of the operator F . On the other hand, recently Kumari and Parida [13] studied Local convergence of Chebyshev method under a new type majorant condition on the operator F which makes a relationship of the majorizing function f and the nonlinear operator F under the above mentioned majorant conditions. This result generalizes results based on Lipschitz and Smale type conditions. Ling and Xu [14] and Argyros and Ren [16] have also presented a new convergence analysis of Halley's method under the above mentioned majorant condition.

Under the assumption that the second derivative of F satisfies the majorant conditions, we establish a local convergence of third order continuation method. Here we will find its convergence ball and present two special cases based on Kantorovich-type and Smale-type conditions which are particular cases of our majorant condition. We conclude the paper with two numerical examples to validate our convergence analysis.

The paper is organized as follows. Some preliminary results are contained in section 2. In Section 3, we study the local convergence analysis of the continuation method. Two special cases of main result are presented in section 3.1. Final remarks and two numerical examples are provided in section 4. Section 5 forms the conclusion part of the paper.

For a positive number a and $w \in \mathbb{W}_1$, we consider $B(w, a)$ to stand for the open ball with radius a and center w and $\bar{B}(w, a)$ is the corresponding close ball.

2. Preliminaries

In this section we provide some basic results which is required for our convergence analysis of the method.

Assume $\varphi > 0$ and $f, f_* : (0, \varphi) \rightarrow \mathbb{R}$ be twice continuously differentiable functions. Let $w, v \in B(\tau, \varphi) \subset \mathbb{D}$, with $\|v - w\| + \|w - \tau\| < \varphi$. We say that the operator F satisfy the majorizing function f and f_* at τ if F'' satisfies the majorant conditions,

$$(6) \quad \|F'(\tau)^{-1}[F''(v) - F''(w)]\| \leq f''(\|v - w\| + \|w - \tau\|) - f''(\|w - \tau\|),$$

and

$$(7) \quad \|F'(\tau)^{-1}[F''(v) - F''(\tau)]\| \leq f''(\|v - \tau\|) - f''(0),$$

with the assumptions:

- (M₁) $f''(0) \geq 0$, $f''(0) > 0$, $f'_*(0) = -1$.
- (M₂) f''_* is convex in $[0, \varphi)$, f'' and f''_* are strictly increasing in $[0, \varphi)$.
- (M₃) f'_* has zeros in $(0, \varphi)$ and assume that the smallest zero of f'_* in $(0, \varphi)$ be φ_0 .
- (M₄) $\|F'(\tau)^{-1} F''(\tau)\| \leq f''(0)$.

It is to be mentioned that f and f_* are called the majorant condition and centered majorant condition of F in $B(\tau, \varphi)$ respectively. Also, we

can conclude that

$$(8) \quad f''_*(\zeta) \leq f''(\zeta), \quad \zeta \in [0, \wp]$$

and $\frac{f''}{f''_*}$ can be arbitrarily large [1].

In the next lemma some basic conclusions about the function f_* has been drawn.

LEMMA 2.1. *Under the assumptions (M₁)-(M₄), the following results hold on the function f_* :*

- (i₁) f'_* is strictly convex and strictly increasing in $[0, \wp]$.
- (i₂) $-1 < f'_*(\zeta) < 0$ for $\zeta \in (0, \wp_0)$.

Proof. From (M₂), we imply that f''_* is a strictly increasing and convex function in $[0, \wp]$ and $f''_*(0) > 0$. Then, f'_* can be concluded as a strictly increasing and strictly convex function [17], which proves (i₁).

By (M₁) and (M₃), $f'_*(0) = -1$ and $f'_*(\wp_0) = 0$. Then, by using (i₁) we obtain $-1 < f'_*(\zeta) < 0$ for $\zeta \in (0, \wp_0)$, which proves (i₂). Hence the Lemma is proved. □

Now on $[0, \wp]$, we define five scalar valued functions J_1, J_2, J_3, J_4 and J_5 .

$$J_1(\zeta) = (2 + f'_*(\zeta))f''_*(\zeta)\zeta - (f'_*(\zeta))^2,$$

Note that by the condition of (M₁)-(M₄), J_1 is a continuous function on $[0, \wp]$ and

$$J_1(0) = -(f'_*(0))^2 = -1 < 0$$

and

$$\begin{aligned} J_1(\wp_0) &= (2 + f'_*(\wp_0))f''_*(\wp_0)\wp_0 - (f'_*(\wp_0))^2 \\ &= f''_*(\wp_0)\wp_0 > 0. \end{aligned}$$

Then, it can be concluded that there exist a minimal zero \wp_1 of J_1 on $(0, \wp_0)$ by intermediate value theorem. Now define J_2, J_3, J_4 on $[0, \wp_1]$ as,

$$J_2(\zeta) = \frac{(2 + f'_*(\zeta))f''_*(\zeta)\zeta}{(f'_*(\zeta))^2},$$

$$J_3(\zeta) = \frac{1}{1 - J_2(\zeta)},$$

and

$$J_4(\zeta) = -\frac{1}{2}J_3(\zeta)\left(\frac{1}{f'_*(\zeta)}\right)\left[\frac{1}{\zeta} - \frac{(f''_*(\zeta))^2}{f'_*(\zeta)} + \frac{1}{2}J_2(\zeta)(1 + (1 + |\alpha|)J_2(\zeta))\frac{f''_*(\zeta)}{\zeta}\right]. \quad \alpha \in [0, 1]$$

Since

$$(9) \quad J_2(0) = 0 \quad \text{and} \quad J_2(\wp_1) = \frac{(2 + f'_*(\wp_1))f''_*(\wp_1)\wp_1}{(f'_*(\wp_1))^2},$$

then by using Lemma 2.1 (i_2), conditions (M_1) and (M_2) we get,

$$(10) \quad J_2(\wp_1) > 0.$$

Also, since \wp_1 is a minimal positive zero of J_1 on $(0, \wp_0)$ i.e $J_1(\wp_1) = 0$, then

$$0 = J_1(\wp_1) = (2 + f'_*(\wp_1))f''_*(\wp_1)\wp_1 - (f'_*(\wp_1))^2$$

or

$$(11) \quad \frac{(2 + f'_*(\wp_1))f''_*(\wp_1)\wp_1}{(f'_*(\wp_1))^2} = 1.$$

Thus, from (9) and (11) we found that

$$(12) \quad J_2(\wp_1) = 1.$$

Then, (9), (10) and (12) implies that for $0 \leq \zeta < \wp_1$

$$(13) \quad 0 \leq J_2(\zeta) < 1.$$

Since $J_3(0) = \frac{1}{1-0} = 1$, then by using (13) we can conclude that for $0 \leq \zeta < \wp_1$

$$(14) \quad J_3(\zeta) > 0.$$

Since,

$$\begin{aligned} J_4(0) &= -\frac{1}{2}J_3(0)\left(\frac{1}{f'_*(0)}\right)\left[\frac{1}{0} - \frac{(f''_*(0))^2}{f'_*(0)} + J_2(0)(1 + |\alpha|J_2(0))\frac{f''_*(0)}{0}\right] \\ &= \frac{1}{2}f''_*(0) > 0, \quad \alpha \in [0, 1] \end{aligned}$$

then, by using Lemma 2.1, (M_2), (13) and (14), we can conclude that for $0 \leq \zeta < \wp_1$

$$J_4(\zeta) > 0.$$

Here, from (M_2) and lemma 2.1, $J_4(\zeta)$ is strictly increasing. Now, we define the function $J_5(\zeta)$ as

$$J_5(\zeta) = J_4(\zeta)\zeta^2 - 1.$$

If $J_5(\zeta)$ has at least one zero in $(0, \wp_1)$, let $\wp_* \in (0, \wp_1)$ be it's minimal zero. Since, \wp_* is the minimal zero of $J_5(\zeta)$ i.e $J_5(\wp_*) = 0$ and $J_5(0) < 0$ and $J_4(\zeta)$ is strictly increasing, then

$$J_5(\zeta) < 0 \text{ for } \zeta \in [0, \wp_*).$$

3. Local convergence results for continuation method

Following lemmas will play important role for the local convergence analysis of our method.

LEMMA 3.1. Assume $\|w - \tau\| \leq \zeta < \wp_0$. If $f_* : [0, \wp_0) \rightarrow \mathbb{R}$ be a twice continuously differentiable function and is the majorizing function to F at τ , then $F'(w)$ is non-singular and

$$\|F'(w)^{-1}F'(\tau)\| \leq -\frac{1}{f'_*(\|w - \tau\|)} \leq -\frac{1}{f'_*(\zeta)}.$$

In particular, F' is non-singular in $\mathcal{BL}(\mathbb{W}_2, \mathbb{W}_1)$.

Proof. Let $w \in \bar{B}(\tau, \wp_0)$ and $0 \leq \zeta < \wp_0$. By Taylor series we have

$$F'(w) = F'(\tau) + \int_0^1 [F''(w^\vartheta) - F''(\tau)](w - \tau)d\vartheta + F''(\tau)(w - \tau)$$

or

$$F'(\tau)^{-1}[F'(w) - F'(\tau)] = \int_0^1 F'(\tau)^{-1}[F''(w^\vartheta) - F''(\tau)](w - \tau)d\vartheta + F'(\tau)^{-1}F''(\tau)(w - \tau)$$

where, $w^\vartheta = \tau + \vartheta(w - \tau)$. Use of conditions (M_1) , (M_4) , Eq. (6), and Lemma 2.1 (i_2) , to imply

$$\begin{aligned} \|F'(\tau)^{-1}(F'(w) - F'(\tau))\| &\leq \int_0^1 \|F'(\tau)^{-1}[F''(w^\vartheta) - F''(\tau)]\| \|w - \tau\| d\vartheta \\ &\quad + \|F'(\tau)^{-1}F''(\tau)\| \|w - \tau\|. \\ &\leq \int_0^1 [f_*''(\vartheta\|w - \tau\|) - f_*''(0)] \|w - \tau\| d\vartheta \\ &\quad + f_*''(0) \|w - \tau\| \\ &\leq f_*'(\|w - \tau\|) - f_*'(0) \leq f_*'(\zeta) - f_*'(0) < 1. \end{aligned}$$

Then Banach lemma on invertible operator [7] imply that $F'(w)^{-1} \in \mathcal{BL}(\mathbb{W}_2, \mathbb{W}_1)$ and

$$\begin{aligned} \|F'(w)^{-1}F'(\tau)\| &\leq \frac{1}{1 - (f_*'(\|w - \tau\|) - f_*'(0))} \\ &\leq -\frac{1}{f_*'(\|w - \tau\|)} \leq -\frac{1}{f_*'(\zeta)}. \end{aligned}$$

which proves the lemma. \square

LEMMA 3.2. Under the assumption of Lemma 3.1, we have

- (i.) $\|F'(\tau)^{-1}F''(w)\| \leq f_*''(\|w - \tau\|) \leq f_*''(\zeta)$.
(ii.) In $\mathcal{BL}(\mathbb{W}_2, \mathbb{W}_1)$, $(I - \mathcal{L}_F(w))$ is invertible and

$$\|(I - \mathcal{L}_F(w))^{-1}\| \leq \frac{1}{1 - J_2(\|w - \tau\|)} \leq \frac{1}{1 - J_2(\zeta)} = J_3(\zeta)$$

Proof. Since f_*'' is strictly increasing, by Eq. (6) and (M_4) , we obtain

$$\begin{aligned} \|F'(\tau)^{-1}F''(w)\| &\leq \|F'(\tau)^{-1}[F''(w) - F''(\tau)]\| + \|F'(\tau)^{-1}F''(\tau)\| \\ &\leq f_*''(\|w - \tau\|) - f_*''(0) + f_*''(0) \\ &\leq f_*''(\|w - \tau\|) \leq f_*''(\zeta), \end{aligned}$$

which proves the first part of the lemma. Now in addition to the results of lemma 3.1, use of above result in the definition of $\mathcal{L}_F(w)$ for $w \in \bar{B}(\tau, \varphi^*)$

imply that

$$\begin{aligned}
 \|\mathcal{L}_F(w)\| &\leq \|F'(w)^{-1}F'(\tau)\| \|F'(\tau)^{-1}F''(w)\| \|F'(w)^{-1}F'(\tau)\| \times \\
 &\quad \left\| \int_0^1 F'(\tau)^{-1}F'(w^\vartheta)d\vartheta(w-\tau) \right\| \\
 &\leq \|F'(w)^{-1}F'(\tau)\|^2 \|F'(\tau)^{-1}F''(w)\| \\
 &\quad \times \left\| \int_0^1 F'(\tau)^{-1}F'(w^\vartheta)d\vartheta(w-\tau) \right\| \\
 &\leq \frac{f''(\|w-\tau\|)}{f'_*(\|w-\tau\|)^2} (2 + f'_*(\|w-\tau\|)) \|w-\tau\| \\
 (15) \quad &\leq J_2(\|w-\tau\|) \leq J_2(\zeta) < 1
 \end{aligned}$$

where, $w^\vartheta = \tau + \vartheta(w - \tau)$. Thus, by Banach Lemma, it follows that $(I - \mathcal{L}_F(w))^{-1} \in \mathcal{BL}(\mathbb{W}_2, \mathbb{W}_1)$ and

$$\begin{aligned}
 \|H_\alpha(w)\| &= \|(I - \mathcal{L}_F(w))^{-1}\| \\
 &\leq \frac{1}{I - \|\mathcal{L}_F(w)\|} \\
 &\leq \frac{1}{1 - J_2(\|w-\tau\|)} \\
 &= J_3(\|w-\tau\|) = J_3(\zeta),
 \end{aligned}$$

which proves second part of the lemma. □

Now the main local convergence result for the continuation method (5) is presented as follows.

THEOREM 3.3. *Under the majorant condition on F , started with $w_0 \in B(\tau, \wp^*)$ the sequence $\{w_n\}$ generated by continuation method is well defined, contained in $B(\tau, \wp_*)$ and converges to the unique solution $\tau \in B(\tau, \wp_0)$ of the Eq.(1). Moreover, the following estimate hold*

$$(16) \quad \|w_{n+1} - \tau\| \leq J_4(\wp_*) \|w_n - \tau\|^3.$$

In particular, the method converges Q-cubically to τ .

Proof. Here note that $F(\tau) = 0$ and by Eq.(5)

$$\begin{aligned}
w_{n+1} - \tau &= w_n - \tau - \left[I + \frac{1}{2} \mathcal{L}_F(w_n) (I + \alpha \mathcal{L}_F(w_n) H_\alpha(w_n)) \right] \\
&\quad \times F'(w_n)^{-1} F(w_n) \\
&= H_\alpha(w_n) \left[(w_n - \tau) F'(w_n) F'(w_n)^{-1} - (w_n - \tau) \mathcal{L}_F(w_n) \right. \\
&\quad - F'(w_n)^{-1} F(w_n) + \mathcal{L}_F(w_n) F'(w_n)^{-1} F(w_n) \\
&\quad - \frac{1}{2} \mathcal{L}_F(w_n) F'(w_n)^{-1} F(w_n) \\
&\quad + \frac{1}{2} \mathcal{L}_F(w_n) \mathcal{L}_F(w_n) F'(w_n)^{-1} F(w_n) \\
&\quad \left. - \frac{1}{2} \alpha \mathcal{L}_F(w_n) \mathcal{L}_F(w_n) F'(w_n)^{-1} F(w_n) \right] \\
&= H_\alpha(w_n) \left[F'(w_n)^{-1} [- (\tau - w_n) F'(w_n) - F(w_n)] \right. \\
&\quad + F'(w_n)^{-1} F''(w_n) F'(w_n)^{-1} F(w_n) (\tau - w_n) \\
&\quad + \frac{1}{2} \mathcal{L}_F(w_n) F'(w_n)^{-1} F(w_n) \\
&\quad + \frac{1}{2} \mathcal{L}_F(w_n) \mathcal{L}_F(w_n) F'(w_n)^{-1} F(w_n) \\
&\quad \left. - \frac{1}{2} \alpha \mathcal{L}_F(w_n) \mathcal{L}_F(w_n) F'(w_n)^{-1} F(w_n) \right] \\
&= H_\alpha(w_n) \left[F'(w_n)^{-1} F'(\tau) \times \right. \\
&\quad \int_0^1 F'(\tau)^{-1} [F''(w_n^\vartheta) - F''(w_n)] (\tau - w_n)^2 (1 - \vartheta) d\vartheta \\
&\quad + F'(w_n)^{-1} F'(\tau) F'(\tau)^{-1} F''(w_n) F'(w_n)^{-1} F'(\tau) \times \\
&\quad \int_0^1 F'(\tau)^{-1} F''(w_n^\vartheta) (\tau - w_n)^2 (1 - \vartheta) d\vartheta (\tau - w_n) \\
&\quad + \frac{1}{2} \mathcal{L}_F(w_n) F'(w_n)^{-1} F'(\tau) \left\{ 1 + \mathcal{L}_F(w_n) - \alpha \mathcal{L}_F(w_n) \right\} \\
&\quad \left. \times \int_0^1 F'(\tau)^{-1} F''(w_n^\vartheta) (\tau - w_n)^2 (1 - \vartheta) d\vartheta \right]
\end{aligned}$$

where, $w_n^\vartheta = w_n + \vartheta(\tau - w_n)$. Thus,

$$\begin{aligned}
 \|w_{n+1} - \tau\| \leq & \|H_\alpha(w_n)\| \left[\|F'(w_n)^{-1}F'(\tau)\| \times \right. \\
 & \int_0^1 \|F'(\tau)^{-1}[F''(w_n^\vartheta) - F''(w_n)]\| \|\tau - w_n\|^2(1 - \vartheta)d\vartheta \\
 & + \|F'(w_n)^{-1}F'(\tau)\| \|F'(\tau)^{-1}F''(w_n)\| \|F'(w_n)^{-1}F'(\tau)\| \\
 & \times \int_0^1 \|F'(\tau)^{-1}F''(w_n^\vartheta)\| \|\tau - w_n\|^2(1 - \vartheta)d\vartheta \|\tau - w_n\| \\
 & + \frac{1}{2} \|\mathcal{L}_F(w_n)\| \|F'(w_n)^{-1}F'(\tau)\| \\
 & \times \left\{ 1 + \|\mathcal{L}_F(w_n)\| + \alpha \|\mathcal{L}_F(w_n)\| \right\} \\
 (17) \quad & \times \left. \int_0^1 \|F'(\tau)^{-1}F''(w_n^\vartheta)\| \|\tau - w_n\|^2(1 - \vartheta)d\vartheta \right]
 \end{aligned}$$

Using lemmas 3.1, 3.2, majorant conditions (6), (7), and (15) in (17), we get

$$\begin{aligned}
 \|w_{n+1} - \tau\| \leq & J_3(\|w_n - \tau\|) \left[-\frac{1}{f'_*(\|w_n - \tau\|)} \times \right. \\
 & \int_0^1 [f''(\vartheta\|w_n - \tau\| + \|w_n - \tau\|) - f''(\|w_n - \tau\|)] \times \\
 & \|w_n - \tau\|^2(1 - \vartheta)d\vartheta + \frac{f''_*(\|w_n - \tau\|)}{(f'_*(\|w_n - \tau\|))^2} \times \\
 & \int_0^1 [f''_*(\vartheta\|w_n - \tau\| + \|w_n - \tau\|)] \|w_n - \tau\|^2(1 - \vartheta)d\vartheta \times \\
 & \|w_n - \tau\| - \frac{1}{2} J_2(\|w_n - \tau\|) \left(\frac{1}{f'_*(\|w_n - \tau\|)} \right) \times \\
 & \int_0^1 [f''_*(\vartheta\|w_n - \tau\| + \|w_n - \tau\|)] \|w_n - \tau\|^2(1 - \vartheta)d\vartheta \\
 & - \frac{1}{2}(1 + |\alpha|) J_2(\|w_n - \tau\|)^2 \left(\frac{1}{f'_*(\|w_n - \tau\|)} \right) \times \\
 & \left. \int_0^1 [f''_*(\vartheta\|w_n - \tau\| + \|w_n - \tau\|)] \|w_n - \tau\|^2(1 - \vartheta)d\vartheta \right].
 \end{aligned}$$

By, the convexity [1] of f'' , we also have

$$\begin{aligned} f''(\vartheta\|w_n - \tau\| + \|w_n - \tau\|) - f''(\|w_n - \tau\|) \\ \leq [f''(\vartheta\wp_* + \wp) - f''(\wp_*)] \frac{\vartheta\|w_n - \tau\|}{\vartheta\wp_*}. \end{aligned}$$

Thus,

$$\begin{aligned} \|w_{n+1} - \tau\| &\leq -\frac{1}{2}J_3(\|w_n - \tau\|) \left(\frac{1}{f'_*(\|w_n - \tau\|)} \right) \frac{\|w_n - \tau\|^3}{\wp_*} \\ &\quad + \frac{1}{2}J_3(\|w_n - \tau\|) \left(\frac{f''_*(\|w_n - \tau\|)}{f'_*(\|w_n - \tau\|)} \right)^2 \|w_n - \tau\|^3 \\ &\quad - \frac{1}{4}J_3(\|w_n - \tau\|)J_2(\|w_n - \tau\|) \left(\frac{f''_*(\|w_n - \tau\|)}{f'_*(\|w_n - \tau\|)} \right) \\ &\quad \times \frac{\|w_n - \tau\|^3}{\wp_*} \\ &\quad - \frac{1}{4}J_3(\|w_n - \tau\|)(1 + |\alpha|) \left(\frac{f''_*(\|w_n - \tau\|)}{f'_*(\|w_n - \tau\|)} \right) \frac{\|w_n - \tau\|^3}{\wp_*} \\ &\leq -\frac{1}{2}J_3(\|w_n - \tau\|) \left(\frac{1}{f'_*(\wp_*)} \right) \left[\frac{1}{\wp_*} - \frac{(f''_*(\wp_*))^2}{f'_*(\wp_*)} \right. \\ &\quad \left. + \frac{1}{2}J_2(\wp_*)(1 + (1 + |\alpha|)J_2(\wp_*) \frac{f''_*(\wp_*)}{\wp_*}) \right] \|w_n - \tau\|^3 \\ &\leq J_4(\wp_*)\|w_n - \tau\|^3 = \frac{\|w_n - \tau\|^3}{(\wp_*)^2} \leq \|w_n - \tau\| < \wp_*, \end{aligned}$$

which shows that Eq. (16) exist. From above expression we conclude that $w_{n+1} \in B(\tau, \wp_*)$ and $\lim_{n \rightarrow \infty} w_n = \tau$.

To show uniqueness part, we have to show that the solution τ of (1) is unique in $\bar{B}(\tau, \wp_0)$. For that, we assume there exists another solution σ in $\bar{B}(\tau, \wp_0)$ of the operator F . Define the linear operator $\mathcal{L} = \int_0^1 F'(\sigma + \vartheta(\tau - \sigma))d\vartheta$. Then by Lemma 2.1, we have

$$\begin{aligned} \left\| F'(\tau)^{-1} \left[\int_0^1 F'(\sigma + \vartheta(\tau - \sigma))d\vartheta - F'(\tau) \right] \right\| \\ \leq f'_*(\|\sigma - \tau\|) - f'_*(0) < 1. \end{aligned}$$

Then, Banach lemma infers that linear operator \mathcal{L} is invertible. Thus from the identity

$$0 = F(\tau) - F(\sigma) = \mathcal{L}(\tau - \sigma)$$

we thus conclude that $\tau = \sigma$. □

Moreover, in the case the function f_* is replaced by f in (M_1) - (M_4) and in the definition of J_1, J_2, J_3, J_4 and J_5 , let these replacement functions are denoted by R_1, R_2, R_3, R_4 and R_5 , respectively. Furthermore \wp_0, \wp_1 and \wp_* are denoted by ζ_0, ζ_1 and ζ_* , respectively. Then, Eq. (16) reduces as:

$$(18) \quad \|w_{n+1} - \tau\| \leq R_4(\zeta_*) \|w_n - \tau\|^3.$$

Again this shows that the continuation method (5) is cubically convergent to τ .

3.1. Special cases and applications. This section consists of two special cases of the convergence results obtained in section 3, namely, convergence results under Kantorovich-type condition and the Smale γ -condition.

3.2. Smale-type. Here, we study the local convergence of the continuation method under Smale-type assumption. Let $\tau \in \mathbb{D}$, $F : \mathbb{D} \subseteq \mathbb{X} \rightarrow \mathbb{Y}$ be analytic, $F'(\tau)^{-1} \in \mathcal{BL}(\mathbb{Y}, \mathbb{X})$ and F satisfies

$$\|F'(\tau)^{-1}F^{(n)}(\tau)\| \leq n!\gamma_*^{n-1} \quad (n \geq 2)$$

where

$$\gamma_* = \sup_{n>1} \left\| \frac{F'(\tau)^{-1}F^{(n)}(\tau)}{n!} \right\|^{\frac{1}{n-1}}.$$

Now, we define two majorant functions f and f^* by

$$(19) \quad f(t) = \frac{\gamma_0 \zeta^2}{1 - \gamma_0 \zeta} - t + a_0, \quad a_0 > 0, 0 \leq \zeta < \frac{1}{\gamma_0}.$$

and

$$(20) \quad f_*(\zeta) = \frac{\gamma_1 \zeta^2}{1 - \gamma_1 \zeta} - \zeta + a_1, \quad a_1 > 0, 0 \leq \zeta < \frac{1}{\gamma_1}.$$

Here, the majorant function f and f^* satisfies all the conditions (M_1) - (M_4) and by using Eq. (19) and (20) in Eq. (6) and Eq. (7), we get

$$\| [F'(\tau)^{-1}[F''(v) - F''(w)]] \| \leq 2\gamma_0 \left[\frac{1}{(1 - \gamma_0\|v - w\| - \gamma_0\|v - \tau\|)^3} - \frac{1}{(1 - \gamma_0\|w - \tau\|)^3} \right]$$

and

$$\| F'(\tau)^{-1}[F''(w) - F''(\tau)] \| \leq 2\gamma_1 \left[\frac{1}{(1 - \gamma_1\|w - \tau\|)^3} - 1 \right].$$

We can see that $\gamma_1 \leq \gamma_0$, and $\gamma_1 = \gamma_0 = \gamma_*$ can certainly be taken as a choice for γ_1 and γ_0 . By the majorant condition (M_3) , we get

$$\wp_0 = \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{\gamma_1} \text{ and } \zeta_0 = \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{\gamma_0}.$$

3.3. Kantorovich-type. For $\mu > 0$, $\beta > 0$, $\lambda > 0$, we define majorant functions f and f_* by

$$f(\zeta) = \frac{\mu}{6}\zeta^3 + \frac{\beta}{2}\zeta^2 - \zeta + a_0, \quad a_0 > 0$$

and

$$f_*(\zeta) = \frac{\lambda}{6}\zeta^3 + \frac{\beta}{2}\zeta^2 - \zeta + a_1, \quad a_1 > 0.$$

With the choice of above majorant functions $f(\zeta)$ and $f_*(\zeta)$, the majorant conditions (6) and (7) reduced to

$$\| F'(\tau)^{-1}[F''(v) - F''(w)] \| \leq \mu\|v - w\|$$

and

$$\| F'(\tau)^{-1}[F''(w) - F''(\tau)] \| \leq \lambda\|w - \tau\|,$$

for all $w, v \in B(\tau, \wp)$. Also, the majorant functions f and f_* satisfy assumptions (M_1) - (M_4) with

$$\| F'(\tau)^{-1}F''(\tau) \| \leq \beta.$$

Thus, from condition (M_3) , we get

$$\wp_0 = \frac{-\beta + \sqrt{\beta^2 + 2\lambda}}{\lambda} \text{ and } \zeta_0 = \frac{-\beta + \sqrt{\beta^2 + 2\mu}}{\mu}.$$

To illustrate the above theoretical results we now provide two numerical examples.

4. Remark and Numerical Examples

REMARK 4.1. Let F be an autonomous differential equation [1] of the form

$$F'(x) = T(F(w))$$

where, T is a known differentiable operator which setup as $T : \mathbb{W}_2 \rightarrow \mathbb{W}_1$. Then the result obtained by Theorem 3.3 and the result given in Eq. (18) can be used to find the zero of an autonomous differential operator F . Here, τ be the zero of (1) thus $F(\tau) = 0$. Since,

$$(21) \quad F''(\tau) = F'(\tau)T'(F(\tau)) = T(F(\tau))T'(F(\tau)) = T(0)T'(0),$$

we can apply the result without actually knowing the solution τ .

EXAMPLE 4.2. From above Remark, Let $\mathbb{D} = \mathbb{W}_1 = \mathbb{W}_2 = \mathbb{R}$ and we define

$$F(w) = e^w - 1.$$

Then, we can choose

$$T(w) = w + 1$$

on $\bar{B}(0, 1)$, with the max norm $\|\cdot\| = \|\cdot\|_\infty$ and norm $\|\cdot\|$ reduces to usual modulus $|\cdot|$. Here, $\tau = 0$ is a zero of function F . Also, $F'(\tau) = T(F(\tau)) = e^\tau = e^0 = 1$. Then, $\|F'(\tau)^{-1}\| = 1$ and by Eq. (21), $F''(\tau) = 1$. Note that,

$$\begin{aligned} \|F'(\tau)^{-1}[F''(v) - F''(w)]\| &\leq e\|v - w\|, \\ \|F'(\tau)^{-1}[F''(w) - F''(\tau)]\| &\leq (e - 1)\|w - \tau\| \end{aligned}$$

and

$$\|F'(\tau)^{-1}F''(\tau)\| \leq 1.$$

Thus, we get

$$\mu = e, \quad \beta = 1 \text{ and } \lambda = e - 1,$$

and we obtain

$$\begin{aligned} \wp_0 &= 0.643849667898780 & \zeta_0 &= 0.565444814154378 \\ \wp_1 &= 0.253825706795416 & \zeta_1 &= 0.230120489416514 . \end{aligned}$$

Now, we find the value of \wp_* and ζ_* for different values of α and summarize in the following tables.

TABLE 1. comparison of the error bound for $\alpha = 0$

n	R.H.S. of (16)	R.H.S. of (18)
0	0.002697174624661	0.003214504245779
1	4.233760117649266e-07	8.541733194380000e-07
2	1.637486750545606e-18	1.602662209127917e-17
3	9.473979400346449e-53	1.058595344683287e-49
4	1.834832230853542e-155	3.050660859982021e-146
5	0	0

TABLE 2. comparison of the error bound for $\alpha = 0.25$

n	R.H.S. of (16)	R.H.S. of (18)
0	0.002720288122176	0.003241658784222
1	4.380761481782595e-07	8.834036391216838e-07
2	1.829589738967638e-18	1.787865475993180e-17
3	1.332803240466948e-52	1.482046399761084e-49
4	5.152322180619548e-155	8.441948304301611e-146
5	0	0

α	ϱ_*	ζ_*
0	0.215278408402431	0.197195956113090
0.25	0.214361878344443	0.196368289382162
0.5	0.213482145913246	0.195574620944516
0.75	0.212636178545174	0.194812081371575
1	0.211821316538971	0.194078162301967

Let us choose the initial value $w_0 = 0.05 \in B(0, 1)$, we can produce sequence $\{w_n\}$ by using continuation method (5). In Table 1, Table 2, Table 3, Table 4 and Table 5 we summarize the comparison or error estimate given in (16) and (18) for different values of α and their corresponding radius ϱ_* and ζ_* . In these tables we show that the error bounds on the distance $\|w_n - \tau\|$ which is obtained by using majorant functions f and f_* rather than f only.

TABLE 3. comparison of the error bound for $\alpha = 0.5$

n	R.H.S. of (16)	R.H.S. of (18)
0	0.002742754231034	0.003268022354721
1	4.527282152227050e-07	9.124940925674993e-07
2	2.036055283220536e-18	1.986393064701195e-17
3	1.852019598601090e-52	2.049135024008290e-49
4	1.393842219825904e-154	2.249503818886918e-145
5	0	0

TABLE 4. comparison of the error bound for $\alpha = 0.75$

n	R.H.S. of (16)	R.H.S. of (18)
0	0.002764621593222	0.003293656016437
1	4.673398018686676e-07	9.414623309126988e-07
2	2.257480424554520e-18	2.198756171010700e-17
3	2.544471917041083e-52	2.800911852238326e-49
4	3.643499118137890e-154	5.789839851590899e-145
5	0	0

TABLE 5. comparison of the error bound for $\alpha = 1$

n	R.H.S. of (16)	R.H.S. of (18)
0	0.002785933123990	0.003318613461156
1	4.819175419940091e-07	9.703237665322237e-07
2	2.494473412055466e-18	2.425474226183148e-17
3	3.459372365484927e-52	3.788232708762449e-49
4	9.226831921791890e-154	1.443299982064361e-144
5	0	0

EXAMPLE 4.3. Let $\mathbb{W}_1 = \mathbb{W}_2 = \mathbb{R}^2$, $\mathbb{D} = \bar{B}(0, 1)$. Here, we define the analytic function $F : \mathbb{W}_1 \rightarrow \mathbb{W}_2$ on \mathbb{D} for $w = (w_1, w_2)^T \in \mathbb{D}$ by

$$(22) \quad F(w) = \left(10e^{w_1} + 5w_1w_2 - 10, 5w_1^2 + \sin w_1 + 10w_2 \right)^T$$

endowed with max norm $\|\cdot\| = \|\cdot\|_\infty$. The first and second Fréchet derivatives of F are:

$$F'(w) = \begin{bmatrix} 10e^{w_1} + 5w_2 & 5w_1 \\ 10w_1 + \cos w_1 & 10 \end{bmatrix}$$

and

$$F''(w) = \begin{bmatrix} 10e^{w_1} & 5 & 10 - \sin w_1 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we find the inverse of $F'(w)$:

$$F'(w)^{-1} = \frac{1}{d} \begin{bmatrix} 10 & -5w_1 \\ -(10w_1 + \cos w_1) & 10e^{w_1} + 5w_2 \end{bmatrix}$$

where,

$$d = \det(F'(w)) = 100e^{w_1} + 50w_2 - 5w_1(10w_1 + \cos w_1).$$

Notice that $\tau = (\tau_1, \tau_2)^T = (0, 0)^T$ is a zero of F . So, we get $\|F'(\tau)^{-1}\| = 0.11$ and $\|F''(\tau)\| = 15$. It is easy to calculate that

$$\|F'(\tau)^{-1}[F''(v) - F''(w)]\| \leq 2.990110011304950 \|v - w\|,$$

$$\|F'(\tau)^{-1}[F''(w) - F''(\tau)]\| \leq 1.890110011304950 \|w - \tau\|$$

and

$$\|F'(\tau)^{-1}F''(\tau)\| \leq \|F'(\tau)^{-1}\| \|F''(\tau)\| \leq 1.6500000000000000.$$

Thus, we get $\mu = 2.990110011304950$, $\beta = 1.6500000000000000$ and $\lambda = 1.890110011304950$. Then, we obtain

$$\begin{aligned} \wp_0 &= 0.476185584934224 & \zeta_0 &= 0.434779083985846 \\ \wp_1 &= 0.176363866650164 & \zeta_1 &= 0.166073493565824. \end{aligned}$$

Now, we find the value of \wp_* and ζ_* for different value α .

n	R.H.S. of (16)	R.H.S. of (18)
0	6.857159777724632e-04	7.636692391211650e-04
1	1.415001988121314e-08	2.176712963036404e-08
2	1.243355710059736e-22	5.040675758054571e-22
3	8.435492090206539e-65	6.259671726968343e-63
4	2.634241034289734e-191	1.198781263075260e-185
5	0	0

TABLE 6. comparison of the error bound for $\alpha = 0$

n	R.H.S. of (16)	R.H.S. of (18)
0	7.113535136238512e-04	7.916421133566884e-04
1	1.638784689719008e-08	2.513596321015180e-08
2	2.003692851317438e-22	8.046285896623789e-22
3	3.662342030684798e-64	2.639339409034319e-62
4	2.236362279545932e-189	9.315251628909988e-184
5	0	0

TABLE 7. comparison of the error bound for $\alpha = 1$

α	\wp_*	ζ_*
0	0.150951654863247	0.143039954827966
0.25	0.150220984208005	0.142360886435521
0.5	0.149521670548422	0.141711221240373
0.75	0.148850973144971	0.141088361453373
1	0.148206507264260	0.140490050383045

Let us choose the initial value $w_0 = 0.025 \in B(0, 1)$. We can produce sequence $\{w_n\}$ by using continuation method (5). In Table 6 and Table 7, we summarize the comparison or error estimate given in (16) and (18) for $\alpha = 0$ and $\alpha = 1$ and their respective radius \wp_* and ζ_* . In these tables we show that the error bounds on the distance $\|w_n - \tau\|$ which is obtained by using majorant functions f and f_* rather than f only.

5. Conclusions

In this paper, the local convergence of a third order continuation method has been studied under majorant conditions on second derivative of F . Convergence ball of the method has also been included. Two special cases: one Kantorovich-type conditions and another Smale-type conditions have also been studied. Two numerical examples are also given here to show applicability of our study.

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