

## SOME PROPERTIES OF BILINEAR MAPPINGS ON THE TENSOR PRODUCT OF $C^*$ -ALGEBRAS

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ABSTRACT. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two unital  $C^*$ -algebras and  $\mathcal{A} \otimes \mathcal{B}$  be their algebraic tensor product. For two bilinear maps on  $\mathcal{A}$  and  $\mathcal{B}$  with some specific conditions, we derive a bilinear map on  $\mathcal{A} \otimes \mathcal{B}$  and study some characteristics. Considering two  $\mathcal{A} \otimes \mathcal{B}$  bimodules, a centralizer is also obtained for  $\mathcal{A} \otimes \mathcal{B}$  corresponding to the given bilinear maps on  $\mathcal{A}$  and  $\mathcal{B}$ . A relationship between orthogonal complements of subspaces of  $\mathcal{A}$  and  $\mathcal{B}$  and their tensor product is also deduced with suitable example.

### 1. Introduction

The characterization of different types of mappings acting on different spaces is an interesting area of research in present times. In 1952, G. J. Wendel [36] first introduced the notion of centralizer in his work on group algebras. Helgason, in 1956 [17] introduced centralizer for Banach algebras. Centralizer for rings was introduced by B. E. Johnson [18] in 1964. Akemann et al. [1], investigated centralizers on  $C^*$ -algebras. In [14], Ghahramani studied about the centralizers and Jordan centralizers on Banach algebras considering bilinear maps satisfying a related condition. Recently a good number of prominent mathematicians have

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studied the behaviour of different maps like homomorphisms, derivations etc. when acting on special products (refer to [2], [31], [35]). Moreover, there are many extensions and generalizations of various existing results regarding the characterization of mappings in different directions with several applications (refer to ([7]- [12]), [15], [21]- [26], [28], [30]).

The theoretical study of tensor product of  $C^*$ -algebras was started in 1952 by T. Turumaru [34]. In 1969 A. Guichardiet [16] discussed about  $C^*$ -tensor norms and tensor product of  $C^*$ -algebras. In 1984, [20] Kaijser and Sinclair studied about the projective tensor product of  $C^*$ -algebras. In [3], Blecher investigated the geometrical properties of algebra norms on the tensor product of  $C^*$ -algebras. Many interesting results in this direction have been developed by different researchers (refer to [5], [6] etc.) time to time.

In this paper, we extend the works of Ghahramani [14] to the tensor product of  $C^*$ -algebras and obtain some specific properties of bilinear maps on such algebras. Using the bilinear map, we also give a characterization of centralizer in the tensor product.

## 2. Some basic definitions

DEFINITION 2.1. [4] Let  $\mathcal{A}$  and  $\mathcal{B}$  be two normed spaces over the field  $\mathbb{F}$  with dual spaces  $\mathcal{A}^*$  and  $\mathcal{B}^*$ . For  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ , let  $a \otimes b$  be the element of  $BL(\mathcal{A}^*, \mathcal{B}^*; \mathbb{F})$  defined by

$$a \otimes b(p, q) = p(a)q(b) \quad (p \in \mathcal{A}^*, q \in \mathcal{B}^*).$$

The algebraic tensor product of  $\mathcal{A}$  and  $\mathcal{B}$ ,  $\mathcal{A} \otimes \mathcal{B}$  is defined as the linear span of  $\{a \otimes b : a \in \mathcal{A}, b \in \mathcal{B}\}$  in  $BL(\mathcal{A}^*, \mathcal{B}^*; \mathbb{F})$ , where  $BL(\mathcal{A}^*, \mathcal{B}^*; \mathbb{F})$  is the set of all bounded bilinear mappings from  $\mathcal{A}^* \times \mathcal{B}^*$  to  $\mathbb{F}$ .

DEFINITION 2.2. [3] Given normed spaces  $\mathcal{A}$  and  $\mathcal{B}$ , the projective tensor norm ( $\gamma$ ) on  $\mathcal{A} \otimes \mathcal{B}$  is defined by

$$\gamma(u) = \inf \left\{ \sum_{i=1}^n \|a_i\| \|b_i\| : u = \sum_{i=1}^n a_i \otimes b_i \right\}$$

where the infimum is taken over all (finite) representations of  $u$ . The completion of  $\mathcal{A} \otimes \mathcal{B}$  with respect to  $\gamma$  is called the projective tensor product of  $\mathcal{A}$  and  $\mathcal{B}$  and it is denoted by  $\mathcal{A} \otimes_{\gamma} \mathcal{B}$ .

For example, Let  $\mu, \nu$  be positive  $\sigma$ -finite measures on measure spaces  $M, N$  respectively, and let  $\mu \times \nu$  be the corresponding product measure on  $M \times N$ . Then there exists an isometric linear isomorphism of  $L^1(\mu) \otimes_\gamma L^1(\nu)$  onto  $L^1(\mu \times \nu)$ . [4].

**DEFINITION 2.3.** [4] A norm  $\alpha$  on  $\mathcal{A} \otimes \mathcal{B}$  is a cross norm if  $\alpha(a \otimes b) = \|a\| \|b\|, \forall a \in \mathcal{A}, b \in \mathcal{B}$ .

For example, projective tensor norm is a cross norms.

**LEMMA 2.4.** [4] Given  $p \in \mathcal{A} \otimes \mathcal{B}$ , there exist linearly independent sets  $\{a_i\}, \{b_i\}$  such that  $p = \sum_{i=1}^n a_i \otimes b_i$ .

**LEMMA 2.5.** [4] Let  $\mathcal{A}$  and  $\mathcal{B}$  be normed algebras over  $\mathbb{F}$ . There exists a unique product on  $\mathcal{A} \otimes \mathcal{B}$  with respect to which  $\mathcal{A} \otimes \mathcal{B}$  is an algebra and

$$(a \otimes b)(c \otimes d) = ac \otimes bd \quad (a, c \in \mathcal{A}, b, d \in \mathcal{B}).$$

**DEFINITION 2.6.** [4] In an algebra  $\mathcal{A}$ , for  $x, x^* \in \mathcal{A}$ , an involution is a map  $x \rightarrow x^*$  such that  $(x + y)^* = x^* + y^*, (x^*)^* = x, (xy)^* = y^*x^*, (\alpha x)^* = \bar{\alpha}x^*, \forall x, y \in \mathcal{A}$  and for all scalar  $\alpha$ , where  $x^*$  is called the adjoint of  $x$ .

An algebra  $\mathcal{A}$  with an involution  $*$  is called a  $*$ -algebra. The most common example of a  $*$ -algebra is the field of complex numbers  $\mathbb{C}$  (over real) where  $*$  is complex conjugation.

If  $\mathcal{A}$  and  $\mathcal{B}$  are two  $*$ -algebras, then  $\mathcal{A} \otimes \mathcal{B}$  is also a  $*$ -algebra where  $(a \otimes b)^* = a^* \otimes b^*$ .

**DEFINITION 2.7.** [19] A norm on a  $*$ -algebra  $\mathcal{A}$  that satisfies  $\|a^*a\| = \|a\|^2$  for all  $a \in \mathcal{A}$  is called a  $C^*$ -norm and the algebra is called  $C^*$ -algebra.

An example of  $C^*$ -algebra is  $B(\mathcal{H})$ , the set of all bounded linear operator on a Hilbert space  $\mathcal{H}$ .

**DEFINITION 2.8.** [14] Let  $\mathcal{A}$  be a  $C^*$ -algebra and  $M$  be an  $\mathcal{A}$ -bimodule. A linear(additive) map  $h : \mathcal{A} \rightarrow M$  is said to be a right (left) centralizer if

$$h(xy) = xh(y) \quad (h(xy) = h(x)y) \quad \forall x, y \in \mathcal{A}.$$

If  $h$  is both right and left centralizer then it is called a centralizer.

For example, let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a mapping defined as  $h(x) = \frac{x}{2}, x \in \mathbb{R}$ . Clearly,  $h$  is a centralizer since  $h(xy) = xh(y) = h(x)y = \frac{xy}{2}, \forall x, y \in \mathbb{R}$ .

DEFINITION 2.9. [14] For a  $C^*$ -algebra  $\mathcal{A}$ , with  $\mathcal{A}$ -bimodule  $M$ , a centralizer  $h : \mathcal{A} \rightarrow M$  is called right (left) Jordan centralizer if

$$h(x^2) = xh(x) \quad (h(x^2) = h(x)x), \text{ for each } x \in \mathcal{A}.$$

$h$  is said to be Jordan centralizer if

$$h(xy + yx) = xh(y) + h(y)x = yh(x) + h(x)y \quad \forall x, y \in \mathcal{A}.$$

Every centralizer is a Jordan centralizer. But the converse is not true in general.

EXAMPLE 2.10. Let  $\mathcal{A}'$  be a  $C^*$ -algebra such that the square of each element in  $\mathcal{A}'$  is zero but the product of some elements in  $\mathcal{A}'$  is non-zero.

$$\text{Let } \mathcal{A} = \left\{ a = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix} : p, q \in \mathcal{A}' \right\}.$$

We define,  $h : \mathcal{A} \rightarrow \mathcal{A}$  such that  $h(a) = \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix}$ .

Then  $h$  is a Jordan centralizer. Also it can be easily verified that  $h$  is a right centralizer, but not a left centralizer.

### 3. Main Results

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two  $C^*$ -algebras with unit elements  $e_1$  and  $e_2$  respectively and  $\mathcal{A} \otimes \mathcal{B}$  be their algebraic tensor product. Then for the unique product as given by Lemma 2.5,  $\mathcal{A} \otimes \mathcal{B}$  is an algebra. Here we consider  $\mathcal{A} \otimes \mathcal{B}$  with the projective tensor norm.

Let  $\circ$  denote the Jordan product on  $\mathcal{A} \otimes \mathcal{B}$  such that for  $\sum_{i=1}^n u_i = \sum_{i=1}^n a_i \otimes b_i$ ,  $\sum_{j=1}^m v_j = \sum_{j=1}^m c_j \otimes d_j$  in  $\mathcal{A} \otimes \mathcal{B}$ ,

$$\left( \sum_{i=1}^n u_i \right) \circ \left( \sum_{j=1}^m v_j \right) = \sum_{i=1}^n \sum_{j=1}^m u_i v_j + \sum_{j=1}^m \sum_{i=1}^n v_j u_i.$$

Let the set of invertible elements of  $\mathcal{A} \otimes \mathcal{B}$  be  $Inv(\mathcal{A} \otimes \mathcal{B})$ . Then  $Inv(\mathcal{A} \otimes \mathcal{B})$  is an open subset of  $\mathcal{A} \otimes \mathcal{B}$  and so, it is a disjoint union of open connected subsets, the components of  $Inv(\mathcal{A} \otimes \mathcal{B})$  (refer to [14]). The component containing  $e_1 \otimes e_2$  is called the principal component of  $Inv(\mathcal{A} \otimes \mathcal{B})$  and it is denoted by  $Inv_0(\mathcal{A} \otimes \mathcal{B})$ .

In [14], Ghahramani used bilinear maps to study centralizers and Jordan centralizers on Banach algebras. From a given bilinear map on Banach algebra with some specific conditions, Ghahramani derived the following linear map.

**THEOREM 3.1.** [14] *Let  $\mathcal{A}$  be a unital Banach algebra and  $X$  be a Banach space. Let  $h : \mathcal{A} \times \mathcal{A} \rightarrow X$  be a continuous bilinear map such that*

$$(1) \quad a \in \text{Inv}_0(\mathcal{A}) \Rightarrow h(a, a^{-1}) = h(e_1, e_1).$$

$$(2) \quad \text{Then, } h(a, a) = h(a^2, e_1) \text{ and } h(a, e_1) = h(e_1, a),$$

where  $a \in \mathcal{A}$  and there exists a continuous linear map  $P : \mathcal{A} \rightarrow X$  such that

$$h(a_1, a_2) + h(a_2, a_1) = P(a_1 \circ a_2), \quad a_1, a_2 \in \mathcal{A}.$$

Here, we extend the above result for the algebraic tensor product  $\mathcal{A} \otimes \mathcal{B}$  with projective tensor norm, and considering Banach spaces  $X$  and  $Y$  with projective tensor product  $X \otimes_\gamma Y$ , (which is also a Banach space).

**THEOREM 3.2.** *Let  $f_1 : \mathcal{A} \times \mathcal{A} \rightarrow X$  and  $f_2 : \mathcal{B} \times \mathcal{B} \rightarrow Y$  be two continuous bilinear maps each satisfying the above property (1). Then there exists a continuous bilinear map*

$$f : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \rightarrow X \otimes_\gamma Y$$

with the following properties:

for  $\sum_{i=1}^n u_i = \sum_{i=1}^n a_i \otimes b_i$ ,  $\sum_{j=1}^m v_j = \sum_{j=1}^m c_j \otimes d_j \in \mathcal{A} \otimes \mathcal{B}$ ,

$$I) u_i \in \text{Inv}_0(\mathcal{A} \otimes \mathcal{B}), \quad i = 1, 2, \dots, n.$$

$$\begin{aligned} \Rightarrow f\left(\sum_{i=1}^n u_i, \sum_{i=1}^n u_i^{-1}\right) &= n f_1(e_1, e_1) \otimes f_2(e_2, e_2) \\ &+ \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j \neq i, j=1}^n (f_1(a_i, a_j^{-1}) \otimes f_2(b_i, b_j^{-1}) + f_1(a_j^{-1}, a_i) \otimes f_2(b_j^{-1}, b_i)) \right], \end{aligned}$$

$$\begin{aligned} II) f\left(\sum_{i=1}^n u_i, \sum_{i=1}^n u_i\right) &= f\left(\sum_{i=1}^n u_i^2, e_1 \otimes e_2\right) \\ &+ \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i, j=1}^n (f(a_i \otimes b_i, a_j \otimes b_j) + f(a_j \otimes b_j, a_i \otimes b_i)), \end{aligned}$$

$$III) f\left(\sum_{i=1}^n u_i, e_1 \otimes e_2\right) = f(e_1 \otimes e_2, \sum_{i=1}^n u_i).$$

Moreover, there exists a continuous linear map

$$S : \mathcal{A} \otimes \mathcal{B} \rightarrow X \otimes_\gamma Y$$

such that

$$\begin{aligned}
 (3) \quad & f\left(\sum_{i=1}^n u_i + \sum_{j=1}^m v_j, \sum_{i=1}^n u_i + \sum_{j=1}^m v_j\right) - \sum_{i=1}^n f\left(u_i, \sum_{j \neq i, j=1}^n u_j + \sum_{j=1}^m v_j\right) \\
 & + \sum_{i=1}^m f\left(v_i, \sum_{j \neq i, j=1}^n u_j + \sum_{j=1}^m v_j\right) \\
 & = S\left(\sum_{i=1}^n u_i^2\right) + S\left(\sum_{j=1}^m v_j^2\right).
 \end{aligned}$$

[Here, the representations  $\sum_{i=1}^n u_i, \sum_{j=1}^m v_j \in \mathcal{A} \otimes \mathcal{B}$  follows by Lemma 2.4.]

*Proof.* Using  $f_1$  and  $f_2$ , we define a map  $f : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \rightarrow X \otimes_{\gamma} Y$  by

$$f\left(\sum_{i=1}^n u_i, \sum_{j=1}^m v_j\right) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m [f_1(a_i, c_j) \otimes f_2(b_i, d_j) + f_1(c_j, a_i) \otimes f_2(d_j, b_i)].$$

First we show that  $f$  is a bilinear map.

For  $\sum_{i=1}^n u_i, \sum_{j=1}^m v_j$  as already defined (without loss of generality let  $m > n$ ), we take,

$x_l \otimes y_l = a_l \otimes b_l, l = 1, 2, \dots, n$  and  $x_{n+j} \otimes y_{n+j} = c_j \otimes d_j, j = 1, 2, \dots, m$ .

Let  $\sum_{k=1}^r w_k = \sum_{k=1}^r p_k \otimes q_k \in \mathcal{A} \otimes \mathcal{B}$  and  $\alpha, \beta$  be scalars. Now,

$$\begin{aligned}
 & f\left(\sum_{i=1}^n u_i + \sum_{j=1}^m v_j, \sum_{k=1}^r w_k\right) = f\left(\sum_{l=1}^{n+m} x_l \otimes y_l, \sum_{k=1}^r w_k\right) \\
 & = \frac{1}{2} \left[ \sum_{l=1}^{n+m} \sum_{k=1}^r (f_1(x_l, p_k) \otimes f_2(y_l, q_k) + f_1(p_k, x_l) \otimes f_2(q_k, y_l)) \right] \\
 & = \frac{1}{2} \left[ \sum_{l=1}^n \sum_{k=1}^r (f_1(x_l, p_k) \otimes f_2(y_l, q_k) + f_1(p_k, x_l) \otimes f_2(q_k, y_l)) \right. \\
 & \quad \left. + \sum_{l=n+1}^m \sum_{k=1}^r (f_1(x_l, p_k) \otimes f_2(y_l, q_k) + f_1(p_k, x_l) \otimes f_2(q_k, y_l)) \right] \\
 & = f\left(\sum_{i=1}^n u_i, \sum_{k=1}^r w_k\right) + f\left(\sum_{j=1}^m v_j, \sum_{k=1}^r w_k\right).
 \end{aligned}$$

Similarly, we can show that

$$f\left(\sum_{i=1}^n u_i, \sum_{j=1}^m v_j + \sum_{k=1}^r w_k\right) = f\left(\sum_{i=1}^n u_i, \sum_{k=1}^r w_k\right) + f\left(\sum_{j=1}^m v_j, \sum_{k=1}^r w_k\right).$$

Also, using the bilinearity of the mapping  $f_1$ ,

$$\begin{aligned} f\left(\sum_{i=1}^n \alpha u_i, \sum_{k=1}^r w_k\right) &= \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^r [f_1(\alpha a_i, p_k) \otimes f_2(b_i, q_k) + f_1(p_k, \alpha a_i) \otimes f_2(q_k, b_i)] \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^r [\alpha f_1(a_i, p_k) \otimes f_2(b_i, q_k) + \alpha f_1(p_k, a_i) \otimes f_2(q_k, b_i)] \\ &= \alpha f\left(\sum_{i=1}^n u_i, \sum_{k=1}^r w_k\right). \end{aligned}$$

Similarly, using the bilinearity of  $f_2$ , we can show that

$$f\left(\sum_{i=1}^n u_i, \sum_{k=1}^r \beta w_k\right) = \beta f\left(\sum_{i=1}^n u_i, \sum_{k=1}^r w_k\right).$$

Now,  $f_1$  and  $f_2$  are continuous (and hence bounded) mappings and the projective tensor norm on  $\mathcal{A} \otimes \mathcal{B}$  is a cross norm. Hence, it follows that

$$\begin{aligned} \left\| f\left(\sum_{i=1}^n u_i, \sum_{j=1}^m v_j\right) \right\| &\leq \sum_{i=1}^n \sum_{j=1}^m \|f_1\| \|f_2\| \|a_i\| \|c_j\| \|b_i\| \|d_j\| \\ &= \|f_1\| \|f_2\| \sum_{i=1}^n \|a_i\| \|b_i\| \sum_{j=1}^m \|c_j\| \|d_j\|. \end{aligned}$$

Using the definition of projective tensor norm,

$$\begin{aligned} \left\| f\left(\sum_{i=1}^n u_i, \sum_{j=1}^m v_j\right) \right\| &\leq \|f_1\| \|f_2\| \left\| \sum_{i=1}^n u_i \right\| \left\| \sum_{j=1}^m v_j \right\|, \\ \text{i.e., } \|f\| &\leq \|f_1\| \|f_2\|, \end{aligned}$$

showing that  $f$  is bounded and hence continuous.

(I) Let  $u_i \in \text{Inv}_0(\mathcal{A} \otimes \mathcal{B})$ ,  $\forall i = 1, 2, \dots, n$ .

$$f\left(\sum_{i=1}^n u_i, \sum_{i=1}^n u_i^{-1}\right) = f\left(\sum_{i=1}^n u_i, \sum_{j=1}^n u_j^{-1}\right).$$

From the definition of  $f$ , the right hand expression equals to

$$\begin{aligned} & \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j=1}^n (f_1(a_i, a_j^{-1}) \otimes f_2(b_i, b_j^{-1}) + f_1(a_j^{-1}, a_i) \otimes f_2(b_j^{-1}, b_i)) \right] \\ &= \sum_{i=1}^n f_1(e_1, e_1) \otimes f_2(e_2, e_2) \\ &+ \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j \neq i, j=1}^n (f_1(a_i, a_j^{-1}) \otimes f_2(b_i, b_j^{-1}) + f_1(a_j^{-1}, a_i) \otimes f_2(b_j^{-1}, b_i)) \right] \\ &= n f_1(e_1, e_1) \otimes f_2(e_2, e_2) \\ &+ \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j \neq i, j=1}^n (f_1(a_i, a_j^{-1}) \otimes f_2(b_i, b_j^{-1}) + f_1(a_j^{-1}, a_i) \otimes f_2(b_j^{-1}, b_i)) \right]. \end{aligned}$$

(II) For  $\sum_{i=1}^n u_i \in \mathcal{A} \otimes \mathcal{B}$ ,

$$\begin{aligned} & f\left(\sum_{i=1}^n u_i, \sum_{i=1}^n u_i\right) \\ &= \sum_{i=1}^n f_1(a_i, a_i) \otimes f_2(b_i, b_i) \\ &+ \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j \neq i, j=1}^n (f_1(a_i, a_j) \otimes f_2(b_i, b_j) + f_1(a_j, a_i) \otimes f_2(b_j, b_i)) \right]. \end{aligned}$$

By property (2) of the Theorem 3.1 of  $f_1$  and  $f_2$ , the above expression equals

$$\begin{aligned} & \sum_{i=1}^n f_1(a_i^2, e_1) \otimes f_2(b_i^2, e_2) \\ &+ \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j \neq i, j=1}^n (f_1(a_i, a_j) \otimes f_2(b_i, b_j) + f_1(a_j, a_i) \otimes f_2(b_j, b_i)) \right] \\ &= f\left(\sum_{i=1}^n a_i^2 \otimes b_i^2, e_1 \otimes e_2\right) \\ &+ \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j \neq i, j=1}^n (f(a_i \otimes b_i, a_j \otimes b_j) + f(a_j \otimes b_j, a_i \otimes b_i)) \right] \end{aligned}$$



$$\begin{aligned}
&= f\left(\sum_{i=1}^n u_i^2, e_1 \otimes e_2\right) \\
(4) \quad &+ \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j \neq i, j=1}^n (f(a_i \otimes b_i, a_j \otimes b_j) + f(a_j \otimes b_j, a_i \otimes b_i)) \right].
\end{aligned}$$

(III) For  $\sum_{i=1}^n u_i \in \mathcal{A} \otimes \mathcal{B}$ ,

$$\begin{aligned}
f\left(\sum_{i=1}^n u_i, e_1 \otimes e_2\right) &= \frac{1}{2} \left[ \sum_{i=1}^n (f_1(a_i, e_1) \otimes f_2(b_i, e_2) + f_1(e_1, a_i) \otimes f_2(e_2, b_i)) \right] \\
&= \frac{1}{2} \left[ \sum_{i=1}^n (f_1(e_1, a_i) \otimes f_2(e_2, b_i) + f_1(e_1, a_i) \otimes f_2(e_2, b_i)) \right], \\
&\quad \text{(using the property (2) of } f_1 \text{ and } f_2.) \\
&= \sum_{i=1}^n f_1(e_1, a_i) \otimes f_2(e_2, b_i) \\
&= f(e_1 \otimes e_2, \sum_{i=1}^n u_i).
\end{aligned}$$

Next, with the help of  $f$  we define a map  $S : \mathcal{A} \otimes \mathcal{B} \rightarrow X \otimes_\gamma Y$  by

$$S\left(\sum_{i=1}^n u_i\right) = f\left(\sum_{i=1}^n u_i, e_1 \otimes e_2\right).$$

Clearly,  $S$  is linear.

Also,  $S$  is bounded and hence continuous, since

$$\begin{aligned}
\|S\left(\sum_{i=1}^n a_i \otimes b_i\right)\| &= \left\| \sum_{i=1}^n f_1(a_i, e_1) \otimes f_2(b_i, e_2) \right\| \\
&\leq \|f_1\| \|f_2\| \sum_{i=1}^n \|a_i\| \|b_i\|.
\end{aligned}$$

For the projective tensor norm on  $\mathcal{A} \otimes \mathcal{B}$ , we get,

$$\|S(\sum_{i=1}^n a_i \otimes b_i)\| \leq \|f_1\| \cdot \|f_2\| \|\sum_{i=1}^n a_i \otimes b_i\|, \text{ i.e., } \|S\| \leq \|f_1\| \|f_2\|.$$

Now, using (3) and taking  $\sum_{i=1}^n u_i + \sum_{j=1}^m v_j = \sum_{k=1}^{m+n} w_k$ , where

$w_k = u_k, k = 1, 2, \dots, n, w_{n+k} = v_k, k = 1, 2, \dots, m$  and  $m > n$  we get,

$$f(\sum_{i=1}^n u_i + \sum_{j=1}^m v_j, \sum_{i=1}^n u_i + \sum_{j=1}^m v_j) = f(\sum_{k=1}^{m+n} w_k, \sum_{k=1}^{m+n} w_k).$$

Since  $f$  satisfies the property (II), so,

$$\begin{aligned} & f(\sum_{k=1}^{m+n} w_k, \sum_{k=1}^{m+n} w_k) \\ &= f(\sum_{k=1}^{n+m} w_k^2, e_1 \otimes e_2) + \frac{1}{2} [\sum_{k=1}^{n+m} \sum_{\substack{l=1 \\ l \neq k}}^{n+m} (f(w_k, w_l) + f(w_l, w_k))] \\ &= f(\sum_{k=1}^n w_k^2, e_1 \otimes e_2) + f(\sum_{k=1}^m w_{n+k}^2, e_1 \otimes e_2) + \frac{1}{2} [\sum_{k=1}^{n+m} \sum_{l=1, l \neq k}^n f(w_k, w_l) \\ &+ \sum_{k=1}^{n+m} \sum_{l=1, l \neq k}^m f(w_k, w_{n+l}) + \sum_{l=1}^n \sum_{\substack{k=1 \\ k \neq l}}^{n+m} f(w_l, w_k) + \sum_{l=1}^m \sum_{\substack{k=1 \\ k \neq l}}^{n+m} f(w_{n+l}, w_k)] \\ &= f(\sum_{k=1}^n w_k^2, e_1 \otimes e_2) + f(\sum_{k=1}^m w_{n+k}^2, e_1 \otimes e_2) \frac{1}{2} [\sum_{k=1}^n \sum_{\substack{l=1 \\ l \neq k}}^n f(w_k, w_l) \\ &+ \sum_{k=1}^m \sum_{\substack{l=1 \\ l \neq k}}^n f(w_{n+k}, w_l) + \sum_{k=1}^n \sum_{\substack{l=1 \\ l \neq k}}^m f(w_k, w_{n+l}) + \sum_{k=1}^m \sum_{\substack{l=1 \\ l \neq k}}^m f(w_{n+k}, w_{n+l}) \\ &+ \sum_{l=1}^n \sum_{\substack{k=1 \\ k \neq l}}^n f(w_l, w_k) + \sum_{l=1}^n \sum_{\substack{k=1 \\ k \neq l}}^m f(w_l, w_{n+k}) + \\ &\sum_{l=1}^m \sum_{\substack{k=1 \\ k \neq l}}^n f(w_{n+l}, w_k) + \sum_{l=1}^m \sum_{\substack{k=1 \\ k \neq l}}^m f(w_{n+l}, w_{n+k})] \end{aligned}$$

After simplification, the above expression reduces to

$$\begin{aligned} & f\left(\sum_{k=1}^n u_k^2, e_1 \otimes e_2\right) + f\left(\sum_{k=1}^m v_k^2, e_1 \otimes e_2\right) + \sum_{k \neq l, k, l=1}^n f(u_k, u_l) \\ & + \sum_{k=1}^m \sum_{k \neq l, l=1}^n f(v_k, u_l) + \sum_{k=1}^n \sum_{k \neq l, l=1}^m f(u_k, v_l) \\ & + \sum_{k \neq l, k, l=1}^m f(v_k, v_l), \end{aligned}$$

which equals to

$$\begin{aligned} & S\left(\sum_{k=1}^n u_k^2\right) + S\left(\sum_{k=1}^m v_k^2\right) + \sum_{k=1}^n f\left(u_k, \sum_{l \neq k, l=1}^n u_l + \sum_{l=1}^m v_l\right) \\ & + \sum_{k=1}^m f\left(v_k, \sum_{l \neq k, l=1}^n u_l + \sum_{l=1}^m v_l\right). \end{aligned}$$

Then,

$$\begin{aligned} & f\left(\sum_{i=1}^n u_i + \sum_{j=1}^m v_j, \sum_{i=1}^n u_i + \sum_{j=1}^m v_j\right) - \sum_{i=1}^n f\left(u_i, \sum_{j \neq i, j=1}^n u_j + \sum_{j=1}^m v_j\right) \\ & + \sum_{i=1}^m f\left(v_i, \sum_{j \neq i, j=1}^n u_j + \sum_{j=1}^m v_j\right) \\ & = S\left(\sum_{i=1}^n u_i^2\right) + S\left(\sum_{j=1}^m v_j^2\right). \end{aligned}$$

□

**EXAMPLE 3.3.** For the unital  $C^*$ -algebras  $l^1$  (over  $\mathbb{R}$ ) and  $\mathbb{R}$ , we define maps  $f_1 : l^1 \times l^1 \rightarrow l^1$  by

$$f_1(\{a_1, a_2, \dots\}, \{b_1, b_2, \dots\}) = \{a_1 b_1, a_2 b_2, 0, 0, \dots\}, \text{ for } \{a_n\}, \{b_n\} \in l^1$$

and  $f_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by  $f_2(x, y) = xy$ , for  $x, y \in \mathbb{R}$ .

Clearly,  $f_1$  and  $f_2$  are continuous and bilinear maps.

Here  $e_1 = \{1, 1, \dots\}$  (the constant sequence)  $\in l^1$  and  $e_2 = 1 \in \mathbb{R}$ .

Also, for  $\{a_n\} \in Inv_0(l^1)$ ,

$$\begin{aligned} &f_1(\{a_1, a_2, \dots\}, \{a_1^{-1}, a_2^{-1}, \dots\}) \\ &= \{a_1 a_1^{-1}, a_2 a_2^{-1}, 0, 0, \dots\} \\ &= \{1, 1, 0, 0, \dots\} \\ &= f_1(e_1, e_1), \end{aligned}$$

and for  $x \in \mathbb{R}$  invertible,

$$f_2(x, x^{-1}) = x x^{-1} = 1 = f_2(e_2, e_2).$$

Now,  $l^1 \otimes \mathbb{R} \cong l^1(\mathbb{R})$ (refer to [29] ).

So, by the Theorem 3.2, there exists  $f : (l^1 \otimes \mathbb{R}) \times (l^1 \otimes \mathbb{R}) \rightarrow l^1(\mathbb{R})$  such that

$$(5) \quad f\left(\sum_{i=1}^n a_i \otimes b_i, \sum_{j=1}^m c_j \otimes d_j\right) = \sum_{i=1}^n \sum_{j=1}^m \{p_{1i}q_{1j}b_i d_j, p_{2i}q_{2j}b_i d_j, 0, 0, \dots\}$$

where  $a_i = \{p_{ki}\}_k, c_j = \{q_{kj}\}_k \in l^1$  and  $b_i, d_j \in \mathbb{R}, i = 1, 2, \dots, n; j = 1, 2, \dots, m$ , and

$$\sum_{i=1}^n a_i \otimes b_i = \sum_{i=1}^n \{p_{ki}b_i\}_k \text{ and } \sum_{j=1}^m c_j \otimes d_j = \sum_{j=1}^m \{q_{kj}d_j\}_k.$$

Now, for the formula defined in Theorem 3.2,

$$\begin{aligned} &\frac{1}{2} \left[ \sum_{i=1}^n \sum_{j=1}^m (f_1(a_i, c_j) \otimes f_2(b_i, d_j) + f_1(c_j, a_i) \otimes f_2(d_j, b_i)) \right] \\ &= \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j=1}^m (\{p_{1i}q_{1j}, p_{2i}q_{2j}, 0, 0, \dots\} \otimes b_i d_j + \{q_{1j}p_{1i}, q_{2j}p_{2i}, 0, 0, \dots\} \otimes d_j b_i) \right] \\ &= \frac{1}{2} \left[ \sum_{i=1}^n \sum_{j=1}^m (\{p_{1i}q_{1j}b_i d_j, p_{2i}q_{2j}b_i d_j, 0, 0, \dots\} + \{q_{1j}p_{1i}d_j b_i, q_{2j}p_{2i}d_j b_i, 0, 0, \dots\}) \right] \\ &= \sum_{i=1}^n \sum_{j=1}^m \{p_{1i}q_{1j}b_i d_j, p_{2i}q_{2j}b_i d_j, 0, 0, \dots\}. \end{aligned}$$

By (4) this is clearly equal to  $f(\sum_{i=1}^n a_i \otimes b_i, \sum_{j=1}^m c_j \otimes d_j)$ , which validates the Theorem 3.2.

Next, we consider two bilinear maps on the individual  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , and derive a centralizer on the algebraic tensor product  $\mathcal{A} \otimes \mathcal{B}$ . For this, let  $M$  and  $N$  be two  $\mathcal{A} \otimes \mathcal{B}$ -bimodules. Then by [32],  $M \otimes N$  is also an  $\mathcal{A} \otimes \mathcal{B}$ -bimodule.

THEOREM 3.4. Suppose  $\phi_1 : \mathcal{A} \times \mathcal{A} \rightarrow M$  and  $\phi_2 : \mathcal{B} \times \mathcal{B} \rightarrow N$  be two bilinear maps satisfying the properties :

$$\begin{aligned} i) & x \in \text{Inv}_0(\mathcal{A}) \Rightarrow \phi_1(x, x^{-1}) = \phi_1(e_1, e_1) \text{ and} \\ & y \in \text{Inv}_0(\mathcal{B}) \Rightarrow \phi_2(y, y^{-1}) = \phi_2(e_2, e_2) \\ ii) & \phi_1(e_1, x_1x_2) + \phi_1(x_1x_2, e_1) = (x_1 \otimes e_2)[\phi_1(e_1, x_2) + \phi_1(x_2, e_1)] \\ & = [\phi_1(x_1, e_1) + \phi_1(e_1, x_1)](x_2 \otimes e_2) \quad \forall x_1, x_2 \in \mathcal{A} \\ iii) & \phi_2(e_2, y_1y_2) + \phi_2(y_1y_2, e_2) = (e_1 \otimes y_1)[\phi_2(e_2, y_2) + \phi_2(y_2, e_2)] \\ & = [\phi_2(y_1, e_2) + \phi_2(e_2, y_1)](e_2 \otimes y_2) \quad \forall y_1, y_2 \in \mathcal{B} \end{aligned}$$

Then corresponding to  $\phi_1$  and  $\phi_2$ , there exists a Jordan centralizer from  $\mathcal{A} \otimes \mathcal{B}$  to  $M \otimes N$ .

*Proof.* Since  $\phi_1$  and  $\phi_2$  satisfy the property (i), so by Theorem 3.1 there exist two linear maps

$$h_1 : \mathcal{A} \rightarrow M \text{ and } h_2 : \mathcal{B} \rightarrow N \text{ given by}$$

$$\begin{aligned} \phi_1(x_1, x_2) + \phi_1(x_2, x_1) &= h_1(x_1 \circ x_2) \text{ and} \\ \phi_2(y_1, y_2) + \phi_2(y_2, y_1) &= h_2(y_1 \circ y_2), \quad x_1, x_2 \in \mathcal{A}, y_1, y_2 \in \mathcal{B}. \end{aligned}$$

Replacing  $x_1$  by  $e_1$  and  $x_2$  by  $x_1x_2$ , we have,

$$\begin{aligned} h_1(x_1x_2) &= \frac{1}{2}[\phi_1(e_1, x_1x_2) + \phi_1(x_1x_2, e_1)] \\ &= \frac{1}{2}[(x_1 \otimes e_2)[\phi_1(e_1, x_2) + \phi_1(x_2, e_1)]] \text{ by property (ii)} \\ (6) \quad &= (x_1 \otimes e_2)h_1(x_2). \\ (7) \quad &h_1(x_1x_2) = h_1(x_1)(x_2 \otimes e_2) \text{ and} \\ (8) \quad &h_2(y_1y_2) = (e_1 \otimes y_1)h_2(y_2) = h_2(y_1)(e_2 \otimes y_2). \end{aligned}$$

Now, we define a map  $h : \mathcal{A} \otimes \mathcal{B} \rightarrow M \otimes N$  by

$$h\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n h_1(a_i) \otimes h_2(b_i), \quad \sum_{i=1}^n a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}.$$

Clearly,  $h$  is linear. Using Lemma 2.4 and by the definition of  $h$ , for

$$\begin{aligned} \sum_{i=1}^n u_i &= \sum_{i=1}^n a_i \otimes b_i, \quad \sum_{j=1}^m v_j = \sum_{j=1}^m c_j \otimes d_j \in \mathcal{A} \otimes \mathcal{B}, \\ h\left(\sum_{i=1}^n u_i \sum_{j=1}^m v_j + \sum_{j=1}^m v_j \sum_{i=1}^n u_i\right) \\ &= \sum_{i=1}^n \sum_{j=1}^m (h_1(a_i c_j) \otimes h_2(b_i d_j) + h_1(c_j a_i) \otimes h_2(d_j b_i)). \end{aligned}$$

Now, using the properties (5), (6), (7) of the mappings  $h_1$  and  $h_2$ , and by the properties of module multiplication with respect to the tensor product [32], the above expression equals to

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^m (a_i \otimes e_2) h_1(c_j) \otimes (e_1 \otimes b_i) h_2(d_j) + \sum_{i=1}^n \sum_{j=1}^m h_1(c_j) (a_i \otimes e_2) \otimes h_2(d_j) (e_1 \otimes b_i) \\ &= \sum_{i=1}^n \sum_{j=1}^m ((a_i \otimes b_i) (h_1(c_j) \otimes h_2(d_j)) + (h_1(c_j) \otimes h_2(d_j)) (a_i \otimes b_i)) \\ &= \left(\sum_{i=1}^n u_i\right) h\left(\sum_{j=1}^m v_j\right) + h\left(\sum_{j=1}^m v_j\right) \left(\sum_{i=1}^n u_i\right). \end{aligned}$$

In a similar way, we can show that

$$h\left(\sum_{i=1}^n u_i \sum_{j=1}^m v_j + \sum_{j=1}^m v_j \sum_{i=1}^n u_i\right) = \left(\sum_{j=1}^m v_j\right) h\left(\sum_{i=1}^n u_i\right) + h\left(\sum_{i=1}^n u_i\right) \left(\sum_{j=1}^m v_j\right),$$

and thus  $h$  is a Jordan centralizer.  $\square$

**Remark:** The mapping  $h$  defined in the above Theorem 3.4 can also be shown to a centralizer, as

$$\begin{aligned} h\left(\sum_{i=1}^n u_i \sum_{j=1}^m v_j\right) &= \sum_{i=1}^n \sum_{j=1}^m (h_1(a_i c_j) \otimes h_2(b_i d_j)) \\ &= \sum_{i=1}^n \sum_{j=1}^m ((a_i \otimes e_2) h_1(c_j) \otimes (e_1 \otimes b_i) h_2(d_j)) \\ &\quad \text{(by property (5) and (7) of } h_1 \text{ and } h_2) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \sum_{j=1}^m (a_i e_1 \otimes e_2 b_i) h(c_j \otimes d_j) \text{ (by Lemma 2.5)} \\
&= \left( \sum_{i=1}^n u_i \right) h \left( \sum_{j=1}^m v_j \right).
\end{aligned}$$

In the same way, it follows that

$$h \left( \sum_{i=1}^n u_i \sum_{j=1}^m v_j \right) = h \left( \sum_{i=1}^n u_i \right) \left( \sum_{j=1}^m v_j \right).$$

□

EXAMPLE 3.5. For the  $C^*$ -algebras  $l^1$  (over  $\mathbb{R}$ ) and  $\mathbb{R}$ , we define  $\phi_1 : l^1 \times l^1 \rightarrow l^1$  by

$$\phi_1(\{a_n\}, \{c_n\}) = \{a_1 c_1, 0, 0, \dots\}, \text{ for } \{a_n\}, \{c_n\} \in l^1$$

and  $\phi_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$\phi_2(b, d) = bd \text{ for } b, d \in \mathbb{R}.$$

Clearly,  $\phi_1$  and  $\phi_2$  are continuous bilinear maps satisfying property (i) of Theorem 3.4.

$$\begin{aligned}
&\phi_1(e_1, \{a_n\}\{c_n\}) + \phi_1(\{a_n\}\{c_n\}, e_1) \\
&= \phi_1(e_1, \{a_n c_n\}) + \phi_1(\{a_n c_n\}, e_1) \\
&= 2\{a_1 c_1, 0, 0, \dots\} \\
&= \{a_1, \dots\}[\{c_1, 0, 0, \dots\} + \{c_1, 0, 0, \dots\}] \\
&= (\{a_n\} \otimes e_2)[\phi_1(e_1, \{c_n\}) + \phi_1(\{c_n\}, e_1)] \\
&= [\phi_1(\{a_n\}, e_1) + \phi_1(e_1, \{a_n\})](\{c_n\} \otimes e_2).
\end{aligned}$$

$$\begin{aligned}
&\text{Also, } \phi_2(1, bd) + \phi_2(bd, 1) = 2bd = (1 \otimes b)(\phi_2(1, d) + \phi_2(d, 1)) \\
(9) \quad &= \phi_2(bd, 1) + \phi_2(1, bd)(1 \otimes d).
\end{aligned}$$

So, properties (ii) and (iii) defined in Theorem 3.4 are also satisfied. Since  $\phi_1$  and  $\phi_2$  satisfy the property (1) of Theorem 3.1, so, there exist two linear maps

$$h_1 : l^1 \rightarrow l^1 \text{ and } h_2 : \mathbb{R} \rightarrow \mathbb{R} \text{ such that}$$

$$\begin{aligned}
&\phi_1(\{a_n\}, \{c_n\}) + \phi_1(\{c_n\}, \{a_n\}) = h_1(\{a_n\} \circ \{c_n\}), \text{ and} \\
&\phi_2(b, d) + \phi_2(d, b) = h_2(b \circ d),
\end{aligned}$$

where  $\{a_n\}, \{c_n\} \in l^1$  and  $b, d \in \mathbb{R}$ .

Replacing  $\{c_n\}$  by  $e_1$  and  $d$  by 1 in (9) we get,

$$\begin{aligned} h_1(\{a_n\}) &= \frac{1}{2}[\phi_1(\{a_n\}, e_1) + \phi_1(e_1, \{a_n\})] \\ &= \frac{1}{2}(\{a_1, 0, 0, \dots\} + \{a_1, 0, 0, \dots\}) \\ &= \{a_1, 0, 0, \dots\}, \end{aligned}$$

$$\text{and } h_2(b) = \frac{1}{2}(\phi_2(b, 1) + \phi_2(1, b)) = \frac{1}{2}(b + b) = b.$$

By Theorem 3.4, the mapping

$h : l^1 \otimes \mathbb{R} \rightarrow l^1 \otimes \mathbb{R}$  is defined by

$$h\left(\sum_{l=1}^n x_l \otimes y_l\right) = \sum_{l=1}^n h_1(x_l) \otimes h_2(y_l), \quad \sum_{l=1}^n x_l \otimes y_l \in l^1 \otimes \mathbb{R}.$$

Let  $\{a_{k_i}\}_k, \{c_{k_j}\}_k \in l^1$  and  $b_i, d_j \in \mathbb{R}, i = 1, 2, \dots, n; j = 1, 2, \dots, m$ .

Now,

$$\begin{aligned} &h\left(\left(\sum_{i=1}^n \{a_{k_i}\}_k \otimes b_i\right)\left(\sum_{j=1}^m \{c_{k_j}\}_k \otimes d_j\right)\right) \\ &= h\left(\sum_{i=1}^n \sum_{j=1}^m \{a_{k_i}\}_k \{c_{k_j}\}_k \otimes b_i d_j\right) \\ &= \sum_{i=1}^n \sum_{j=1}^m h_1(\{a_{k_i} c_{k_j}\}_k) \otimes h_2(b_i d_j) \\ &= \sum_{i=1}^n \sum_{j=1}^m \{a_{k_i} c_{k_j} b_i d_j, 0, 0, \dots\}. \end{aligned}$$

Again,

$$\begin{aligned} &\left(\sum_{i=1}^n \{a_{k_i}\}_k \otimes b_i\right) h\left(\sum_{j=1}^m \{c_{k_j}\}_k \otimes d_j\right) \\ &= \sum_{i=1}^n \{a_{k_i} b_i\}_k \sum_{j=1}^m \{c_{k_j} d_j, 0, 0, \dots\} \end{aligned}$$



$$\begin{aligned}
&= \sum_{i=1}^n \sum_{j=1}^m \{a_{1_i} b_i c_{1_j} d_j, 0, 0, \dots\} \\
&= h\left(\sum_{i=1}^n \{a_{k_i}\}_k \otimes b_i\right) \left(\sum_{j=1}^m \{c_{k_j}\}_k \otimes d_j\right).
\end{aligned}$$

Similarly,

$$h\left(\sum_{i=1}^n \{a_{k_i}\}_k \otimes b_i\right) \left(\sum_{j=1}^m \{c_{k_j}\}_k \otimes d_j\right) = h\left(\sum_{i=1}^n \{a_{k_i}\}_k \otimes b_i\right) \left(\sum_{j=1}^m \{c_{k_j}\}_k \otimes d_j\right),$$

which shows that  $h$  is a centralizer and hence a Jordan centralizer, which is guaranteed by Theorem 3.4.

**COROLLARY 3.6.** *Let  $\phi_1$  and  $\phi_2$  be as defined in Theorem 3.4.*

- I) *If the maps  $\phi_1$  and  $\phi_2$  are symmetric i.e.,  $\phi_1(a_1, a_2) = \phi_1(a_2, a_1)$  and  $\phi_2(b_1, b_2) = \phi_2(b_2, b_1)$ , for  $a_1, a_2 \in \mathcal{A}$ ,  $b_1, b_2 \in \mathcal{B}$ , then  $h(a \otimes b) = \phi_1(a, e_1) \otimes \phi_2(b, e_2)$   $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$ .*
- II) *If any of the maps  $\phi_1$  and  $\phi_2$  is skew-symmetric i.e.,  $\phi_1(a_1, a_2) = -\phi_1(a_2, a_1)$  or  $\phi_2(b_1, b_2) = -\phi_2(b_2, b_1)$ , then  $h$  is a zero mapping.*
- III) *If the maps  $\phi_1$  and  $\phi_2$  are alternating i.e.,  $\phi_1(a, a) = 0$  and  $\phi_2(b, b) = 0$  then,  $h_1(a^2) = 0$  and  $h_2(b^2) = 0$ .*

*Proof.* I) is obvious.

II) If  $\phi_1$  or  $\phi_2$  is skew-symmetric then

$$\phi_1(a, e_1) = -\phi_1(e_1, a) \text{ or } \phi_2(b, e_2) = -\phi_2(e_2, b).$$

Then  $h_1(a) = 0$  or  $h_2(b) = 0$ , which implies that  $h$  is a zero mapping.

III) If  $\phi_1(a, a) = 0$  then

$$h_1(a \circ a) = \phi_1(a, a) + \phi_1(a, a) = 0 \Rightarrow h_1(a^2) = 0.$$

Similarly,  $h_2(b^2) = 0$  if,  $\phi_2(b, b) = 0$ . □

The following result gives some characteristics of the centralizer  $h$  on  $\mathcal{A} \otimes \mathcal{B}$ .

**THEOREM 3.7.** *The mapping  $h$  defined in Theorem 3.4 satisfies the following properties:*

$$(I) \ h\left(\sum_{i=1}^n a_i \otimes b_i\right) - \left(\sum_{i=1}^n a_i \otimes b_i\right)(h_1(e_1) \otimes h_2(e_2)) = 0, \text{ where}$$

$$\sum_{i=1}^n a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}.$$

$$(II) \ \text{For } \sum_{i=1}^n a_i \otimes b_i, \sum_{i=1}^n c_i \otimes d_i \in \mathcal{A} \otimes \mathcal{B}, \text{ if, } a_i c_i \otimes b_i d_i = e_1 \otimes e_2,$$

$$i = 1, 2, 3, \dots, n, \text{ then}$$

$$\sum_{i=1}^n (a_i \otimes b_i)(h_1(c_i) \otimes h_2(d_i)) - n(h_1(e_1) \otimes h_2(e_2)) = 0.$$

$$(III) \ a_i \otimes b_i \in \text{Inv}(\mathcal{A} \otimes \mathcal{B}), i = 1, 2, \dots, n \Rightarrow \sum_{i=1}^n (a_i \otimes b_i)h\left(\sum_{i=1}^n (a_i \otimes b_i)^{-1}\right)$$

$$= n(h_1(e_1) \otimes h_2(e_2)) + \sum_{i,j=1, i \neq j}^n h_1(a_i a_j^{-1}) \otimes h_2(b_i b_j^{-1}).$$

*Proof.* (I) Since  $h$  is a centralizer so, for any  $\sum_{i=1}^n a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}$ , we have,

$$h\left(\sum_{i=1}^n a_i \otimes b_i\right) = h\left(\left(\sum_{i=1}^n a_i \otimes b_i\right)(e_1 \otimes e_2)\right)$$

$$= \left(\sum_{i=1}^n a_i \otimes b_i\right)h(e_1 \otimes e_2)$$

$$= \left(\sum_{i=1}^n a_i \otimes b_i\right)(h_1(e_1) \otimes h_2(e_2)), \text{ (using the definition of } h\text{).}$$

$$(II) \ \text{For } \sum_{i=1}^n a_i \otimes b_i, \sum_{i=1}^n c_i \otimes d_i \in \mathcal{A} \otimes \mathcal{B},$$

if  $a_i c_i \otimes b_i d_i = e_1 \otimes e_2$ ,  $i = 1, 2, 3, \dots, n$ , then using (I),

$$\begin{aligned} h\left(\sum_{i=1}^n a_i c_i \otimes b_i d_i\right) &= \sum_{i=1}^n (a_i c_i \otimes b_i d_i)(h_1(e_1) \otimes h_2(e_2)) \\ \text{i.e., } h\left(\sum_{i=1}^n e_1 \otimes e_2\right) &= \sum_{i=1}^n (a_i \otimes b_i)(c_i \otimes d_i)h(e_1 \otimes e_2). \end{aligned}$$

Using definition of  $h$  and the centralizer property, it follows that

$$\begin{aligned} n(h_1(e_1) \otimes h_2(e_2)) &= \sum_{i=1}^n (a_i \otimes b_i)h((c_i \otimes d_i)(e_1 \otimes e_2)), \\ \text{i.e., } n(h_1(e_1) \otimes h_2(e_2)) &- \sum_{i=1}^n (a_i \otimes b_i)(h_1(c_i) \otimes h_2(d_i)) = 0. \end{aligned}$$

(III) For  $a_i \otimes b_i \in \text{Inv}(\mathcal{A} \otimes \mathcal{B})$ ,  $i = 1, 2, \dots, n$ ,

$$\begin{aligned} &\sum_{i=1}^n (a_i \otimes b_i)h\left(\sum_{i=1}^n (a_i \otimes b_i)^{-1}\right) \\ &= \sum_{i=1}^n (a_i \otimes b_i)\left(\sum_{i=1}^n h_1(a_i^{-1}) \otimes h_2(b_i^{-1})\right) \\ &= \sum_{i=1}^n (a_i \otimes b_i)(h_1(a_i^{-1}) \otimes h_2(b_i^{-1})) + \sum_{i,j=1, i \neq j}^n (a_i \otimes b_i)(h_1(a_j^{-1}) \otimes h_2(b_j^{-1})). \end{aligned}$$

By the definition of  $h$ , the above expression equals

$$\begin{aligned} &n(h_1(e_1) \otimes h_2(e_2)) + \sum_{i,j=1, j \neq i}^n (a_i \otimes b_i)h\left(\sum_j a_j^{-1} \otimes b_j^{-1}\right) \\ &= n(h_1(e_1) \otimes h_2(e_2)) + \left(\sum_{i=1}^n a_i \otimes b_i\right)h\left(\sum_{\substack{j=1 \\ j \neq i}}^n a_j^{-1} \otimes b_j^{-1}\right) \\ &= n(h_1(e_1) \otimes h_2(e_2)) + h\left(\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n a_i a_j^{-1} \otimes b_i b_j^{-1}\right) \text{ (since } h \text{ is a centralizer)} \\ &= n(h_1(e_1) \otimes h_2(e_2)) + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n h_1(a_i a_j^{-1}) \otimes h_2(b_i b_j^{-1}). \end{aligned}$$

□

Our next aim is to discuss the converse part of Theorem 3.4, i.e., from a centralizer on  $\mathcal{A} \otimes \mathcal{B}$  we define a bilinear map for each of the individual  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ . For this, we use the concept of Jordan product (which we denote by  $\bullet$  in case of modules) on  $\mathcal{A} \otimes \mathcal{B}$ -bimodules  $M, N$  and  $M \otimes N$ . Here, also  $\mathcal{A} \otimes \mathcal{B}$  is equipped with the projective tensor norm. For  $\sum_{i=1}^n u_i \in \mathcal{A} \otimes \mathcal{B}$  and  $\sum_{j=1}^m p_j \in M \otimes N$ , where  $u_i = a_i \otimes b_i$ ,  $p_j = m_j \otimes n_j$ , the Jordan product  $\bullet$  on  $M \otimes N$  is defined by

$$\sum_{i=1}^n u_i \bullet \sum_{j=1}^m p_j = \sum_{j=1}^m p_j \bullet \sum_{i=1}^n u_i = \sum_{i=1}^n \sum_{j=1}^m u_i p_j + \sum_{i=1}^n \sum_{j=1}^m p_j u_i.$$

**THEOREM 3.8.** *Corresponding to the centralizer  $h : \mathcal{A} \otimes \mathcal{B} \rightarrow M \otimes N$ , there exist two continuous bilinear maps  $g_1 : \mathcal{A} \times \mathcal{A} \rightarrow M$  and  $g_2 : \mathcal{B} \times \mathcal{B} \rightarrow N$  satisfying the property (1) of Theorem 3.1. Moreover, there exist a bilinear map  $g : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \rightarrow M \otimes N$  with  $\|g\| \leq \|h\|^2$ , if  $h$  is also continuous.*

*Proof.* We define  $g_1 : \mathcal{A} \times \mathcal{A} \rightarrow M$  and  $g_2 : \mathcal{B} \times \mathcal{B} \rightarrow N$  by

$$g_1(a_1, a_2) = \frac{1}{2}(a_1 \otimes e_2) \bullet h(a_2 \otimes e_2) \text{ and}$$

$$g_2(b_1, b_2) = \frac{1}{2}(e_1 \otimes b_1) \bullet h(e_1 \otimes b_2),$$

where  $a_1, a_2 \in \mathcal{A}$  and  $b_1, b_2 \in \mathcal{B}$ .

By the property of module multiplication and Jordan product,  $g_1$  and  $g_2$  are well defined. Also, clearly these are bilinear mappings. For  $a \in \text{Inv}_0(\mathcal{A})$ ,

$$\begin{aligned} g_1(a, a^{-1}) &= \frac{1}{2}(a \otimes e_2) \bullet h(a^{-1} \otimes e_2) \\ &= \frac{1}{2}[(a \otimes e_2)h(a^{-1} \otimes e_2) + h(a^{-1} \otimes e_2)(a \otimes e_2)] \\ &= h(e_1 \otimes e_2) \\ &= \frac{1}{2}(e_1 \otimes e_2) \bullet h(e_1 \otimes e_2) \\ &= g_1(e_1, e_1). \end{aligned}$$

Similarly we can show that  $g_2(b, b^{-1}) = g_2(e_2, e_2)$  for  $b \in \text{Inv}_0(\mathcal{B})$ . Hence,  $g_1$  and  $g_2$  satisfy the condition (1) of Theorem 3.1. Again,

$$\begin{aligned} \|g_1(a_1, a_2)\| &= \left\| \frac{1}{2}(a_1 \otimes e_2) \bullet h(a_2 \otimes e_2) \right\| \\ &= \left\| \frac{1}{2}(a_1 \otimes e_2)h(a_2 \otimes e_2) + h(a_2 \otimes e_2)\frac{1}{2}(a_1 \otimes e_2) \right\| \\ &= \left\| \frac{1}{2}[h(a_1a_2 \otimes e_2) + h(a_2a_1 \otimes e_2)] \right\| \\ &\leq \frac{1}{2}(\|h(a_1a_2 \otimes e_2)\| + \|h(a_2a_1 \otimes e_2)\|) \\ &\leq \|h\|\|a_1\|\|a_2\| \\ &\Rightarrow \|g_1\| \leq \|h\|, \end{aligned}$$

and similarly,  $\|g_2\| \leq \|h\|$ .

Thus,  $g_1$  and  $g_2$  are continuous.

So, by the Theorem 3.2, we have a continuous bilinear mapping

$g : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \rightarrow M \otimes N$  such that

$$g\left(\sum_{i=1}^n u_i, \sum_{j=1}^m v_j\right) = \frac{1}{2}\left[\sum_{i=1}^n \sum_{j=1}^m (g_1(a_i, c_j) \otimes g_2(b_i, d_j) + g_1(c_j, a_i) \otimes g_2(d_j, b_i))\right].$$

Also, it follows that

$$\|g\| \leq \|g_1\|\|g_2\| \leq \|h\|\|h\| = \|h\|^2.$$

□

Now, we show an application of the Theorem 3.2 in deriving a relationship between orthogonal complements of subspaces of the  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  and the algebraic tensor product  $\mathcal{A} \otimes \mathcal{B}$ . In [33], Shenoi studied some basic properties and characteristics of bilinear forms. We note that a bilinear mapping  $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$  is called a bilinear form on  $\mathcal{A}$ . A bilinear form  $f$  on a  $C^*$ -algebra  $\mathcal{A}$  is called reflexive if for  $a_1, a_2 \in \mathcal{A}$ ,  $f(a_1, a_2) = 0$  implies  $f(a_2, a_1) = 0$ . An element  $a \in \mathcal{A}$  is said to be orthogonal to  $a' \in \mathcal{A}$  with respect to a bilinear form  $f$  if  $f(a, a') = 0$ .

For the  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , let  $f_1 : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}^+ \cup \{0\}$  and  $f_2 : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{R}^+ \cup \{0\}$  be two bilinear forms and  $A_1$  and  $B_1$  be subspaces of

$\mathcal{A}$  and  $\mathcal{B}$  respectively. Then the orthogonal complements of  $A_1$  and  $B_1$  with respect to  $f_1$  and  $f_2$  respectively are given by

$$(A_1^\perp)_{f_1} = \{a \in \mathcal{A} : f_1(a, c) = 0 \forall c \in A_1\} \text{ and}$$

$$(B_1^\perp)_{f_2} = \{b \in \mathcal{B} : f_2(b, d) = 0 \forall d \in B_1\}.$$

Now, using Theorem 3.2, corresponding to  $f_1$  and  $f_2$  there exists a bilinear form

$$f : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \rightarrow \mathbb{R} \otimes_\gamma \mathbb{R} (\cong \mathbb{R}) \text{ (refer to [32])}.$$

The following result gives a relation between  $((A_1^\perp)_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^\perp)_{f_2})$  and  $((A_1 \otimes B_1)^\perp)_f$ .

**THEOREM 3.9.** *If  $f_1$  and  $f_2$  are reflexive then*

$$((A_1^\perp)_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^\perp)_{f_2}) \subseteq ((A_1 \otimes B_1)^\perp)_f.$$

*Considering  $f_1$  and  $f_2$  as symmetric bilinear forms, the above inclusion can be replaced by equality.*

*Proof.* Let  $\sum_{i=1}^n a_i \otimes b_i \in ((A_1^\perp)_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^\perp)_{f_2})$ .

Then,  $a_i \otimes b_i \in (A_1^\perp)_{f_1} \otimes \mathcal{B}$  or  $a_i \otimes b_i \in \mathcal{A} \otimes (B_1^\perp)_{f_2}$ ,  $\forall i = 1, 2, \dots, n$ ,

*i.e.*,  $a_i \in (A_1^\perp)_{f_1}$  or  $b_i \in (B_1^\perp)_{f_2}$ ,  $\forall i = 1, 2, \dots, n$ ,

*i.e.*,  $f_1(a_i, c) = 0$  or  $f_2(b_i, d) = 0 \forall c \in A_1, \forall d \in B_1, \forall i = 1, 2, \dots, n$ .

Let  $\sum_{j=1}^m c_j \otimes d_j \in A_1 \otimes B_1$  be arbitrary. Then  $c_j \in A_1$  and  $d_j \in B_1$ ,  $j = 1, \dots, m$

So,  $f_1(a_i, c_j) = 0$  or  $f_2(b_i, d_j) = 0$ , for all  $i = 1, 2, \dots, n; j = 1, 2, \dots, m$ .

If  $f_1$  and  $f_2$  are reflexive, this implies

$$f_1(c_j, a_i) = 0 \text{ or } f_2(d_j, b_i) = 0, \text{ for all } i = 1, 2, \dots, n; j = 1, 2, \dots, m.$$

Hence,

$$f_1(a_i, c_j) \otimes f_2(b_i, d_j) + f_1(c_j, a_i) \otimes f_2(d_j, b_i) = 0, \forall i = 1, 2, \dots, n; j = 1, 2, \dots, m,$$

$$\text{i.e., } \sum_{i=1}^n \sum_{j=1}^m (f_1(a_i, c_j) \otimes f_2(b_i, d_j) + f_1(c_j, a_i) \otimes f_2(d_j, b_i)) = 0.$$

Using Theorem 3.2, by definition of  $f$ , this means that

$$f\left(\sum_{i=1}^n a_i \otimes b_i, \sum_{j=1}^m c_j \otimes d_j\right) = 0, \text{ i.e., } \sum_{i=1}^n a_i \otimes b_i \in ((A_1 \otimes B_1)^\perp)_f,$$

showing that  $((A_1^\perp)_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^\perp)_{f_2}) \subseteq ((A_1 \otimes B_1)^\perp)_f$ .

Conversely, let  $\sum_{i=1}^n a_i \otimes b_i \in ((A_1 \otimes B_1)^\perp)_f$ . Then,

$$f\left(\sum_{i=1}^n a_i \otimes b_i, \sum_{j=1}^m c_j \otimes d_j\right) = 0, \quad \forall \sum_{j=1}^m c_j \otimes d_j \in A_1 \otimes B_1,$$

$$i.e., \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m [f_1(a_i, c_j) \otimes f_2(b_i, d_j) + f_1(c_j, a_i) \otimes f_2(d_j, b_i)] = 0, \quad \forall c_j \in A_1, d_j \in B_1$$

$$i.e., \sum_{i=1}^n \sum_{j=1}^m f_1(a_i, c_j) \otimes f_2(b_i, d_j) = 0, \quad \forall c_j \in A_1, d_j \in B_1, \quad \text{since } f_1 \text{ and } f_2 \text{ are symmetric.}$$

By the definition of tensor product in  $(\mathbb{R}^+ \cup \{0\}) \otimes (\mathbb{R}^+ \cup \{0\}) \cong \mathbb{R}^+ \cup \{0\}$ , (refer to [32]), it follows that

$$\sum_{i=1}^n \sum_{j=1}^m f_1(a_i, c_j) f_2(b_i, d_j) = 0, \quad \forall i = 1, 2, \dots, n; \forall j = 1, 2, \dots, m.$$

$$i.e., f_1(a_i, c_j) f_2(b_i, d_j) = 0, \quad \forall i = 1, 2, \dots, n; j = 1, 2, \dots, m.$$

Thus,  $a_i \in (A_1^\perp)_{f_1}$  or  $b_i \in (B_1^\perp)_{f_2} \forall i = 1, 2, \dots, n; j = 1, 2, \dots, m$  and so,

$$\sum_{i=1}^n a_i \otimes b_i \in ((A_1^\perp)_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^\perp)_{f_2}).$$

This shows that  $((A_1^\perp)_{f_1} \otimes \mathcal{B}) \cup (\mathcal{A} \otimes (B_1^\perp)_{f_2}) = ((A_1 \otimes B_1)^\perp)_f$ . □

EXAMPLE 3.10. For the  $C^*$ -algebra  $\mathbb{R}$  and  $\mathbb{R}^2$ , we define the bilinear forms:

$$f_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \text{ by } f(x, y) = xy, \quad x, y \in \mathbb{R} \text{ and}$$

$$f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \text{ by } f_2((a_1, a_2), (b_1, b_2)) = a_1 b_1 + a_2 b_2, \quad ((a_1, a_2), (b_1, b_2)) \in \mathbb{R}^2.$$

Clearly,  $f_1$  and  $f_2$  are symmetric and satisfy the property (1) of Theorem 3.1.

So, by Theorem 3.2, there exists a bilinear form

$$f : \mathbb{R} \otimes \mathbb{R}^2 \times \mathbb{R} \otimes \mathbb{R}^2 \rightarrow \mathbb{R}$$

such that for  $\alpha_i, \beta_j \in \mathbb{R}$ ,  $(a_i, b_i), (x_i, y_i) \in \mathbb{R}^2$ ;  $i = 1, 2, \dots, n$ ;  $j = 1, 2, \dots, m$ ,

$$\begin{aligned} & f\left(\sum_{i=1}^n \alpha_i \otimes (a_i, b_i), \sum_{j=1}^m \beta_j \otimes (x_j, y_j)\right) \\ &= \sum_{i=1}^n \sum_{j=1}^m f_1(\alpha_i, \beta_j) \otimes f_2((a_i, b_i), (x_j, y_j)) \\ &= \sum_{i=1}^n \sum_{j=1}^m (\alpha_i \beta_j \otimes (a_i x_j + b_i y_j)) \\ &= \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j (a_i x_j + b_i y_j) \end{aligned}$$

We take  $A_1 = \langle \frac{1}{2} \rangle$  and  $B_1 = \langle (1, 1) \rangle$  which are subspaces of  $\mathbb{R}$  and  $\mathbb{R}^2$  respectively.  $(A_1^\perp)_{f_1} = \{0\}$  and  $(B_1^\perp)_{f_2} = \langle (-1, 1) \rangle$  are their corresponding orthogonal complements.

Then,  $(A_1^\perp)_{f_1} \otimes \mathbb{R}^2 = \{0\}$  and  $((A_1^\perp)_{f_1} \otimes \mathbb{R}^2) \cup (\mathbb{R} \otimes (B_1^\perp)_{f_2}) = \langle (-1, 1) \rangle$ . Also,  $((A_1 \otimes B_1)^\perp)_f = \langle (-1, 1) \rangle$ .

Thus Theorem 3.9 holds for  $A_1$  and  $B_1$  and the bilinear forms  $f_1, f_2$ .

#### 4. Concluding Remarks

In this paper, we have extended the works of Ghahramani [14] to the algebraic tensor product of two  $C^*$ -algebras and derived different properties of bilinear maps and centralizers on such algebras. Using bilinear maps on the individual  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , we have obtained a centralizer on the algebraic tensor product  $\mathcal{A} \otimes \mathcal{B}$ , and conversely. In the last part, we apply the result to derive a relationship between orthogonal complements of subspaces of the  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  and then algebraic tensor product  $\mathcal{A} \otimes \mathcal{B}$  and a suitable example is provided.

Now a days, there is a wide application of bilinear maps in different areas of cryptography like encryption, signature and key agreement. (refer to [27], [13] etc.). The practical application of the results obtained in the paper in this direction is a scope for future study.

#### Competing interests

The authors declare that they have no competing interests.



**Authors' contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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