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NONLINEAR ξ-LIE-$*$-DERIVATIONS ON VON NEUMANN ALGEBRAS

Aili Yang

Abstract. Let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ and $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ be a von Neumann algebra without central abelian projections. Let ξ be a non-zero scalar. In this paper, it is proved that a mapping $\varphi : \mathscr{M} \to \mathscr{B}(\mathscr{H})$ satisfies $\varphi([A,B]_*^\xi) = [\varphi(A), B]_*^\xi + [A, \varphi(B)]_*^\xi$ for all $A, B \in \mathscr{M}$ if and only if φ is an additive $*$-derivation and $\varphi(\xi A) = \xi\varphi(A)$ for all $A \in \mathscr{M}$.

1. Introduction

Let $\mathscr{A}$ be an associative $*$-algebra over the complex field $\mathbb{C}$ and ξ be a non-zero scalar. For $A, B \in \mathscr{A}$, define the ξ-Lie-$*$ product of A and B as $[A,B]_*^\xi = AB - \xi BA^*$. A mapping φ between $*$-algebras A and B is said to preserve the ξ-Lie-$*$ product if $\varphi([A,B]_*^\xi) = [\varphi(A), B]_*^\xi + [A, \varphi(B)]_*^\xi$ for all $A, B \in \mathscr{M}$. A map: $\mathscr{A} \to \mathscr{A}$ is said to be an additive $*$-derivation if it is an additive derivation and satisfies $\delta(A^*) = \delta(A)^*$ for all $A \in \mathscr{A}$. Let $\phi : \mathscr{A} \to \mathscr{A}$ be a map (without the additivity assumption). We say that ϕ is a nonlinear $*$-Lie derivation if $\phi([A,B]_*) = [\phi(A), B]_* + [A, \phi(B)]_*$ for all $A, B \in \mathscr{A}$, where $[A,B]_* = AB - BA^*$.

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The structure of linear Lie derivations on C^*-algebras has attracted some attention over past years. Johnson [1] proved that every continuous linear Lie derivation from a C^*-algebra A into a Banach $\mathscr{A}$-bimodule $\mathscr{E}$ can be decomposed as $\delta + h$, Where $\delta : \mathscr{A} \to \mathscr{E}$ is a derivation and h is a linear mapping from $\mathscr{A}$ into the center of $\mathscr{E}$. Mathieu and Villena [2] proved that every linear Lie derivation on a C^*-algebra can be decomposed into the sum of a derivation and a center-valued trace. In [3], Zhang proved the same result for nest subalgebras of factor von Neumann algebras. Cheung gave in [4] a characterization of linear Lie derivations on triangular algebras. Qi and Hou [5] discussed additive ξ-Lie derivations on nest algebras. The most interesting result on additive Lie derivations of prime rings was obtained in [6]. However, the structure of nonlinear Lie derivations or nonlinear $*$-Lie derivations on operator algebras is not clear, it needs to be discussed further. In [7], Cheng and Zhang investigated nonlinear Lie derivations on upper triangular matrix algebras. Yu and Zhang [8] proved that every nonlinear Lie derivations of triangular algebras is the sum of an additive derivation and a map into its centers ending commutators to zero. Motivated by these study, we consider nonlinear $*$-Lie derivations on von Neumann algebras.

As usual, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field. Let $\mathscr{H}$ be a complex Hilbert space. We denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. Recall that $\mathscr{M}$ is a factor if its center is $\mathbb{C}I$ where I is the identity of $\mathscr{M}$.

2. Main result and the proof

In this section, our main result is the following theorem.

Main Theorem. Let $\mathscr{M}$ be a von Neumann algebra without central abelian projections, and ξ be a non-zero scalar. Then, a mapping $\varphi : \mathscr{M} \to \mathscr{B}(\mathscr{H})$ satisfies $\varphi([A,B]_*^\xi) = [\varphi(A), B]_*^\xi + [A, \varphi(B)]_*^\xi$ for all $A, B \in \mathscr{M}$ if and only if φ is an additive $*$-derivation.

Before proving the theorem, we need some notations and preliminaries about von Neumann algebras. A von Neumann algebra $\mathscr{M}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathscr{H}$ containing the identity I. The set $\mathscr{Z}_{\mathscr{M}} = \{Z \in \mathscr{M} : ZM = MZ, \forall M \in \mathscr{M}\}$ is called the centre of $\mathscr{M}$. A projection P is called the central abelian projection if $P \in \mathscr{Z}_{\mathscr{M}}$ and $P\mathscr{M}P$ is abelian. Recall that the central carrier of M, denoted by $\overline{M}$, is the smallest central projection P satisfying $PM = M$. It is not difficult that the central carrier of M is the projection onto the closed subspace span by $\{NM(h) : h \in \mathscr{H}\}$. If M is self-adjoint, then the core Q satisfying $Q \leq P$. A projection P is said to be core-free if $\underline{P} = 0$. It is clear that $\underline{P} = 0$ if and only if $\overline{I - P} = I$.

LEMMA 2.1([9, Lemma 4]) Let $\mathscr{M}$ be a von Neumann algebra without central abelian projections, and ξ be a non-zero scalar. Then each non-zero cental projection in $\mathscr{M}$ is the central carrier of a core-free projection in $\mathscr{M}$.

LEMMA 2.2 Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$. Let $A \in \mathscr{B}(\mathscr{H})$ and $P \in \mathscr{M}$ is a projection with $\overline{P} = I$.

(a) If $ABP = 0$ for all $B \in \mathscr{M}$, then $A = 0$;

(b) If $[PT(I-P), A]_*^\xi = 0$ for all $T \in \mathscr{M}$, then $A(I-P) = 0$.

Proof. (a) It follows from $\overline{P} = I$ that the linear span of $\{BP(x) : x \in \mathscr{H}\}$ is dense in $\mathscr{H}$. So $ABP = 0$ for all $B \in \mathscr{M}$ implies $A = 0$.

(b) Since $[PT(I-P), A]_*^\xi = PT(I-P)A - \xi A(I-P)T^*P = 0$, by replacing iT by T, we get $PT(I-P)A + \xi A(I-P)T^*P = 0$ and hence $A(I-P)T^*P = 0$ for all $A \in \mathscr{M}$. By (a), $A(I-P) = 0$. □

By Lemma 2.1, there exists a projection P such that $\underline{P} = 0$ and $\overline{P} = I$. Throughout the paper, $P_1 = P$ is fixed, and let $P_2 = I - P$. Set $\mathscr{M}_{ij} = P_i \mathscr{M} P_j$. Then $\mathscr{M} = \sum_{i,j}^2 \mathscr{M}_{ij}$.

LEMMA 2.3 Let $\mathscr{M}$ be a von Neumann algebra without central abelian projections, and ξ be a non-zero scalar. Then, a mapping $\varphi : \mathscr{M} \to \mathscr{B}(\mathscr{H})$ satisfies $\varphi([A,B]_*^\xi) = [\varphi(A), B]_*^\xi + [A, \varphi(B)]_*^\xi$ for all $A, B \in \mathscr{M}$, then φ is additive.

Proof. We shall organize the proof in a series of claims.

Claim 1 $\varphi(0) = 0$.

Indeed, $\varphi(0) = \varphi([0,0]_*^\xi) = [\varphi(0), 0]_*^\xi + [0, \varphi(0)]_*^\xi = 0$.

Claim 2 For $i, j, k \in \{1, 2\}, i \neq j, A_{kk} \in \mathscr{M}_{kk}, B_{ij} \in \mathscr{M}_{ij}$, we have

$$\varphi(A_{kk} + B_{ij}) = \varphi(A_{kk}) + \varphi(B_{ij}).$$

We only prove the case $i = k = 1, j = 2$, the proof of the other cases is similar. Let $T = T_{11} + T_{12} + T_{21} + T_{22} = \varphi(A_{kk} + B_{ij}) - \varphi(A_{kk}) - \varphi(B_{ij})$. We only need to prove $T = 0$.

For any $\alpha \in \mathbb{C}$, since $[\alpha P_2, A_{11}]_*^\xi = 0$ and $[\alpha P_2, A_{11}+B_{12}]_*^\xi = [\alpha P_2, B_{12}]_*^\xi$, it follows from Claim 1 that

$$\begin{aligned}
&[\varphi(\alpha P_2), A_{11} + B_{12}]_*^\xi + [\alpha P_2, \varphi(A_{11} + B_{12})]_*^\xi \\
&= \varphi([\alpha P_2, A_{11} + B_{12}]_*^\xi) \\
&= \varphi([\alpha P_2, B_{12}]_*^\xi) \\
&= \varphi([\alpha P_2, A_{11}]_*^\xi) + \varphi([\alpha P_2, B_{12}]_*^\xi) \\
&= [\varphi(\alpha P_2), A_{11}]_*^\xi + [\alpha P_2, \varphi(A_{11})]_*^\xi + [\varphi(\alpha P_2), B_{12}]_*^\xi + [\alpha P_2, \varphi(B_{12})]_*^\xi \\
&= [\varphi(\alpha P_2), A_{11} + B_{12}]_*^\xi + [\alpha P_2, \varphi(A_{11}) + \varphi(B_{12})]_*^\xi.
\end{aligned}$$

Hence $[\alpha P_2, \varphi(A_{11}+B_{12}) - \varphi(A_{11}) - \varphi(B_{12})]_*^\xi = 0$, that is, $[\alpha P_2, T]_*^\xi = 0$, so $\alpha P_2 T - \overline{\alpha}\xi T P_2 = 0$ for any $\alpha \in \mathbb{C}$. Let $\alpha - \overline{\alpha}\xi \neq 0$, we have $T_{12} = T_{21} = T_{22} = 0$.

Similarly, since $[\alpha\xi P_1 + \overline{\alpha} P_2, B_{12}]_*^\xi = 0$ and $[\alpha\xi P_1 + \overline{\alpha} P_2, A_{11} + B_{12}]_*^\xi = [\alpha\xi P_1 + \overline{\alpha} P_2, A_{11}]_*^\xi$, it follows that

$$\begin{aligned}
&[\varphi(\alpha\xi P_1 + \overline{\alpha} P_2), A_{11} + B_{12}]_*^\xi + [\alpha\xi P_1 + \overline{\alpha} P_2, \varphi(A_{11} + B_{12})]_*^\xi \\
&= \varphi([\alpha\xi P_1 + \overline{\alpha} P_2, A_{11} + B_{12}]_*^\xi) \\
&= \varphi([\alpha\xi P_1 + \overline{\alpha} P_2, A_{11}]_*^\xi) \\
&= \varphi([\alpha\xi P_1 + \overline{\alpha} P_2, A_{11}]_*^\xi) + \varphi([\alpha\xi P_1 + \overline{\alpha} P_2, B_{12}]_*^\xi) \\
&= [\varphi(\alpha\xi P_1 + \overline{\alpha} P_2), A_{11}]_*^\xi + [\alpha\xi P_1 + \overline{\alpha} P_2, \varphi(A_{11})]_*^\xi \\
&+ [\varphi(\alpha\xi P_1 + \overline{\alpha} P_2), B_{12}]_*^\xi + [\alpha\xi P_1 + \overline{\alpha} P_2, \varphi(B_{12})]_*^\xi \\
&= [\varphi(\alpha\xi P_1 + \overline{\alpha} P_2), A_{11} + B_{12}]_*^\xi + [\alpha\xi P_1 + \overline{\alpha} P_2, \varphi(A_{11}) + \varphi(B_{12})]_*^\xi.
\end{aligned}$$

Hence $[\alpha\xi P_1 + \overline{\alpha} P_2, \varphi(A_{11}+B_{12}) - \varphi(A_{11}) - \varphi(B_{12})]_*^\xi = 0$, that is, $[\alpha\xi P_1 + \overline{\alpha} P_2, T]_*^\xi = 0$, from which and the result $T_{12} = T_{21} = T_{22} = 0$ we have $(\alpha - \overline{\alpha\xi})T_{11} = 0$ for any $\alpha \in \mathbb{C}$, so $T_{11} = 0$, hence $\varphi(A_{11} + B_{12}) = \varphi(A_{11}) + \varphi(B_{12})$.

Claim 3 For $A_{11} \in \mathscr{M}_{11}, B_{22} \in \mathscr{M}_{22}$, we have

$$\varphi(A_{11} + B_{22}) = \varphi(A_{11}) + \varphi(B_{22}).$$

We let $T = T_{11} + T_{12} + T_{21} + T_{22} = \varphi(A_{11} + B_{22}) - \varphi(A_{11}) - \varphi(B_{22})$, then, we only need to prove that $T = 0$.

For any $\alpha \in \mathbb{C}$, since $[\alpha P_1, B_{22}]_*^\xi = 0$ and $[\alpha P_1, A_{11}+B_{22}]_*^\xi = [\alpha P_1, A_{11}]_*^\xi$, it follows that

$$\begin{aligned}
&[\varphi(\alpha P_1), A_{11} + B_{22}]_*^\xi + [\alpha P_1, \varphi(A_{11} + B_{22})]_*^\xi \\
&= \varphi([\alpha P_1, A_{11} + B_{22}]_*^\xi) \\
&= \varphi([\alpha P_1, B_{22}]_*^\xi) \\
&= \varphi([\alpha P_1, A_{11}]_*^\xi) + \varphi([\alpha P_1, B_{22}]_*^\xi) \\
&= [\varphi(\alpha P_1), A_{11}]_*^\xi + [\alpha P_1, \varphi(A_{11})]_*^\xi + [\varphi(\alpha P_1), B_{22}]_*^\xi + [\alpha P_1, \varphi(B_{22})]_*^\xi \\
&= [\varphi(\alpha P_1), A_{11} + B_{22}]_*^\xi + [\alpha P_1, \varphi(A_{11}) + \varphi(B_{22})]_*^\xi.
\end{aligned}$$

Consequently, $[\alpha P_1, \varphi(A_{11}+B_{22})-\varphi(A_{11})-\varphi(B_{22})]_*^\xi = 0$, that is, $[\alpha P_1, T]_*^\xi = 0$, so $\alpha P_1 T - \overline{\alpha}\xi T P_1 = 0$ for any $\alpha \in \mathbb{C}$. Let $\alpha - \overline{\alpha}\xi \neq 0$, we have $T_{11} = T_{12} = T_{21} = 0$. Similarly, we have $T_{22} = 0$. Hence $T = 0$, that is, $\varphi(A_{11}+B_{22}) = \varphi(A_{11}) + \varphi(B_{22})$.

Claim 4 For $A_{12} \in \mathscr{M}_{12}, B_{21} \in \mathscr{M}_{21}$, we have

$$\varphi(A_{12}+B_{21}) = \varphi(A_{12}) + \varphi(B_{21}).$$

We let $T = T_{11}+T_{12}+T_{21}+T_{22} = \varphi(A_{12}+B_{21}) - \varphi(A_{12}) - \varphi(B_{21})$, then we only need to prove that $T = 0$. Since $[\alpha\xi P_1 + \overline{\alpha}P_2, A_{12}]_*^\xi = 0$ and $[\alpha\xi P_1 + \overline{\alpha}P_2, A_{12}+B_{21}]_*^\xi = [\alpha\xi P_1 + \overline{\alpha}P_2, B_{21}]_*^\xi$, it follows that

$$\begin{aligned}
&[\varphi(\alpha\xi P_1 + \overline{\alpha}P_2), A_{12}+B_{21}]_*^\xi + [\alpha\xi P_1 + \overline{\alpha}P_2, \varphi(A_{12}+B_{21})]_*^\xi \\
&= \varphi([\alpha\xi P_1 + \overline{\alpha}P_2, A_{12}+B_{21}]_*^\xi) \\
&= \varphi([\alpha\xi P_1 + \overline{\alpha}P_2, B_{21}]_*^\xi) \\
&= \varphi([\alpha\xi P_1 + \overline{\alpha}P_2, A_{12}]_*^\xi) + \varphi([\alpha\xi P_1 + \overline{\alpha}P_2, B_{21}]_*^\xi) \\
&= [\varphi(\alpha\xi P_1 + \overline{\alpha}P_2), A_{12}]_*^\xi + [\alpha\xi P_1 + \overline{\alpha}P_2, \varphi(A_{12})]_*^\xi \\
&+[\varphi(\alpha\xi P_1 + \overline{\alpha}P_2), B_{21}]_*^\xi + [\alpha\xi P_1 + \overline{\alpha}P_2, \varphi(B_{21})]_*^\xi \\
&= [\varphi(\alpha\xi P_1 + \overline{\alpha}P_2), A_{12}+B_{21}]_*^\xi + [\alpha\xi P_1 + \overline{\alpha}P_2, \varphi(A_{12})+\varphi(B_{21})]_*^\xi.
\end{aligned}$$

Therefore, $[\alpha\xi P_1 + \overline{\alpha}P_2, \varphi(A_{12}+B_{21}) - \varphi(A_{12}) - \varphi(B_{21})]_*^\xi = 0$, that is, $[\alpha\xi P_1 + \overline{\alpha}P_2, T]_*^\xi = 0$, from which we get $T_{11} = T_{22} = 0$.

And since $[A_{12}, P_1]_*^\xi = 0$, it follows that $\varphi([A_{12}+B_{21}, P_1]_*^\xi) = \varphi([A_{12}, P_1]_*^\xi) + \varphi([B_{21}, P_1]_*^\xi)$. Hence $[T, P_1]_*^\xi$, from which we get $T_{21} = 0$. Similarly, $T_{12} = 0$. Therefore, $\varphi(A_{12}+B_{21}) = \varphi(A_{12}) + \varphi(B_{21})$.

Claim 5 For $A_{11} \in \mathscr{M}_{11}, B_{12} \in \mathscr{M}_{12}, C_{21} \in \mathscr{M}_{21}, D_{22} \in \mathscr{M}_{22}$, we have

$$\varphi(A_{11}+A_{12}+C_{21}) = \varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21})$$

and

$$\varphi(D_{22}+A_{12}+C_{21}) = \varphi(D_{22}) + \varphi(B_{12}) + \varphi(C_{21}).$$

We only need to prove that $T = \varphi(A_{11}+A_{12}+C_{21}) - \varphi(A_{11}) - \varphi(B_{12}) - \varphi(C_{21}) = 0$. Similarly, we can prove $\varphi(D_{22}+A_{12}+C_{21}) = \varphi(D_{22}) + \varphi(B_{12}) + \varphi(C_{21})$. For any $\alpha \in \mathbb{C}$, since $[\alpha P_2, A_{11}]_*^\xi = 0$ and $[\alpha P_2, A_{11} + B_{12}]_*^\xi = [\alpha P_2, B_{12}]_*^\xi$, it follows from Claim 4 that

$$\begin{aligned}
&[\varphi(\alpha P_2), A_{11}+B_{12}+C_{21}]_*^\xi + [\alpha P_2, \varphi(A_{11}+B_{12}+C_{21})]_*^\xi \\
&= \varphi([\alpha P_2, A_{11}+B_{12}+C_{21}]_*^\xi) \\
&= \varphi([\alpha P_2, B_{12}]_*^\xi) \\
&= \varphi([\alpha P_2, A_{11}]_*^\xi) + \varphi([\alpha P_2, B_{12}+C_{21}]_*^\xi) \\
&= [\varphi(\alpha P_2), A_{11}]_*^\xi + [\alpha P_2, \varphi(A_{11})]_*^\xi \\
&\quad +[\varphi(\alpha P_2), B_{12}+C_{21}]_*^\xi + [\alpha P_2, \varphi(B_{12}+C_{21})]_*^\xi \\
&= [\varphi(\alpha P_2), A_{11}+B_{12}+C_{21}]_*^\xi + [\alpha P_2, \varphi(A_{11}) + \varphi(B_{12}+\varphi(C_{21}))]_*^\xi.
\end{aligned}$$

Hence $[\alpha P_2, T]_*^\xi = 0$ for any $\alpha \in \mathbb{C}$, from which we get $T_{12} = T_{21} = T_{22} = 0$.

Since $[\overline{\alpha} P_1 + \alpha\xi P_2, C_{21}]_*^\xi = 0$, it follows from Claim 2 that

$$\begin{aligned}
&[\varphi(\overline{\alpha} P_1 + \alpha\xi P_2), A_{11} + B_{12} + C_{21}]_*^\xi + [\alpha P_2, \varphi(A_{11} + B_{12} + C_{21})]_*^\xi \\
&= \varphi([\overline{\alpha} P_1 + \alpha\xi P_2, A_{11} + B_{12} + C_{21}]_*^\xi) \\
&= \varphi([\overline{\alpha} P_1 + \alpha\xi P_2, A_{11} + B_{12}]_*^\xi) + \varphi([\overline{\alpha} P_1 + \alpha\xi P_2, C_{21}]_*^\xi) \\
&= [\varphi(\overline{\alpha} P_1 + \alpha\xi P_2), A_{11} + B_{12} + C_{21}]_*^\xi + [\overline{\alpha} P_1 \\
&\qquad + \alpha\xi P_2, \varphi(A_{11}) + \varphi(B_{12} + \varphi(C_{21}))]_*^\xi.
\end{aligned}$$

Hence $[\overline{\alpha} P_1 + \alpha\xi P_2, T]_*^\xi = 0$ for any $\alpha \in \mathbb{C}$, from which we get $T_{11} = 0$. So $T = 0$. Therefore, $\varphi(A_{11} + A_{12} + C_{21}) = \varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21})$. Similarly, we have $\varphi(D_{22} + A_{12} + C_{21}) = \varphi(D_{22}) + \varphi(B_{12}) + \varphi(C_{21})$.

Claim 6 For $A_{ij}, B_{ij} \in \mathscr{M}_{ij}, 1 \leq i \neq j \leq 2$, we have

$$\varphi(A_{ij} + B_{ij}) = \varphi(A_{ij}) + \varphi(B_{ij}).$$

Compute $[P_i + A_{ij}, P_j + B_{ij}]_*^\xi = A_{ij} + B_{ij} - \xi A_{ij}^* - \xi B_{ij} A_{ij}^*$. It follows from Claim 5 and Claim 2 that

$$\begin{aligned}
&\varphi(A_{ij} + B_{ij}) - \varphi(\xi A_{ij}^*) - \varphi(\xi B_{ij} A_{ij}^*) \\
&= \varphi([P_i + A_{ij}, P_j + B_{ij}]_*^\xi) \\
&= [\varphi(P_i + A_{ij}), P_j + B_{ij}]_*^\xi + [P_i + A_{ij}, \varphi(P_j + B_{ij})]_*^\xi \\
&= [\delta(P_i) + \varphi(A_{ij}), P_j + B_{ij}]_*^\xi + [P_i + A_{ij}, \varphi(P_j) + \varphi(B_{ij})]_*^\xi \\
&= \varphi(A_{ij}) + \varphi(B_{ij}) - \varphi(\xi A_{ij}^*) - \varphi(\xi B_{ij} A_{ij}^*).
\end{aligned}$$

Consequently, $\varphi(A_{ij} + B_{ij}) = \varphi(A_{ij}) + \varphi(B_{ij})$.

Claim 7 For $A_{ii}, B_{ii} \in \mathscr{M}_{ii}, i = 1, 2$, we have

$$\varphi(A_{ii} + B_{ii}) = \varphi(A_{ii}) + \varphi(B_{ii}).$$

Let $T = \varphi(A_{ii} + B_{ii}) - \varphi(A_{ii}) - \varphi(B_{ii})$. We only need to prove $T = 0$.

For any $\alpha \in \mathbb{C}$, since $[\alpha P_j, A_{ii}]_*^\xi = [\alpha P_j, B_{ii}]_*^\xi = [\alpha P_j, A_{ii} + B_{ii}]_*^\xi = 0 (i \neq j)$, it follows that

$$\varphi([\alpha P_j, A_{ii} + B_{ii}]_*^\xi) = \varphi([\alpha P_j, A_{ii}]_*^\xi) + \varphi([\alpha P_j, B_{ii}]_*^\xi).$$

Hence, $[\alpha P_j, T]_*^\xi) = 0$, from which we get that $T_{ij} = T_{ji} = T_{jj} = 0$.

For any $C_{ij} \in \mathscr{M}_{ij} (i \neq j)$, it follows from Claim 6 that

$$\begin{aligned}
&[\varphi(A_{ii}+B_{ii}), C_{ij}]_*^\xi + [A_{ii}+B_{ii}, \varphi(C_{ij})]_*^\xi \\
&= \varphi([(A_{ii}+B_{ii}), C_{ij}]_*^\xi) \\
&= \varphi(A_{ii}C_{ij} + B_{ii}C_{ij}) \\
&= \varphi(A_{ii}C_{ij}) + \varphi(B_{ii}C_{ij}) \\
&= \varphi([A_{ii}, C_{ij}]_*^\xi) + \varphi([B_{ii}, C_{ij}]_*^\xi) \\
&= [(\varphi(A_{ii}) + \varphi(B_{ii})), C_{ij}]_*^\xi + [A_{ii}+B_{ii}, \varphi(C_{ij})].
\end{aligned}$$

Consequently, $[T_{ii}, C_{ij}]_*^\xi = 0$, that is, $T_{ii}C_{ij} = 0$ for any $C_{ij} \in \mathscr{M}_{ij}$. Note that $\overline{I-P} = I$. It follows from Lemma 2.2 (1) that $T_{ii} = 0$. So $\varphi(A_{ii}+B_{ii}) = \varphi(A_{ii}) + \varphi(B_{ii})$.

Claim 8 For $A_{11} \in \mathscr{M}_{11}, B_{12} \in \mathscr{M}_{12}, C_{21} \in \mathscr{M}_{21}, D_{22} \in \mathscr{M}_{22}$, we have

$$\varphi(A_{11}+A_{12}+C_{21}+D_{22}) = \varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21}) + \varphi(D_{22}).$$

Let $T = \varphi(A_{11}+A_{12}+C_{21}+D_{22}) - \varphi(A_{11}) - \varphi(B_{12}) - \varphi(C_{21}) - \varphi(D_{22})$. We only need to prove $T = 0$.

For any $\alpha \in \mathbb{C}$, since $[\alpha P_1, D_{22}]_*^\xi = 0$, It follows from Claim 5 that

$$\begin{aligned}
&[\varphi(\alpha P_1), A_{11}+A_{12}+C_{21}+D_{22}]_*^\xi + [\alpha P_1, \varphi(A_{11}+A_{12}+C_{21}+D_{22})]_*^\xi \\
&= \varphi([\alpha P_1, A_{11}+A_{12}+C_{21}+D_{22}]_*^\xi) \\
&= \varphi([\alpha P_1, A_{11}+A_{12}+C_{21}]_*^\xi) + \varphi([\alpha P_1, D_{22}]_*^\xi) \\
&= [\varphi(\alpha P_1), A_{11}+A_{12}+C_{21}+D_{22}]_*^\xi \\
&\quad + [\alpha P_1, \varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21}) + \varphi(D_{22})]_*^\xi
\end{aligned}$$

Hence, $[\alpha P_1, T]_*^\xi = 0$, from which we have $T_{11} = T_{12} = T_{21} = 0$. Similarly, we can get $T_{22} = 0$. Hence, $\varphi(A_{11}+A_{12}+C_{21}+D_{22}) = \varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21}) + \varphi(D_{22})$.

Claim 9 φ is additive.

It is an immediate consequence of Claims 6, 7 and 8. □

LEMMA 2.4 For any $A \in \mathscr{M}$, we have $\varphi(\xi A) = \xi\varphi(A)$ and $\varphi(A^*) = \varphi(A)^*$.

Proof. For any $A \in \mathscr{M}$, it follows from $\varphi(I) = 0$ that

$$\varphi(A) - \varphi(\xi A) = \varphi([I, A]_*^\xi) = [I, \varphi(A)]_*^\xi = \varphi(A) - \xi\varphi(A).$$

On the other hand, we have

$$\varphi(A) - \xi\varphi(A^*) = \varphi([A, I]_*^\xi) = [\varphi(A), I]_*^\xi = \varphi(A) - \xi\varphi(A)^*.$$

□

Proof of Main Theorem By Lemma 2.2, Lemma 2.3 and Lemma 2.4, we get that if $\varphi([A,B]_*^\xi) = [\varphi(A),B]_*^\xi + [A,\varphi(B)]_*^\xi$ for all $A, B \in \mathscr{M}$, then φ is an additive $*$-derivation and $\varphi(\xi A) = \xi\varphi(A)$ for all $A \in \mathscr{M}$.

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Aili Yang
College of Science
Xi'an University of Science and Technology
Xi'an 710054, P. R. China
E-mail: yangaili@xust.edu.cn