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INVESTIGATION OF SOME FIXED POINT THEOREMS IN HYPERBOLIC SPACES FOR A THREE STEP ITERATION PROCESS

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Abstract. In the present paper, we investigate the convergence, equivalence of convergence, rate of convergence and data dependence results using a three step iteration process for mappings satisfying certain contractive condition in hyperbolic spaces. Also we give nontrivial examples for the rate of convergence and data dependence results to show effciency of three step iteration process. The results obtained in this paper may be interpreted as a refinement and improvement of the previously known results.

1. Introduction and Preliminaries

The relationship between the geometric properties of a space and fixed point theory makes it possible to obtain very effective and useful results. In particular, geometric properties of a space play an important role in metric fixed point theory. Since every Banach space is a vector space, it is easier to give these spaces a convex structure. For this reason, the geometry of the Banach spaces has been worked intensively with their convex structures (see [6, 8, 9, 13, 21, 42]).

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However, it is more difficult to gain convex structure to metric spaces. This difficulty has been overcome by Takahashi's describing convex structure in metric spaces (see [40]) and after this point, many authors have studied fixed point theory in convex metric spaces and obtained very efficient results (see [10, 11, 28, 29]) and the references cited therein. Since many problems encountered in real life can be expressed in nonlinear form, it will be more realistic approach to study these problems in nonlinear structures instead of linear structures such as Banach spaces. At this point, hyperbolic space due to its nonlinear structure and rich geometrical properties is a good mathematical framework for metric fixed point theory in the study of these problems.

In 2004, Kohlenbach in [30] gave the definiton of hyperbolic space as follows:

Definition 1.1. A (H, d, W) is called a hyperbolic space if (H, d) is a metric space and $W : H \times H \times I \to H$ satisfying

(H_1) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$;
(H_2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| \, d(x, y)$;
(H_3) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$;
(H_4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$;

for all $x, y, z, w \in H$ and $\alpha, \beta \in [0, 1]$.

If a metric space (H, d) with a mapping $W : H \times H \times [0, 1] \to H$ satisfies only condition (H_1), then it is a convex metric space in the sense of Takahashi [40], if (H, d) satisfies conditions $(H_1) - (H_3)$ then it being a space of hyperbolic type in Goebel and Kirk [14]. The condition (H_4) is used in [37] to define the class of hyperbolic spaces. This class contains normed linear spaces and convex subsets therefore the open unit ball in complex Hilbert spaces. After that many authors have studied fixed point problems in hyperbolic spaces (see [2, 12, 27]) and the references cited therein.

While the existence theorems in the fixed point theory guarantee the existence of the solutions, the iterative approximation methods are significant tools for determining what is the solution. For this purpose many iteration processes have been introduced and analyzed by a great number of researches in the sense of their convergence, equivalence of convergence and rate of convergence etc. (see [1, 16, 18, 19, 24, 35, 38]).

In 2014, Gursoy [17] introduced the Picard-S iteration process in a Banach space as follows:

$$(1)\qquad \begin{cases} k_0 \in K \\ k_{n+1} = Tm_n \\ m_n = (1-\alpha_n)\, Tk_n + \alpha_n Tb_n \\ b_n = (1-\beta_n)\, k_n + \beta_n Tk_n \end{cases}$$

where K is a nonempty subset of a real Banach space X and T is a self-mapping on K and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0,1]$.

The iteration process (1) can be expressed in hyperbolic as

$$(2)\qquad \begin{cases} k_0 \in K \\ k_{n+1} = Tm_n \\ m_n = W\,(Tk_n, Tb_n, \alpha_n) \\ b_n = W\,(k_n, Tk_n, \beta_n) \end{cases}$$

Faster and simpler iteration process have been defined to reach solution by doing less processing. In accordance with this purpose, we used the following iteration process which is defined by Karakaya et. al [24]:

$$(3)\qquad \begin{cases} h_0 \in K \\ h_{n+1} = Ts_n \\ s_n = (1-\alpha_n)\, r_n + \alpha_n Tr_n \\ r_n = Th_n \end{cases}$$

The iteration process (3) can be expressed in hyperbolic space as

$$(4)\qquad \begin{cases} h_0 \in K \\ h_{n+1} = Ts_n \\ s_n = W\,(r_n, Tr_n, \alpha_n) \\ r_n = Th_n \end{cases}$$

Definition 1.2. [22] Let T be a self operator on a metric space X. The operator T is called a contractive-like operator if there exists a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$, with $\varphi(0) = 0$, such that for each $x, y \in X$,

$$(5)\qquad d(Tx,Ty) \le \delta d(x,y) + \varphi\,[d(x,Tx)]\,.$$

This definiton is more general than the definitons of by Berinde [3], [4], Harder and Hicks [20], Zamfirescu [43], Osilike and Udomene [34]. Several mathematicians have established some fixed points results for this class of mappings under the assumption that this mapping has a unique fixed point (see [7, 23, 26, 31–33]). However, as we show in the following example, this mapping need not has a fixed point even if X is a complete:

EXAMPLE 1.3. Let $X = [0,1]$ be endowed with the usual metric. Define an operator $T : [0,1] \to [0,1]$ by

$$Tx = e^{-x} + \frac{1}{2}\sin x^2 + 0.75.$$

We have to show the operator T satisfies the condition (5). Define the function $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \frac{t}{40}$. Then φ is increasing, continuous and $\varphi(0) = 0$.

We have the following equalities:

$$|Tx - Ty| = \left| e^{-x} - e^{-y} + \frac{1}{2}\left(\sin x^2 - \sin y^2\right)\right|$$

and

$$\varphi\left[|x - Tx|\right] = 0.025\left| x - e^{-x} - \frac{1}{2}\left(\sin x^2\right) - 0.75\right|$$

If $\delta = 0.7$, then we have

$$\begin{aligned}\left| e^{-x} - e^{-y} + \frac{1}{2}\left(\sin x^2 - \sin y^2\right)\right| &\leq 0.7\,|x - y| \\ &\quad +0.025\left| x - e^{-x} - \frac{1}{2}\left(\sin x^2\right) - 0.75\right|\end{aligned}$$

for all $x, y \in [0,1]$. That is T satisfies the condition (5). But the operator T has no fixed point in $[0,1]$ as shown in the following figure:

Bosede and Rhoades [5] observed that if a contractive-like operator T has a fixed point then it satisfies the following contractive condition:

$$d(x_*, Tx) \leq \delta d(x_*, x) \tag{6}$$

for some $0 \leq \delta < 1$ and for each $x \in X$.

In our opinion it is better to work with the contractive condition defined by (6) than with (5), because if we suppose that T has a fixed point, then (5) implies (6) and using (6), we avoid doing unnecessary calculations.

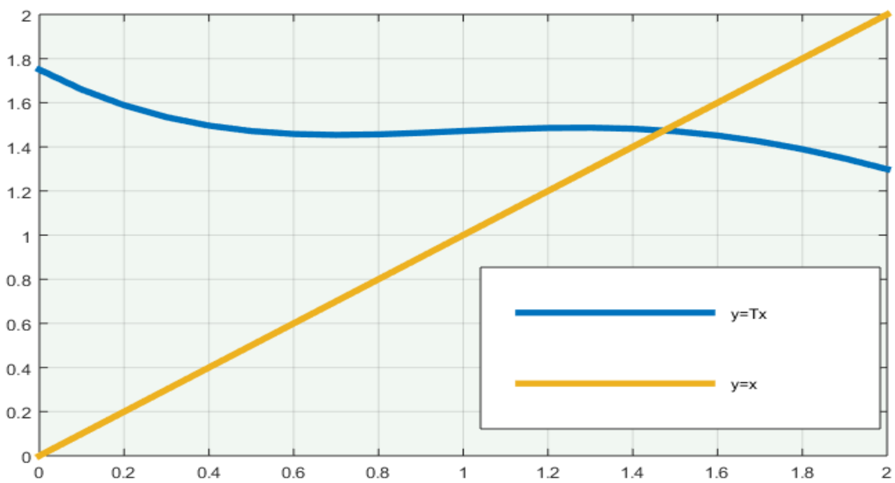

FIGURE 1. Graph of T and $y = x$

Also, from (6), we obtain

$$\begin{aligned} d(Tx, Ty) &\leq d(Tx, x_*) + d(x_*, Ty) \\ &\leq \delta d(x_*, x) + \delta d(x_*, y) \\ &\leq \delta d(x, y) + 2\delta d(x_*, y) \end{aligned} \tag{7}$$

In this work, we prove that the iteration process (4) converges to fixed point x_* of a mapping, which satisfies (6), under suitable control conditions. Further, we show that there is an equivalency between iteration processes (4) and (2) in the sense of their convergence. Moreover, we prove that the iteration process (4) has a better convergence speed when compared the iteration process (2). Also, we show that a data dependence result can be obtained for the mappings which satisfy (6) by using the iteration process (4). Finally, we give numerical examples to support rate of convergence and data dependence results.

DEFINITION 1.4 ([36]). Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences converging to the same point x_*. We say that $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ to x_*, if

$$\lim_{n\to\infty} \frac{d(a_n, x_*)}{d(b_n, x_*)} = 0.$$

Lemma 1.5. [41] *Let* $\{c_n\}_{n=0}^{\infty}$ *and* $\{d_n\}_{n=0}^{\infty}$ *be nonnegative real sequences satisfying the following inequality:*

$$c_{n+1} \leq \rho c_n + d_n$$

where $\rho \in [0,1)$ *and* $\lim_{n\to\infty} d_n = 0$, *then* $\lim_{n\to\infty} c_n = 0$.

Lemma 1.6. [41] *Let* $\{c_n\}_{n=0}^{\infty}$ *and* $\{d_n\}_{n=0}^{\infty}$ *be nonnegative real sequences satisfying the following inequality:*

$$c_{n+1} \leq (1-\xi_n)\, c_n + d_n,$$

where $\xi_n \in (0,1)$ *for all* $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \xi_n = \infty$ *and* $\frac{d_n}{\xi_n} \to 0$ *as* $n \to \infty$, *then* $\lim_{n\to\infty} c_n = 0$.

Lemma 1.7. [39] *Let* $\{c_n\}_{n=0}^{\infty}$ *be a nonnegative real sequence and there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *satisfying the following inequality*

$$c_{n+1} \leq (1-\xi_n)\, c_n + \xi_n \mu_n.$$

where $\xi_n \in (0,1)$ *such that* $\sum\limits_{n=0}^{\infty} \xi_n = \infty$ *and* $\{\mu_n\}_{n=0}^{\infty} \geq 0$. *Then the following inequality holds:*

$$0 \leq \limsup_{n\to\infty} c_n \leq \limsup_{n\to\infty} \mu_n.$$

Definition 1.8. [39] L*et* $T, S : C \to C$ *be two operators. We say that* S *is an approximate operator of* T *for all* $x \in C$ *and a fixed* $\varepsilon > 0$ *if* $d(Tx, Sx) \leq \varepsilon$.

2. Main Results

Theorem 2.1. *Let* C *be a nonempty, closed and convex subset of a hyperbolic metric space* H *and let* $\{h_n\}_{n=0}^{\infty}$ *be the iteration process (4) with a real sequence* $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$ *satisfying* $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. *If* $T : C \to C$ *is a quasi-contractive operator satisfying (6) then* $\{h_n\}_{n=0}^{\infty}$ *converges to* x_*.

Proof. By (H_1), (4) and (6) we have

$$d(h_{n+1}, x_*) = d(Ts_n, x_*) \leq \delta d(s_n, x_*), \tag{8}$$

and

$$\begin{aligned} d(s_n, x_*) &= d(W(r_n, Tr_n, \alpha_n), x_*) \\ &\leq (1-\alpha_n) d(r_n, x_*) + \alpha_n d(Tr_n, x_*) \\ &\leq (1-\alpha_n) d(r_n, x_*) + \alpha_n \delta d(r_n, x_*) \\ &= [1-\alpha_n(1-\delta)] d(r_n, x_*). \end{aligned} \tag{9}$$

and

$$d(r_n, x_*) = d(Th_n, x_*) \leq \delta d(h_n, x_*). \tag{10}$$

Substituting (10) in (9) and (9) in (8) respectively, we obtain

$$d(h_{n+1}, x_*) \leq \delta^2 [1-\alpha_n(1-\delta)] d(h_n, x_*).$$

By repeating this process n times, we get

$$\begin{aligned} d(h_n, x_*) &\leq \delta^2 [1-\alpha_{n-1}(1-\delta)] d(h_{n-1}, x_*) \\ d(h_{n-1}, x_*) &\leq \delta^2 [1-\alpha_{n-2}(1-\delta)] d(h_{n-2}, x_*) \\ &\vdots \\ d(h_1, x_*) &\leq \delta^2 [1-\alpha_0(1-\delta)] d(h_0, x_*). \end{aligned}$$

From the above inequalities, we have

$$d(h_{n+1}, x_*) \leq d(h_0, x_*) \delta^{2(n+1)} \prod_{i=0}^{n} [1-\alpha_i(1-\delta)]. \tag{11}$$

From classical analysis, we know that $1-x \leq e^{-x}$ for all $x \in [0,1]$. By using this inequality with (11), we obtain

$$\begin{aligned} d(h_{n+1}, x_*) &\leq d(h_0, x_*) \delta^{2(n+1)} \prod_{i=0}^{n} e^{-(1-\delta)\alpha_i} \\ &= d(h_0, x_*) \delta^{2(n+1)} e^{-(1-\delta)\sum_{i=0}^{n} \alpha_i}. \end{aligned} \tag{12}$$

Taking the limit in both sides of inequality (12), it can be seen that $h_n \to x_*$ as $n \to \infty$. □

THEOREM 2.2. *Let C, H and T with fixed point x_* be the same as in Theorem 2.1. Let $\{h_n\}_{n=0}^{\infty}$ be defined by the iteration process (4) for $h_0 \in C$ and let $\{k_n\}_{n=0}^{\infty}$ be defined by the iteration process (2) for $k_0 \in C$. Then the following statements are equivalent:*

(i) $\{h_n\}_{n=0}^{\infty}$ converges to x_*

(ii) $\{k_n\}_{n=0}^{\infty}$ converges to x_*

Proof. We will prove (i)⇒(ii). Suppose that $\{h_n\}_{n=0}^{\infty}$ converges to x_*. It follows from (H_1), (2), (4), (7), (9) and (10) that

$$d(s_n, x_*) \le \delta [1 - \alpha_n (1 - \delta)] d(h_n, x_*) \tag{13}$$

and

$$d(h_n, Th_n) \le (1 + \delta) d(h_n, x_*). \tag{14}$$

Also

$$\begin{aligned} d(b_n, r_n) &= d(W(k_n, Tk_n, \beta_n), Th_n) \\ &\le (1 - \beta_n) d(k_n, Th_n) + \beta_n d(Tk_n, Th_n) \\ &\le (1 - \beta_n) d(k_n, h_n) + (1 - \beta_n) d(h_n, Th_n) \\ &\quad + \beta_n \delta d(k_n, h_n) + 2\beta_n \delta d(x_*, h_n). \end{aligned} \tag{15}$$

Substituting (14) in (15), we obtain

$$\begin{aligned} d(b_n, r_n) &= [1 - \beta_n (1 - \delta)] d(k_n, h_n) \\ &\quad + [1 - \beta_n (1 - \delta) + \delta] d(h_n, x_*). \end{aligned} \tag{16}$$

Moreover, using (H_4), (6) and (7), we obtain

$$\begin{aligned} d(m_n, s_n) &= d(W(Tk_n, Tb_n, \alpha_n), W(r_n, Tr_n, \alpha_n)) \\ &\le (1 - \alpha_n) d(Tk_n, r_n) + \alpha_n d(Tb_n, Tr_n) \\ &\le (1 - \alpha_n) d(Tk_n, Th_n) + \alpha_n \delta d(b_n, r_n) \\ &\quad + 2\alpha_n \delta d(x_*, r_n) \\ &\le (1 - \alpha_n) \delta d(k_n, h_n) + 2(1 - \alpha_n) \delta d(x_*, h_n) \\ &\quad + \alpha_n \delta d(b_n, r_n) + 2\alpha_n \delta d(x_*, r_n). \end{aligned} \tag{17}$$

Substituting (10) and (16) in (17), and using $\delta \in (0, 1)$ and $[1 - \beta_n (1 - \delta)] \le 1$, we obtain

$$\begin{aligned} d(m_n, s_n) &\le \delta d(k_n, h_n) \\ &\quad + [2 + 2\alpha_n - \alpha_n \beta_n (1 - \delta)] d(h_n, x_*) \end{aligned} \tag{18}$$

and also using (7), we obtain

$$\begin{aligned} d(k_{n+1}, h_{n+1}) &= d(Tm_n, Ts_n) \\ &\le \delta d(m_n, s_n) + 2\delta d(x_*, s_n). \end{aligned}$$

Substituting (13) and (18) in the above inequality, we obtain

$$\begin{aligned} d(k_{n+1}, h_{n+1}) \leq\ & \delta^2 d(k_n, h_n) \\ & + \{[2\delta + 2\alpha_n\delta - \alpha_n\beta_n\delta(1-\delta)] \\ & + 2\delta^2 [1 - \alpha_n (1-\delta)]\} d(h_n, x_*). \end{aligned}$$

Denote that

$$\begin{aligned} c_n &= d(k_n, h_n), \\ \rho &= \delta^2 \in (0,1), \\ d_n &= \{[2\delta + 2\alpha_n\delta - \alpha_n\beta_n\delta(1-\delta)] \\ & \quad + 2\delta^2 [1 - \alpha_n (1-\delta)]\} d(h_n, x_*). \end{aligned} \tag{19}$$

It is clear that (19) satisfies all the conditions in Lemma 1.5 and hence it follows from its conclusion that $\lim_{n\to\infty} d(k_n, h_n) = 0$. Hence, we obtain $\lim_{n\to\infty} d(k_n, x_*) = 0$.

Secondly, we will prove (ii)⇒(i). Suppose that $\{k_n\}_{n=0}^{\infty}$ converges to x_*. It follows from (2), (4), (7), and (H_1) that

$$\begin{aligned} d(b_n, x_*) &= d(W(k_n, Tk_n, \beta_n), Tx_*) \\ &\leq (1-\beta_n) d(k_n, x_*) + \beta_n d(Tk_n, Tx_*) \\ &\leq (1-\beta_n) d(k_n, x_*) + \beta_n \delta d(k_n, x_*) \\ &= [1 - \beta_n (1-\delta)] d(k_n, x_*) \end{aligned} \tag{20}$$

and

$$\begin{aligned} d(m_n, x_*) &= d(W(Tk_n, Tb_n, \alpha_n), Tx_*) \\ &\leq (1-\alpha_n) d(Tk_n, Tx_*) + \alpha_n d(Tb_n, Tx_*) \\ &\leq (1-\alpha_n) \delta d(k_n, x_*) + \alpha_n \delta d(b_n, x_*). \end{aligned} \tag{21}$$

Substituting (20) in (21), we obtain

$$d(m_n, x_*) \leq \delta [1 - \alpha_n\beta_n (1-\delta)] d(k_n, x_*). \tag{22}$$

Also using (6) and (H_1) we get

$$d(k_n, Tk_n) \leq (1+\delta) d(k_n, x_*) \tag{23}$$

and

$$\begin{aligned} d(r_n, b_n) &= d(Th_n, W(k_n, Tk_n, \beta_n)) \\ &\leq (1-\beta_n)\, d(Th_n, k_n) + \beta_n d(Th_n, Tk_n) \\ &\leq (1-\beta_n)\, d(Th_n, Tk_n) + (1-\beta_n)\, d(Tk_n, k_n) \\ &\quad +\beta_n d(Th_n, Tk_n) \\ &= d(Th_n, Tk_n) + (1-\beta_n)\, d(Tk_n, k_n). \end{aligned}$$

Substituting (23) in the above inequality, we obtain

$$\begin{aligned} d(r_n, b_n) &\leq \delta d(h_n, k_n) \\ &\quad + [2\delta + (1-\beta_n)(1+\delta)]\, d(k_n, x_*). \end{aligned} \tag{24}$$

Moreover using (6), (H_4) and (7), we obtain

$$\begin{aligned} d(s_n, m_n) &= d(W(r_n, Tr_n, \alpha_n), W(Tk_n, Tb_n, \alpha_n)) \\ &\leq (1-\alpha_n)\, d(r_n, Tk_n) + \alpha_n d(Tr_n, Tb_n) \\ &\leq (1-\alpha_n)\, d(Th_n, Tk_n) + \alpha_n d(Tr_n, Tb_n) \\ &\leq (1-\alpha_n)\, \delta d(h_n, k_n) + 2(1-\alpha_n)\, \delta d(x_*, k_n) \\ &\quad +\alpha_n \delta d(r_n, b_n) + 2\alpha_n \delta d(x_*, b_n). \end{aligned} \tag{25}$$

Substituting (20) and (24) in (25) and using $\delta \in (0,1)$, we obtain

$$\begin{aligned} d(s_n, m_n) &\leq \delta d(h_n, k_n) \\ &\quad + \left\{2(1-\alpha_n)\delta + \left[2\alpha_n\delta^2 + \alpha_n(1-\beta_n)\delta(1+\delta)\right]\right. \\ &\quad \left. + 2\alpha_n\delta[1-\beta_n(1-\delta)]\right\} d(k_n, x_*) \end{aligned} \tag{26}$$

and also using (7), we get

$$\begin{aligned} d(h_{n+1}, k_{n+1}) &= d(Ts_n, Tm_n) \\ &\leq \delta d(s_n, m_n) + 2\delta d(x_*, m_n). \end{aligned}$$

Substituting (22) and (26) in the above inequality, we obtain

$$\begin{aligned} d(h_{n+1}, k_{n+1}) &\leq \delta^2 d(h_n, k_n) \\ &\quad + \left\{ \begin{array}{c} 2\delta^2(1-\alpha_n) + 2\alpha_n\delta^3 \\ +\alpha_n(1-\beta_n)\delta^2(1+\delta) \\ +2\alpha_n\delta^2[1-\beta_n(1-\delta)] \\ +2\delta^2[1-\alpha_n\beta_n(1-\delta)] \end{array} \right\} d(k_n, x_*) \end{aligned}$$

Denote that

$$\begin{aligned} c_n &= d(h_n, k_n), \\ \rho &= \delta^2 \in (0,1), \\ d_n &= \left\{ \begin{array}{c} 2\delta^2(1-\alpha_n) + 2\alpha_n\delta^3 \\ +\alpha_n(1-\beta_n)\delta^2(1+\delta) \\ +2\alpha_n\delta^2[1-\beta_n(1-\delta)] \\ +2\delta^2[1-\alpha_n\beta_n(1-\delta)] \end{array} \right\} d(k_n, x_*). \end{aligned}$$

It is clear that the above equalities satisfies all the conditions in Lemma 1.5 and hence it follows from its conclusion that $\lim_{n\to\infty} d(h_n, k_n) = 0$. Hence, we obtain $\lim_{n\to\infty} d(h_n, x_*) = 0$. □

THEOREM 2.3. *Let C, H, and T with fixed point x_* be the same as in Theorem 2.1. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences in $[0,1]$ satisfying $\alpha_1 < \alpha_n \leq 1$, and $\beta_1 < \beta_n \leq 1$ for all $n \in \mathbb{N}$. For given $h_0 = k_0 \in C$, consider the iterative sequences $\{h_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$ defined by (4) and (2) respectively. Then, $\{h_n\}_{n=0}^{\infty}$ converges to x_* faster than $\{k_n\}_{n=0}^{\infty}$.*

Proof. From (22), we have

$$d(m_n, x_*) \leq \delta[1-\alpha_n\beta_n(1-\delta)]\, d(k_n, x_*). \tag{27}$$

Also, using (2) and (6) we get

$$\begin{aligned} d(k_{n+1}, x_*) &= d(Tm_n, x_*) \\ &\leq \delta d(m_n, x_*). \end{aligned} \tag{28}$$

Substituting (27) in (28), we obtain

$$d(k_{n+1}, x_*) \leq \delta^2[1-\alpha_n\beta_n(1-\delta)]\, d(k_n, x_*). \tag{29}$$

By repeating this process n times, we get

$$\begin{aligned} d(k_n, x_*) &\leq \delta^2[1-\alpha_{n-1}\beta_{n-1}(1-\delta)]\, d(k_{n-1}, x_*) \\ d(k_{n-1}, x_*) &\leq \delta^2[1-\alpha_{n-2}\beta_{n-2}(1-\delta)]\, d(k_{n-2}, x_*) \\ &\vdots \\ d(k_1, x_*) &\leq \delta^2[1-\alpha_0\beta_0(1-\delta)]\, d(k_0, x_*). \end{aligned}$$

From the above inequalities, we have

$$d(k_{n+1}, x_*) \leq d(k_0, x_*)\, \delta^{2(n+1)} \prod_{i=0}^{n}[1-\alpha_i\beta_i(1-\delta)]. \tag{30}$$

Also, from Theorem 2.1, we have

$$d(h_{n+1}, x_*) \leq d(h_0, x_*)\, \delta^{2(n+1)} \prod_{i=0}^{n} [1 - \alpha_i(1-\delta)]. \tag{31}$$

Applying assumptions $\alpha_1 < \alpha_n \leq 1$, *and* $\beta_1 < \beta_n \leq 1$ to (30) and (31) respectively, we obtain

$$\begin{aligned} d(k_{n+1}, x_*) &\leq d(k_0, x_*)\, \delta^{2(n+1)} [1 - \alpha_1\beta_1(1-\delta)]^{n+1} \\ d(h_{n+1}, x_*) &\leq d(h_0, x_*)\, \delta^{2(n+1)} [1 - \alpha_1(1-\delta)]^{n+1}. \end{aligned}$$

Define

$$\begin{aligned} a_n &= d(h_0, x_*)\, \delta^{2(n+1)} [1 - \alpha_1(1-\delta)]^{n+1} \\ b_n &= d(k_0, x_*)\, \delta^{2(n+1)} [1 - \alpha_1\beta_1(1-\delta)]^{n+1} \end{aligned}$$

and

$$\begin{aligned} \Delta_n &= \frac{a_n}{b_n} = \frac{d(h_0, x_*)\, \delta^{2(n+1)} [1 - \alpha_1(1-\delta)]^{n+1}}{d(k_0, x_*)\, \delta^{2(n+1)} [1 - \alpha_1\beta_1(1-\delta)]^{n+1}} \\ &= \left[\frac{1 - \alpha_1(1-\delta)}{1 - \alpha_1\beta_1(1-\delta)}\right]^{n+1}. \end{aligned}$$

Since δ and $\beta_1 \in (0,1)$, we have

$$\begin{aligned} \beta_1 &< 1 \\ &\Rightarrow \alpha_1\beta_1 < \alpha_1 \\ &\Rightarrow \alpha_1\beta_1(1-\delta) < \alpha_1(1-\delta) \\ &\Rightarrow \frac{[1 - \alpha_1(1-\delta)]}{[1 - \alpha_1\beta_1(1-\delta)]} < 1. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \Delta_n = 0$. Hence from Definition 1.4, we obtain that $\{h_n\}_{n=1}^{\infty}$ converges faster than $\{k_n\}_{n=1}^{\infty}$. □

In the following we give a non-trivial example to show iteration process (4) has higher convergence speed when compared to iteration process (2):

Example 2.4. Let $H = [0,1]$ be endowed with the usual metric. Define operator $T : H \to H$ by $Tx = \frac{1}{4}\exp(0.025 - x^2) - \frac{1}{2}\sin x$ with a unique fixed point $x_* = 0.166471116$. The operator T satisfies the condition (6) with $\delta \in [0.75, 1)$. For $h_0 = k_0 = 1$ and $\alpha_n = 0.40$, $\beta_n = 0.30$, the following table shows that iteration process (4) converges to $x_* = 0.166471116$ faster than iteration process (2).

TABLE 1. Comparison the convergence speed of iteration process (4) and iteration process (2).

Number of iter.	**iteration (4)**	**iteration (2)**
x_0	1	1
x_1	$0,275703121$	$0,275703121$
x_2	$0,180394601$	$0,218930031$
$\vdots$	$\vdots$	$\vdots$
x_{10}	$0,166471117$	$0,166472569$
x_{11}	$0,166471116$	$0,166471222$
$\vdots$	$\vdots$	$\vdots$
x_{17}		$0,166471117$
x_{18}		$0,166471116$

THEOREM 2.5. *Let S be an approximate operator of T. Let $\{h_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (4) for T and define an iterative sequence $\{u_n\}_{n=1}^{\infty}$ as follows:*

$$\left\{\begin{array}{c} u_0 \in C, \\ u_{n+1} = Sv_n \\ v_n = W(w_n, Tw_n, \alpha_n) \\ w_n = Su_n. \end{array}\right. \tag{32}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying $\frac{1}{2} \leq \alpha_n$ for all $n \in \mathbb{N}$. If $Tx_* = x_*$ and $Su_* = u_*$ such that $u_n \to u_*$ as $n \to \infty$, then we have

$$d(x_*, u_*) \leq \frac{5\varepsilon}{1-\delta}$$

where $\varepsilon > 0$ is a fixed number.

Proof. It follows from $(H4)$,(4), (6), (9), (10) and (32), that

$$d(r_n, x_*) \leq \delta d(h_n, x_*) \tag{33}$$

and

$$d(s_n, x_*) \leq \delta [1 - \alpha_n (1-\delta)] d(h_n, x_*) \tag{34}$$

and

$$\begin{aligned} d(w_n, r_n) &= d(Su_n, Th_n) \\ &\leq d(Su_n, Tu_n) + d(Tu_n, Th_n) \\ &\quad \varepsilon + \delta d(u_n, h_n) + 2\delta d(x_*, h_n). \end{aligned} \tag{35}$$

Also

$$\begin{aligned} d(v_n, s_n) &= d(W(w_n, Sw_n, \alpha_n), W(r_n, Tr_n, \alpha_n)) \\ &\leq (1-\alpha_n) d(w_n, r_n) + \alpha_n d(Sw_n, Tr_n) \\ &\leq (1-\alpha_n) d(w_n, r_n) + \alpha_n d(Sw_n, Tw_n) \\ &\quad + \alpha_n d(Tw_n, Tr_n) \\ &\leq (1-\alpha_n) d(w_n, r_n) + \alpha_n \varepsilon \\ &\quad + \alpha_n \delta d(w_n, r_n) + 2\alpha_n \delta d(x_*, r_n) \\ &= [1-\alpha_n(1-\delta)] d(w_n, r_n) \\ &\quad + 2\alpha_n \delta d(r_n, x_*) + \alpha_n \varepsilon. \end{aligned} \tag{36}$$

Substituting (33) and (35) in (36), we obtain

$$\begin{aligned} d(v_n, s_n) &\leq [1-\alpha_n(1-\delta)] \varepsilon \\ &\quad + \delta [1-\alpha_n(1-\delta)] d(u_n, h_n) \\ &\quad + 2\delta [1-\alpha_n(1-\delta)] d(h_n, x_*) \\ &\quad + 2\alpha_n \delta^2 d(h_n, x_*) + \alpha_n \varepsilon. \end{aligned}$$

Using $\delta \in (0,1)$ and the above inequality, we obtain

$$\begin{aligned} d(v_n, s_n) &\leq [1-\alpha_n(1-\delta)] d(u_n, h_n) \\ &\quad + \left\{2\delta [1-\alpha_n(1-\delta)] + 2\alpha_n \delta^2\right\} d(h_n, x_*) \\ &\quad + \{[1-\alpha_n(1-\delta)] + \alpha_n\} \varepsilon. \end{aligned} \tag{37}$$

Moreover

$$\begin{aligned} d(u_{n+1}, h_{n+1}) &= d(Sv_n, Ts_n) \\ &\leq d(Sv_n, Tv_n) + d(Tv_n, Ts_n) \\ &\leq \varepsilon + \delta d(v_n, s_n) + 2\delta d(x_*, s_n). \end{aligned} \tag{38}$$

Substituting (34) and (37) in (38) and using $\delta \in (0,1)$, we obtain

$$\begin{aligned} d(u_{n+1}, h_{n+1}) &\leq [1-\alpha_n(1-\delta)] d(u_n, h_n) \\ &\quad + \left\{4\delta^2 + 2\alpha_n \delta^2\right\} d(h_n, x_*) \\ &\quad + (2+\alpha_n)\varepsilon. \end{aligned} \tag{39}$$

From hypothesis, we obtain

$$1-\alpha_n \leq \alpha_n.$$

Applying the above inequality to (39), we get

$$\begin{aligned} d\left(u_{n+1}, h_{n+1}\right) &\leq \left[1-\alpha_n\left(1-\delta\right)\right] d\left(u_n, h_n\right) \\ &\quad \alpha_n\left(1-\delta\right)\left[\frac{10\delta^2 d\left(h_n, x_*\right)+5\varepsilon}{\left(1-\delta\right)}\right]. \end{aligned}$$

Denote

$$\begin{aligned} c_n &= d\left(u_n, h_n\right) \\ \xi_n &= \alpha_n\left(1-\delta\right) \in (0,1), \\ \mu_n &= \frac{10\delta^2 d\left(h_n, x_*\right)+5\varepsilon}{\left(1-\delta\right)}. \end{aligned}$$

Hence, all conditions in Lemma 1.7 are satisfied. Therefore,

$$0 \leq \limsup_{n\to\infty} d\left(u_n, h_n\right) \leq \limsup_{n\to\infty} \frac{10\delta^2 d\left(h_n, x_*\right)+5\varepsilon}{\left(1-\delta\right)}.$$

Since $h_n \to x_*$ and $u_n \to u_*$ as $n \to \infty$, then we have

$$d\left(x_*, u_*\right) \leq \frac{5\varepsilon}{1-\delta}.$$

□

EXAMPLE 2.6. Let $H = [0,1]$ be endowed with the usual metric. Define operator $T : H \to H$ by $Tx = \frac{1}{3}\cos(2x)$ with a unique fixed point $x_* = 0.2818$. It is easy to check that T satisfies (6) with $\delta \in [0.50, 1)$. Define operator $S : H \to H$ by

$$Su = \frac{1}{2} - \frac{2}{3}(u-0.01)^3 + \frac{2}{15}(u-0.05)^5 - \frac{4}{283}(u+0.02)^8 \tag{40}$$

By utilizing Wolfram Mathematica 9 software package, we get $\max_{x\in H}|T-S| = 0.1667$. Hence for all $x \in H$ and for a fixed $\varepsilon = 0.1667 > 0$, we have $|Tx - Sx| \leq 0.1667$. Thus S is an approximate operator of T in the sense of Definition 1.8. Also $u_* = 0.446009101$ is the unique fixed point for the operator S in $H = [0,1]$. Therefore $|x_* - u_*| = 0.178$. If we put

$\alpha_n = 0.15$ in (32) for the approximate operator S (40), we obtain
(41)

$$\begin{cases} u_{n+1} = \frac{1}{2} - \frac{2}{3}(v_n - 0.01)^3 \\ +\frac{2}{15}(v_n - 0.05)^5 - \frac{4}{283}(v_n + 0.02)^8 \\ v_n = (0.85)\, w_n \\ +(0.15)\left[\frac{1}{2} - \frac{2}{3}(w_n - 0.01)^3 + \frac{2}{15}(w_n - 0.05)^5 - \frac{4}{283}(w_n + 0.02)^8\right] \\ w_n = \frac{1}{2} - \frac{2}{3}(u_n - 0.01)^3 + \\ \frac{2}{15}(u_n - 0.05)^5 - \frac{4}{283}(u_n + 0.02)^8 \end{cases}$$

The following table shows that the sequence $\{u_n\}_{n=0}^{\infty}$ generated by (41) converges to the fixed point $u_* = 0.446009101$.

TABLE 2. Convergence of iteration process (41)

Number of iter.	**iteration (41)**
x_0	1
x_1	$0,499998240$
x_2	$0,452178384$
$\vdots$	$\vdots$
x_9	$0,446009102$
x_{10}	$0,446009101$

Then we can find the following estimate

$$|x_* - u_*| = 0.178 \leq \frac{5(0.1667)}{1 - 0.50} = 1.667.$$

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