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FIXED POINT THEOREMS FOR ASYMPTOTICALLY REGULAR MAPPINGS IN FUZZY METRIC SPACES

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Abstract. The aim of this paper is to extend some existing fixed point results for asymptotically regular mappings to fuzzy metric spaces. For this purpose some contractive type conditions with respect to an altering distance function are used. Some new common fixed point results have been derived for such mappings. We provide suitable examples to justify our study.

1. Introduction and preliminaries

In 1966, Browder and Petryshyn introduced the concept of asymptotic regularity of self mapping at a point in a metric space [1]. After that several researchers (refer to [6, 8, 12, 13, 15]) derived different results in fixed point theory for such type of mappings.

After the initiation of the study of fixed point theory in fuzzy metric spaces by Grabiec in 1988 [3], extensions of many existing fixed point results as well as new fixed point results have been established by several research workers in fuzzy metric spaces (refer to [8, 10, 11, 14–18]). The introduction of the concept of altering distance function by Khan et al. [5] enhanced the fixed point theory in different spaces.

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In this paper, we prove some fixed point theorems for asymptotically regular mappings in fuzzy metric spaces which extend some existing fixed point results. Moreover using the concept of compatibility, we derive some common fixed point results for asymptotically regular mappings.

First we provide some basic concepts.

Definition 1.1. [4] A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a triangular norm or t-norm if

i) $T(x_1, 1) = x_1 \ \forall \ x_1 \in [0,1]$,
ii) $T(x_1, x_2) = T(x_2, x_1) \ \forall \ x_1, x_2 \in [0,1]$,
iii) $x_1 \geq x_2, \ x_3 \geq x_4 \Rightarrow T(x_1, x_3) \geq T(x_2, x_4)$, $\forall \ x_1, x_2, x_3, x_4 \in [0,1]$,
iv) $T(x_1, T(x_2, x_3)) = T(T(x_1, x_2), x_3), \ \forall \ x_1, x_2, x_3 \in [0,1]$.

Some basic examples of t-norms are $T_m(x_1, x_2) = \min(x_1, x_2)$, $T_p(x_1, x_2) = x_1.x_2$, $T_L(x_1, x_2) = \max(x_1 + x_2 - 1, 0)$.

Definition 1.2. [2] For an arbitrary set X, let T be a continuous t-norm and M be a fuzzy set on $X \times X \times (0, \infty)$. The 3-tuple (X, M, T) is called a fuzzy metric space if the following conditions are satisfied :

a) $M(x_1, x_2, t) > 0, \ \forall \ x_1, x_2 \in X, t > 0$,
b) $M(x_1, x_2, t) = 1 \Leftrightarrow x_1 = x_2, \ \forall \ t > 0$,
c) $M(x_1, x_2, t) = M(x_2, x_1, t), \ \forall \ x_1, x_2 \in X, t > 0$,
d) $T(M(x_1, x_2, t), M(x_2, x_3, s)) \leq M(x_1, x_3, t + s)$, $\forall \ x_1, x_2, x_3 \in X, t, s > 0$,
e) $M(x_1, x_2, .) : (0, \infty) \longrightarrow [0,1]$ is continuous $\forall \ x_1, x_2 \in X$.

For a fuzzy metric space (X, M, T), the function M is a continuous function on $X \times X \times (0, \infty)$ (refer to [7]).

Example 1.3. Let $X = [0,1]$. For a continuous function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $t \in (0, \infty)$, let $M(x, y, t) = e^{(\frac{-d(x,y)}{g(t)})}$ where $d(x, y) = |x - y|$. Then (X, M, T) is a fuzzy metric space with respect to the t-norm $T_p(x, y) = x.y, \ \ x, y \in X$ (refer to [4]).

Definition 1.4. [3] Let (X, M, T) be a fuzzy metric space and $\{x_n\}$ be a sequence in X.

$\{x_n\}$ is a Cauchy sequence in X if $\forall \ \varepsilon \in (0,1), \ \exists \ n_0 \ \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon, \ \forall \ n, m \geq n_0$, or equivalently, if $\lim_{n,m \to \infty} M(x_n, x_m, t) = 1, \ \forall \ \ t > 0$.

$\{x_n\}$ converges to x if $\forall \ \varepsilon \in (0,1), \ \exists \ n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon, \ \forall \ \ n \geq n_0$.

X is said to be complete if and only if every Cauchy sequence converges in X.

DEFINITION 1.5. [14] A mapping $\phi : [0,1] \longrightarrow [0,1]$ is called an altering distance function if
(i) ϕ is strictly decreasing and left continuous.
(ii) $\phi(\lambda) = 0$ if and only if $\lambda = 1$
i.e, $\lim_{\phi \longrightarrow 1-} \phi(1) = 0$.

DEFINITION 1.6. [9,13] Let f and g be self mappings on a fuzzy metric space (X, M, T) and $\{x_n\}$ be a sequence in X.

f is said to be asymptotically regular at a point $x_0 \in X$ if $\lim_{n\to\infty} M(f^n(x_0), f^{n+1}(x_0), t) = 1, \forall \ t > 0$.

Also the sequence $\{x_n\}$ is said to be asymptotically regular with respect to the pair (f, g) if $\lim_{n\to\infty} M(f(x_n), g(x_n), t) = 1, \forall \ t > 0$.

DEFINITION 1.7. [9] Two self mappings f and g on a fuzzy metric space (X, M, T) are said to be compatible if $\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = 1, \forall \ t > 0$, where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = x$, for some $x \in X$.

2. Results and discussion

In 2013, Shukla et al. [15] proved a fixed point result in a complete partial metric space using asymptotic regularity. Following is an extension of this result in the setting of fuzzy metric space.

THEOREM 2.1. *Let (X, M, T) be a complete fuzzy metric space, ϕ be the altering distance function and $f : X \longrightarrow X$ be such that the following condition is satisfied:*

$$\begin{aligned}\phi(M(f(x), f(y), t)) \le b_1(x,y)\theta[\min\{\phi(M(x, f(x), t)), \phi(M(y, f(y), t))\}] \\ +b_2(x,y)\psi[\phi(M(x, f(x), t)).\phi(M(y, f(y), t))] + b_3(x,y)\phi(M(x, y, t)) \\ + b_4(x,y)(\phi(M(x, f(x), t)) + \phi(M(y, f(y), t))) \\ + b_5(x,y)[\phi(M(x, f(y), t)) + \phi(M(f(x), y, t))] \qquad (1)\end{aligned}$$

$\forall \ x, y \in X, t > 0$ where $b_i : X \times X \longrightarrow [0, \infty), \ i = 1, 2, 3, 4, 5$ are such that for some arbitrarily fixed $\lambda_1 > 0, 0 < \lambda_2 < 1$,

$$b_1(x,y) + b_2(x,y) \le \lambda_1, \qquad (2)$$

$$b_3(x,y) + b_4(x,y) + 2b_5(x,y) \le \lambda_2 \qquad (3)$$

and $\theta, \psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *are continuous functions at* 0 *and* $\theta(0) = \psi(0) = 0$. *If* f *is asymptotically regular at some point* $x_0 \in X$*, then* f *has a unique fixed point in* X.

Proof. Suppose that $\{x_n\}$ is a sequence in X where $x_0 \in X$ and $x_{n+1} = f(x_n)\ \forall\ n \geq 0$. Now if for some $n \geq 0$, $x_n = x_{n+1}$, then x_n is a fixed point of f. Suppose that $x_n \neq x_{n+1}\ \forall\ n$. We show that the sequence $\{x_n\}$ is Cauchy.
Suppose to the contrary $\exists\ 0 < \varepsilon < 1, t > 0$ and two sequences of integers $\{r_n\}$ and $\{s_n\}$ such that $r_n > s_n > n$,

$$\begin{aligned} &M(x_{r_n}, x_{s_n}, t) \leq 1 - \varepsilon, \\ &M(x_{r_n-1}, x_{s_n-1}, t) > 1 - \varepsilon, \\ &M(x_{r_n-1}, x_{s_n}, t) > 1 - \varepsilon, \quad \forall\ n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{4}$$

Now we have

$$\begin{aligned} 1 - \varepsilon \geq M(x_{r_n}, x_{s_n}, t) &\geq T(M(x_{r_n}, x_{r_n-1}, \frac{t}{2}), M(x_{r_n-1}, x_{s_n}, \frac{t}{2})) \\ &\geq T(M(x_{r_n}, x_{r_n-1}, t), 1 - \varepsilon) \\ \Rightarrow 1 - \varepsilon \geq \lim_{n\to\infty} M(x_{r_n}, x_{s_n}, t) &\geq T(1, 1 - \varepsilon) \\ &\qquad \text{(since } f \text{ asymptotically regular at } x_0) \end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} M(x_{r_n}, x_{s_n}, t) = 1 - \varepsilon \tag{5}$$

Again,

$$\begin{aligned} &M(x_{r_n}, x_{s_n-1}, t) \geq T(M(x_{r_n}, x_{s_n}, \frac{t}{2}), M(x_{s_n}, x_{s_n-1}, \frac{t}{2})) \\ &\Rightarrow \lim_{n\to\infty} M(x_{r_n}, x_{s_n-1}, t) > T(1 - \varepsilon, 1) \end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} M(x_{r_n}, x_{s_n-1}, t) > 1 - \varepsilon \tag{6}$$

Taking $x = x_{r_n-1}$ and $y = x_{s_n-1}$ in (1), we have

$$\begin{aligned} \phi(M(x_{r_n}, x_{s_n}, t)) \leq\ & b_1(x, y)\theta(\min\{\phi(M(x_{r_n-1}, x_{r_n}, t)), \phi(M(x_{s_n-1}, x_{s_n}, t))\}) \\ & + b_2(x, y)\psi(\phi(M(x_{r_n-1}, x_{r_n}, t)).\phi(M(x_{s_n-1}, x_{s_n}, t))) \\ & + b_3(x, y)\phi(M(x_{r_n-1}, x_{s_n-1}, t)) + b_4(x, y)[\phi(M(x_{r_n-1}, x_{r_n}, t)) \\ & + \phi(M(x_{s_n-1}, x_{s_n}, t))] + b_5(x, y)[\phi(M(x_{r_n-1}, x_{s_n}, t)) \\ & + \phi(M(x_{r_n}, x_{s_n-1}, t))] \end{aligned}$$

Taking $n \longrightarrow \infty$ and by (4), (5), (6) and using the fact that f is asymptotically regular at x_0 we have,

$$\phi(1-\varepsilon) \le b_3(x,y)\phi(1-\varepsilon) + 2b_5(x,y)\phi(1-\varepsilon) < \phi(1-\varepsilon)$$

which is a contradiction.
Thus $\{x_n\}$ is a Cauchy sequence. Since (X, M, T) is a complete fuzzy metric space, $\exists\, z \in X$ such that $x_n \longrightarrow z$.

$$\begin{aligned}
&\text{Now, } \phi(M(f(x_n), f(z), t))\\
&\le b_1(x,y)\theta(\min\{\phi(M(x_n, x_{n+1}, t)), \phi(M(z, f(z), t))\})\\
&+ b_2(x,y)\psi(\phi(M(x_n, x_{n+1}, t)).\phi(M(z, f(z), t)))\\
&+ b_3(x,y)\phi(M(x_n, z, t)) + b_4(x,y)[\phi(M(x_n, x_{n+1}, t))\\
&+ \phi(M(z, f(z), t))] + b_5(x,y)[\phi(M(x_n, f(z), t))\\
&+ \phi(M(z, x_{n+1}, t))]
\end{aligned}$$

For $n \longrightarrow \infty$,

$$\begin{aligned}
&\lim_{n\to\infty} \phi(M(z, f(z), t)) \le (b_4(x,y) + b_5(x,y)) \lim_{n\to\infty} \phi(M(z, f(z), t))\\
\Rightarrow &(1 - b_4(x,y) - b_5(x,y)) \lim_{n\to\infty} \phi(M(z, f(z), t)) \le 0\\
\Rightarrow &\lim_{n\to\infty} \phi(M(z, f(z), t)) = 0 \qquad (\text{since } 0 < b_3(x,y) + b_4(x,y) + 2b_5(x,y) < 1)\\
\Rightarrow &f(z) = z.
\end{aligned}$$

If u is another fixed point of f in X, then

$$\begin{aligned}
&\phi(M(f(u), f(z), t))\\
&\le b_1(x,y)\theta(\min\{\phi(M(u, f(u), t)), \phi(M(z, f(z), t))\})\\
&+ b_2(x,y)\psi(\phi(M(u, f(u), t)).\phi(M(z, f(z), t)))\\
&+ b_3(x,y)\phi(M(u, z, t)) + b_4(x,y)[\phi(M(u, f(u), t))\\
&+ \phi(M(z, f(z), t))] + b_5(x,y)[\phi(M(u, f(z), t))\\
&+ \phi(M(z, f(u), t))]\\
\Rightarrow\ &\phi(M(u, z, t)) \le b_3(x,y)\phi(M(u, z, t)) + 2b_5(x,y)\phi(M(u, z, t))\\
\Rightarrow\ &(1 - b_3(x,y) - 2b_5(x,y))\phi(M(u, z, t)) \le 0\\
\Rightarrow\ &\phi(M(u, z, t)) = 0 \qquad (\text{since } 0 < b_3(x,y) + b_4(x,y) + 2b_5(x,y) < 1)\\
\Rightarrow\ &u = z,
\end{aligned}$$

establishes that the fixed point of f in X is unique. □

EXAMPLE 2.2. Consider the complete metric space (X, d) where $X = [0,1]$ and $d(x,y) = |x-y|\ \forall\ x, y \in X$. Let M be a fuzzy set on $X \times X \times (0,\infty)$ given by $M(x,y,t) = \dfrac{t}{t+d(x,y)}$, if $t > 0$.

Then (X, M, T) is a complete fuzzy metric space with respect to the t-norm $T(a,b) = \min\{a,b\}$, $a, b \in [0,1]$. Define $f : X \longrightarrow X$ by
$f(x) = \begin{cases} \dfrac{x}{3}, & x \in [0, \frac{1}{2}] \\ \frac{1}{6}, & x \in [\frac{1}{2}, 1] \end{cases}$ and $\phi(\lambda) = 1-\lambda$, $\lambda \in [0,1]$.
Let $\theta(p) = \sqrt{p}$ and $\psi(q) = q^2$, $\forall\ p, q \in\ \mathbb{R}^+$.

Also let $b_1(x,y) = |x-y|, b_2(x,y) = |x^2-y^2|, b_3(x,y) = \begin{cases} \dfrac{1}{|x-y|}, x \neq y \\ 0, \quad x = y \end{cases}$,

$b_4(x,y) = b_5(x,y) = \dfrac{1-b_3(x,y)}{3}$.
Then f satisfies (1).
Hence from Theorem 2.1 we can say that f has a unique fixed point.

COROLLARY 2.3. *Let $f, g : X \longrightarrow X$ be mappings on a complete fuzzy metric space (X, M, T) and ϕ be the altering distance function. Let f and g be asymptotically regular at a point $x_0 \in X$ and both satisfy the inequality* (1)*. Moreover, if*

(7)
$$\phi(M(f(x), g(y), t)) \leq k(\phi(M(x,y,t)) + \phi(M(x, f(x), t)) + \phi(M(y, g(y), t))),$$
where $0 < k < 1$ and $x, y \in X$,
then f and g have a unique common fixed point in X.

Proof. From Theorem 2.1, both f and g have unique fixed points say, z and v respectively.
Since f and g satisfy (7),

$$\phi(M(f(z), g(v), t)) \leq k(\phi(M(z,v,t)) + \phi(M(z, f(z), t)) + \phi(M(v, g(v), t)))$$
$$\Rightarrow \phi(M(z,v,t)) \leq k(\phi(M(z,v,t)) + \phi(M(z,z,t)) + \phi(M(v,v,t)))$$
$$\Rightarrow (1-k)\phi(M(z,v,t)) \leq 0$$
$$\Rightarrow \phi(M(z,v,t)) \leq 0 \quad (\text{since } k < 1)$$
$$\Rightarrow z = v$$
i.e., f and g have a unique common fixed point. □

EXAMPLE 2.4. Let $X = \{P, Q, R, S\} \subseteq \mathbb{R}^2$, where $P = (0,0), Q = (0,1), R = (1,0)$ and $S = (1,1)$. Let $M(x,y,t) = \frac{t}{t+d(x,y)}$, $t > 0$, where

$d(x, y)$ is the Euclidean distance in $\mathbb{R}^2$. Let $\phi(\lambda) = 1 - \lambda, \quad \lambda \in [0, 1]$. Then (X, M, T) is a complete fuzzy metric space with respect to the t-norm $T(a, b) = \min\{a, b\}$.
Let $f, g : X \longrightarrow X$ be defined by
$f(P) = f(Q) = P, f(R) = Q, f(S) = R$, and
$g(P) = g(R) = P, g(Q) = R, g(S) = Q$.
we can be shown easily that, for $k = \frac{4}{5}$ every pair of points in X satisfies the condition (7).
Hence by Theorem 2.3, f and g have a unique common fixed point which is P in this case.

In the following result, we use a minimum condition.

THEOREM 2.5. *Let $f : X \longrightarrow X$ be a mapping on a complete fuzzy metric space (X, M, T) and ϕ is the altering distance function. If f is asymptotically regular at a point $x_0 \in X$ and f satisfies,*

$$\phi(M(f(x), f(y), t)) \leq h_1 \min\{\phi(M(x, y, t)), \phi(M(f(x), x, t)), \phi(M(f(x), y, t))\}$$
$$+h_2 \min\{\phi(M(x, y, t)), \phi(M(f(y), y, t)), \phi(M(x, f(y), t))\} \tag{8}$$

for all $x, y \in X, t > 0$, where $h_1, h_2 > 0$ are constants such that $h_1 + h_2 < 1$,
then f has a unique fixed point in X.

Proof. As in Theorem 2.1, we construct a sequence $\{x_n\}$ in X by $x_{n+1} = f(x_n) \ \forall \ n \in \mathbb{N} \cup \{0\}$ where $x_0 \in X$. If there exists n with $x_n = x_{n+1}$, then x_n is a fixed point of f. Suppose that $x_n \neq x_{n+1}$ for all n.
To show that $\{x_n\}$ is a Cauchy sequence.
Let $m, n \in \mathbb{N} \cup \{0\}$. From (8),

$$\phi(M(f(x_n), f(x_m), t)) \leq h_1 \min\{\phi(M(x_n, x_m, t)), \phi(M(f(x_n), x_n, t)), \phi(M(f(x_n), x_m, t))\} + h_2 \min\{\phi(M(x_n, x_m, t)), \phi(M(f(x_m), x_m, t)), \phi(M(x_n, f(x_m), t))\}.$$

Since, f is asymptotically regular at $x_0 \in X$, taking $n, m \longrightarrow \infty$,

$$\lim_{n,m\to\infty} \phi(M(f(x_n), f(x_m), t)) = 0$$
$$\Rightarrow \lim_{n,m\to\infty} M(f(x_n), f(x_m), t) = 1,$$

i.e. $\{x_n\}$ is a Cauchy sequence in (X, M, T). Since (X, M, T) is complete, therefore $x_n \longrightarrow z$(say) in X.

Using (8),

$$\begin{aligned}\phi(M(x_{n+1}, f(z), t)) =&\phi(M(f(x_n), f(z), t))\\ &\leq h_1 \min\{\phi(M(x_n,z,t)), \phi(M(f(x_n), x_n, t)), \phi(M(f(x_n), z, t))\}\\ &+h_2 \min\{\phi(M(x_n,z,t)), \phi(M(f(z), z, t)), \phi(M(x_n, f(z), t))\}\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} \phi(M(x_{n+1}, f(z), t)) = 0$$

$$\Rightarrow \phi(M(z, f(z), t)) = 0$$

$$\Rightarrow f(z) = z, \text{ establishes that } z \text{ is a fixed point for } f.$$

Uniqueness can be shown easily.
Hence, z is the unique fixed point of f. □

Prudhvi [12] proved a common fixed point theorem for asymptotically regular self mappings in cone metric spaces. We obtain an extension of this result for fuzzy metric spaces.

THEOREM 2.6. *Let (X, M, T) be a fuzzy metric space, ϕ be the altering distance function and f and g be two commutative self mappings on X such that*

$$\begin{aligned}\phi(M(f(x), f(y), t)) \leq k_1[\phi(M(g(x), g(y), t)) + k_2(\phi(M(g(x), f(x), t))\\ +\phi(M(g(y), f(y), t)))]\end{aligned} \tag{9}$$

where $x, y \in X$, $t > 0$ and $k_1 : \mathbb{R} \longrightarrow [0, 1)$, $0 < k_1, k_2 < 1$.
Moreover if

(i) *f and g are asymptotically regular at x_0,*
(ii) *$f(X) \subseteq g(X)$,*
(iii) *$f(X)$ or $g(X)$ is a complete subspace of X,*

then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq g(X)$, define a sequence $\{u_n\}$ by $u_{n+1} = f(x_n) = g(x_{n+1})$, $n \in \mathbb{N} \cup \{0\}$. Again since f and g are asymptotically regular at x_0,

$$\lim_{n\to\infty} \phi(M(u_n, u_{n+1}, t)) = 0 \tag{10}$$

To show that the sequence $\{u_n\}$ is Cauchy.
Suppose, there exists $0 < \varepsilon < 1$ and two sequences of integers $\{r_n\}$ and

$\{s_n\}$ such that $r_n > s_n > n$,

$$M(u_{r_n}, u_{s_n}, t) \leq 1 - \varepsilon,$$
$$M(u_{r_n-1}, u_{s_n-1}, t) > 1 - \varepsilon,$$
$$M(u_{r_n-1}, u_{s_n}, t) > 1 - \varepsilon, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{11}$$

Following the technique applied in Theorem 2.2 we can show that

$$\lim_{n\to\infty} M(u_{r_n}, u_{s_n}, t) = 1 - \varepsilon, \quad t > 0 \tag{12}$$

Now from (9)

$$\begin{aligned}\phi(M(u_{r_n+1}, u_{s_n+1}, t)) =& \phi(M(f(x_{r_n}), f(x_{s_n}), t)) \\ &\leq k_1[\phi(M(g(x_{r_n}), g(x_{s_n}), t)) + k_2(\phi(M(g(x_{r_n}), f(x_{r_n}), t)) \\ &\quad + \phi(M(g(x_{s_n}), f(x_{s_n}), t)))]\end{aligned}$$

Taking $n \to \infty$ and using (10) and (12) we have
$\phi(1-\varepsilon) \leq k_1\phi(1-\varepsilon) < \phi(1-\varepsilon)$
a contradiction. Hence $\{u_n\}$ is Cauchy sequence.
Suppose that $g(X)$ is complete, then there exists $v \in g(X)$ such that $\lim_{n\to\infty} u_n = v$. Also, for some $z \in X$ we have $g(z) = v$.
Now,

$$\begin{aligned}\phi(M(f(z), u_{n+1}, t)) =& \phi(M(f(z), f(x_n), t)) \\ \leq& k_1[\phi(M(g(z), g(x_n), t)) + k_2(\phi(M(f(z), g(z), t)) \\ &+ \phi(M(f(x_n), g(x_n), t)))]\end{aligned}$$

For $n \longrightarrow \infty$,

$$\phi(M(f(z), v, t)) \leq k_1[k_2\phi(M(f(z), v, t))]$$
$$\Rightarrow (1 - k_1.k_2)\phi(M(f(z), v, t)) = 0$$
$$\Rightarrow \phi(M(f(z), v, t)) = 0$$
$$\Rightarrow f(z) = v$$

Therefore $f(z) = v = g(z)$ i.e., z is the coincident point of f and g.
Next, from (9),

$$\begin{aligned}\phi(M(f(f(z)), f(z), t)) \leq & k_1[\phi(M(g(f(z)), g(z), t)) + k_2(\phi(M(f(f(z)), \\ & g(f(z)), t)) + \phi(M(f(z), g(z), t)))] \\ = & k_1[\phi(M(f(g(z)), g(z), t)) + k_2(\phi(M(f(f(z)), f(g(z)), t)) \\ & +\phi(M(f(z), g(z), t)))] \quad (\text{Since} \quad fg = gf) \\ = & k_1[\phi(M(f(f(z)), f(z), t)) + k_2(\phi(M(f(f(z)), f(f(z)), t)) \\ & +\phi(M(f(z), f(z), t)))] \\ = & k_1\phi(M(f(f(z)), f(z), t))\end{aligned}$$

$$\Rightarrow (1 - k_1)\phi(M(f(f(z)), f(z), t)) = 0$$

$$\Rightarrow \phi(M(f(f(z)), f(z), t)) = 0$$

$$\Rightarrow f(f(z)) = f(z) = v$$

Similarly, $g(g(z)) = g(z) = v$.

Hence v is a common fixed point of f and g. If v_1 is another common fixed point of f and g, then

$$\begin{aligned}\phi(M(f(v), f(v_1), t)) \leq & k_1[\phi(M(g(v), g(v_1), t)) + k_2(\phi(M(f(v), g(v), t)) \\ & +\phi(M(f(v_1), g(v_1), t)))]\end{aligned}$$

$$\Rightarrow \phi(M(v, v_1, t)) \leq k_1[\phi(M(v, v_1, t)) + k_2(\phi(M(v, v, t)) + \phi(M(v_1, v_1, t)))]$$

$$\Rightarrow (1 - k_1)\phi(M(v, v_1, t)) = 0$$

$$\Rightarrow v = v_1$$

Hence f and g have a unique common fixed point in X. □

EXAMPLE 2.7. For $X = \{1, 2, 3, 4, 5\}$, let M be a fuzzy set on $X \times X \times [0, \infty[$ given by $M(x, y, t) = \dfrac{t}{t + d(x, y)}$, $t > 0$, where $d(x, y) = |x - y| \ \forall \ x, y \in X$.

Then (X, M, T) is a complete fuzzy metric space with respect to the t-norm $T(a, b) = \min\{a, b\}$, $a, b \in [0, 1]$. Define $f, g : X \longrightarrow X$ by
$f(1) = f(2) = 2, f(3) = f(4) = f(5) = 1$ and
$g(1) = g(2) = 2, g(3) = g(4) = 3, g(5) = 1$.
Also let $\phi(\lambda) = 1 - \lambda$. Then f and g satisfy (9), and hence from Theorem 2.6, we have f and g have a unique common fixed point.

In view of Theorem 2.6, we formulate the following result without proof.

THEOREM 2.8. *Let (X, M, T) be a fuzzy metric space, ϕ be the altering distance function and f and g be two self mappings on X such that*

$$\begin{aligned}\phi(M(f(x), g(y), t)) \leq k_1[&\phi(M(f(x), g(x), t)) + \phi(M(f(y), g(y), t))]+\\ k_2[&\phi(M(f(x), f(y), t)) + \phi(M(g(x), g(y), t))]\\ &+ k_3\phi(M(f(y), g(x), t))\end{aligned} \tag{13}$$

where $x, y \in X$, $t > 0$ and $k_1 + 2k_2 + k_3 < 1$, $0 < k_1, k_2, k_3 < 1$. Moreover (f, g) is asymptotically regular at x_0 and commutes with each other.If the range of f is a subset of range of g and if $f(X)$ or $g(X)$ is a complete subspace of X then f and g have a unique common fixed point.

Now we derive a common fixed point theorem for three compatible self mappings.

THEOREM 2.9. *Let f, g and h be self mappings on the complete fuzzy metric space (X, M, T) and ϕ be altering distance function such that the following conditions are satisfied:*

(I) $\phi(M(f(x), f(y), t))$
$\leq k_1\phi(M(f(x), g(y), t))[\phi(M(g(x), h(x), t)) + \phi(M(g(y), h(y), t))]$
$+k_2\phi(M(g(x), h(y), t))[\phi(M(h(x), f(x), t))+\phi(M(h(y), f(y), t))]$
$+k_3\phi(M(h(x), f(y), t))[\phi(M(f(x), g(x), t))+\phi(M(f(y), g(y), t))]$
for all $x, y \in X$ and $0 < k_1, k_2, k_3 < 1$,

(II) *g and h are continuous mappings, and each pair (f, g), (f, h) and (g, h) are compatible,*

(III) *there exists a sequence $\{x_n\}$ in X which is asymptotically regular with respect to each pair (f, g), (f, h) and (g, h).*

Then f, g and h have a common unique fixed point in X.

Proof. From condition (I), for $n, m \in \mathbb{N}, t > 0$

$$\begin{aligned}\phi(M(f(x_n), f(x_m), t)) \leq{}& k_1\phi(M(f(x_n), g(x_m), t))[\phi(M(g(x_n), h(x_n), t)) \\ &+\phi(M(g(x_m), h(x_m), t))] + k_2\phi(M(g(x_n), h(x_m), t)) \\ &[\phi(M(h(x_n), f(x_n), t)) + \phi(M(h(x_m), f(x_m), t))] \\ &+k_3\phi(M(h(x_n), f(x_m), t))[\phi(M(f(x_n), g(x_n), t)) \\ &+\phi(M(f(x_m), g(x_m), t))],\end{aligned}$$

which shows that

$$\lim_{n,m\to\infty} \phi(M(f(x_n), f(x_m), t)) = 0,$$
$$\text{i.e.,} \lim_{n,m\to\infty} M(f(x_n), f(x_m), t) = 1$$

So, the sequence $\{f(x_n)\}$ is a Cauchy sequence.
Since X is complete, $\{f(x_n)\}$ converges to some $z \in X$ (say).
Now,

$$M(g(x_n), z, t) \leq T(M(g(x_n), f(x_n), t), M(f(x_n), z, t))$$

Since $\{x_n\}$ is asymptotically regular with respect to the pair (f, g) and $f(x_n)$ converges to z,

$$\lim_{n\to\infty} M(g(x_n), z, t) \leq T(1, 1) = 1,$$
$$\text{i.e.,} \lim_{n\to\infty} M(g(x_n), z, t) = 1,$$

shows that $\{g(x_n)\}$ converges to z.
Similarly, we can show that $\{h(x_n)\}$ converges to z.
Again,

$$M(f(g(x_n)), g(z), t) \geq T(M(f(g(x_n)), g(f(x_n)), t), M(g(f(x_n)), g(z), t))$$

Since f and g are compatible and g is continuous,

$$\lim_{n\to\infty} M(f(g(x_n)), g(z), t) \geq T(1, 1)$$
$$\Rightarrow \lim_{n\to\infty} M(f(g(x_n)), g(z), t) = 1$$
$$\text{i.e.,} f(g(x_n)) \longrightarrow g(z).$$

Similarly, $f(h(x_n)) \longrightarrow h(z)$.
Now from condition (I), using the continuity of g and h,

$$\phi(M(f(g(x_n)), f(h(x_n)), t)) \leq k_1\phi(M(f(g(x_n)), g(h(x_n)), t))$$
$$[\phi(M(g(g(x_n)), h(g(x_n)), t)) + \phi(M(g(h(x_n)), h(h(x_n)), t))]$$
$$+k_2\phi(M(g(g(x_n)),h(h(x_n)), t))[\phi(M(h(g(x_n)), f(g(x_n)), t))$$
$$+\phi(M(h(h(x_n)), f(h(x_n)), t))] + k_3\phi(M(h(g(x_n)), f(h(x_n)), t))$$
$$[\phi(M(f(g(x_n)), g(g(x_n)), t)) + \phi(M(f(h(x_n)), g(h(x_n)), t))]$$

Taking limit as $n \longrightarrow \infty$,

$$\phi(M(g(z), h(z), t)) \leq k_2\phi(M(g(z), h(z), t)).\phi(M(g(z), h(z), t))$$

If $\phi(M(g(z), h(z), t)) \neq 0$, then

$$1 \leq k_2\phi(M(g(z), h(z), t))$$

which is a contradiction as $k_2 < 1$. Therefore

$$\phi(M(g(z), h(z), t)) = 0$$
$$\Rightarrow M(g(z), h(z), t) = 1$$
$$\Rightarrow g(z) = h(z)$$

Again,

$$\phi(M(f(g(x_n)), f(z), t)) \tag{14}$$
$$\leq k_1\phi(M(f(g(x_n)), g(z), t))[\phi(M(g(g(x_n)), h(g(x_n)), t))$$
$$+\phi(M(g(z), h(z), t))] + k_2\phi(M(g(g(x_n)), h(z), t))$$
$$[\phi(M(h(g(x_n)), f(g(x_n)), t)) + \phi(M(h(z), f(z), t))]+$$
$$k_3\phi(M(h(g(x_n)), f(z), t))[\phi(M(f(g(x_n)), g(g(x_n)), t))$$
$$+ \phi(M(f(z), g(z), t))]$$

Taking limit as $n \longrightarrow \infty$

$$\Rightarrow\phi(M(g(z), f(z), t))$$
$$\leq k_3\phi(M(h(z), f(z), t)).\phi(M(f(z), g(z), t)) \text{ (since } g(z) = h(z)) \tag{15}$$

Now, $M(h(z), f(z), t) \leq T(M(h(z), g(z), t), M(g(z), f(z), t))$

$$\Rightarrow M(h(z), f(z), t) \leq T(1, M(g(z), f(z), t)) = M(g(z), f(z), t) \tag{16}$$

Hence from (14) and (16),

$$\begin{aligned} \phi(M(g(z),f(z),t)) \leq& k_3\phi(M(g(z),f(z),t)).\phi(M(g(z),f(z),t)) \\ \Rightarrow \phi(M(g(z),f(z),t)) =& 0 \\ \Rightarrow g(z) = f(z) \end{aligned}$$

Similarly,

$$\begin{aligned} \phi(M(f(z),f(x_n),t)) \leq\ & k_1\phi(M(f(z),g(x_n),t))[\phi(M(g(z),h(z),t)) \\ & + \phi(M(g(x_n),h(x_n),t))] + k_2\phi(M(g(z),h(x_n),t)) \\ & [\phi(M(h(z),f(z),t)) + \phi(M(h(x_n),f(x_n),t))]+ \\ & k_3\phi(M(h(z),f(x_n),t))[\phi(M(f(z),g(z),t))+ \\ & \phi(M(f(x_n),g(x_n),t))] \end{aligned}$$

Taking limit as $n \longrightarrow \infty$ and using asymptotic regularity of x_n with respect to each pairsof mappings (f,g), (f,h) and (g,h),

$$\begin{aligned} &\phi(M(f(z),z,t)) = 0 \\ &\Rightarrow f(z) = z \end{aligned}$$

Hence $f(z) = g(z) = h(z) = z$ i.e., z is a common fixed point of f, g and h.

For the uniqueness, suppose there is another common fixed point of f, g and h, say v. Then from condition (I)

$$\begin{aligned} \phi(M(f(z),f(v),t)) \leq\ & k_1\phi(M(f(z),g(v),t))[\phi(M(g(z),h(z),t)) \\ & + \phi(M(g(v),h(v),t))] + k_2\phi(M(g(z),h(v),t)) \\ & [\phi(M(h(z),f(z),t)) + \phi(M(h(v),f(v),t))]+ \\ & k_3\phi(M(h(z),f(x_n),t))[\phi(M(f(z),g(z),t))+ \\ & \phi(M(f(v),g(v),t))] \end{aligned}$$

$$\begin{aligned} &\Rightarrow \phi(M(z,v,t)) = 0 \\ &\Rightarrow M(z,v,t) = 1 \\ &\Rightarrow z = v \end{aligned}$$

Therefore z is a unique common fixed point of f, g and h. □

EXAMPLE 2.10. Let $X = [0,2]$, $d(x,y) = |x-y|$ and $M(x,y,t) = 1 - \frac{d(x,y)}{g(t)}$ where $g : (0,\infty) \longrightarrow]2,\infty[$. Then (X,M,T) is a complete fuzzy metric space with respect to the t-norm $T(a,b) = \min\{a,b\}$, $a,b \in [0,1]$.

Consider the self mappings on f, g and h on X defined by $f(x) = \begin{cases} 2, \text{ if } x = 0, \\ 1, \text{ if } x \neq 0 \end{cases}$, $g(x) = x, \ \forall\, x \in X$ and $h(x) = \begin{cases} 2 - x, \text{ if } x \in [0,1], \\ 1, \text{ if } x \in (1,2] \end{cases}$.
Then each pair (f, g), (f, h) and (g, h) are compatible. Let the sequence $\{x_n\} \in X$ defined by $x_n = 1 - \frac{1}{3^n}$. Then $\{x_n\}$ is asymptotically regular with respect to (f, g), (f, h) and (g, h).
Moreover for $k_1 = k_2 = k_3 = \frac{3}{4}$, the condition (I) of Theorem 2.9 is satisfied.
Hence from Theorem 2.9, f, g and h have a common unique fixed point in X which is 1 in this case.

3. Conclusion

Throughout this paper, we have discussed some fixed point results for the class of asymptotically regular mappings in fuzzy metric spaces. The practical applications of these results for solving systems of fuzzy differential as well as integral equations is a scope for future study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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