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Investigating Children's Informal Knowledge and Strategies: The Case of Fraction Division

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This paper investigates what informal knowledge and strategies fifth-grade students brought to a classroom and how much they had potential to solve fraction division story problems. The findings show that most of the participants were engaged to understand the meaning of fraction division prior to their formal instruction at school. In order to solve the story problems, the informal knowledge related to fractions as well as division was actively utilized in student's strategies and justification. Students also used various informal strategies from mental calculation, direct modeling, to relational thinking. Formal instructions about fraction division at schools can be facilitated for sense-making of this complex fraction division conception by unpacking informal knowledge and thinking they might bring to the classrooms.

Keywords: fraction division, informal knowledge, story problems
MESC Classification: C32, F42
MSC2010 Classification: 97C30

I. INTRODUCTION

Research has provided meaningful evidence about the informal thinking of young children. Their knowledge is acquired through construction and transmission (Resnick, 1989). The construction of mathematical knowledge is affected by children's personal informal knowledge and understanding. These ideas were theorized as the development of mathematical understanding, which is part of a continuous process of organizing student's knowledge structures (Hatano, 1996; Pirie & Kieren, 1994; Von Glasersfeld, 1987). However, the connection and development of children's knowledge about fraction division are not strong from a constructivist perspective. As one example, the fraction division algorithm does not emerge from meaningful explorations with manipulatives (Borko et al., 1992).

Fraction division is often regarded as the most difficult domain for elementary students to understanding the meaning as well as for pre-service and in-service teachers to teach (Flores, 2002; Ma, 1999; Siebert, 2002; Sinicrope, Mick, & Kolb, 2002). To solve the fraction division problem, students have to use complicated knowledge from various content areas and it is hard to find everyday situations as a problem context (Ma, 1999). After formal instruction about invert-and-multiply (i.e., inverting the divisor fraction and then multiplying that reciprocal by the dividend fraction) or common denominator algorithm (determining how many times to be subtracted from common denominators), students are getting fixed how to use the algorithm appropriately, rather than understand the meaning of fraction division (Flores, 2002). Also, memorizing such algorithms doesn't ensure students would make a sense of fraction division based on informal understanding. That is, the gap between informal understanding and algorithms becomes bigger.

Researchers have provided the empirical findings of fraction operation to attempt to gain an understanding of children's fraction strategies. They found out children could solve the fraction story problem with informal understanding without formal instruction (Empson et al., 2005; Empson & Levi, 2011; Kent et al., 2015; Mack, 1990, 2001; Sharp & Adams, 2002). However, little research has been performed to explore students' informal understanding when they are engaged in the fraction division problem.

The purpose of this study was to evidence how students use the informal understanding of fraction division with story problems before formal instruction. In particular, this study sought to examine the following questions: (a) how is children's informal thinking to solve fraction division problems? (b) what mathematical strategies do children choose to solve the problems? (c) how does informal thinking affect their strategies? Our overarching premise is when students have the opportunity to question their mathematical ideas before problem-solving, they are able to move beyond their initial or intermediate conceptualizations about the mathematical ideas involved. As students reflect on their thinking in response to questions that are posed by the interviewer, students have the opportunity to revise, refine, and extend their ways of thinking about fraction division and an interview explores the student's understanding and knowledge. I would stress the role of representations in this dynamic since the ways in which mathematical ideas are represented are related to how a student can understand and use those ideas (Pirie & Kieren, 1994).

II. CONCEPTUAL FRAMEWORK

1. FOUNDATION OF MATHEMATICAL UNDERSTANDING

In many research projects, how students understand mathematics and how understanding is developed have been an issue for a long time. Mathematical understandings also have been approached in various ways such as instrumental and relational, intuitive and formal, and conceptual and procedural understanding. Researchers theorized that the development of understanding is the non-linear and dynamic process (Hatano, 1996; Mack, 2001; Pirie & Kieren, 1994). For example, Pirie and Kieren (1994) described eight leveled domains within the growth of mathematical understanding, on any specific topic: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventing. The most basic understanding in the process was called primitive knowing, a starting place for the development of any particular mathematical understanding and what a student can do initially (e.g. to solve fraction addition problem, a student has at least usable knowledge of fraction words, equivalence, and part-part-whole reasoning). When students become stuck with some higher-level understanding, they can return to prior types of thinking to one's current, inadequate understanding. Student's flowing back and forth between and among informal, intuitive, and symbolic types of understanding is not extraordinary in any specific mathematical contents (Sharp & Adams, 2002). When students return their initial understanding, they have a chance to connect complex ideas and restructure their prior knowledge. However, students might need guided assistance to stimulate their informal knowledge or to return to initial understanding during the instructions.

2. INFORMAL KNOWLEDGE

Researchers have focused on student's informal knowledge. Mathematical knowledge begins with individual knowledge of a specific example (concrete knowledge) and gradually broadens the specific example to a generalization (abstract knowledge) (Resnick, 1992). Ginsburg (1977) described the development of mathematical knowledge as three successive phases: (1) direct perception (concrete knowledge based on appearances), (2) informal knowledge (concrete knowledge based on everyday experiences), and (3) formal knowledge (school-taught symbolic knowledge). Informal knowledge is contrasted terminology to formal knowledge, students have informal knowledge before formal instruction in school (Baroody & Coslick, 1998). Mack (2001) emphasized informal knowledge is "applied, circumstantial knowledge constructed by an individual's response to their real-life experience" (p.267). Also, informal knowledge can be easily connected to other knowledge regardless of the accuracy of knowledge (Leinhardt, 1988).

Students feel comfortable struggling with problems using their informal mathematical knowledge (Sharp & Adams, 2002). When solving story problems of fraction division, the majority of children resolved the situations using division concepts built on knowledge from their existing whole-number knowledge about division. However, such informal knowledge could cause misconceptions about fraction division (Streefland, 1993). For example, they wonder how a quotient could be larger than the dividend because they understand the iterative subtractions technique of division in their collective conceptual knowledge base.

1) Informal Strategies

In mathematics, informal knowledge has been studied from arithmetic skills and concepts such as counting or the whole number operation. However, procedural fluency builds from an initial exploration and discussion of number concepts to using informal reasoning strategies and the properties of operations to develop general methods for solving problems (NCTM, 2014). Teachers give opportunities for students to realize their unnoticed informal knowledge and build up formal knowledge by discussing what they already know. Researchers emphasized the informal knowledge in fraction division before the instruction about the formal algorithms (Baroody & Coslick, 1998; Kent, Empson, & Nielsen, 2015; Sharp & Adams, 2002; Siebert, 2002; Van de Walle, 2004).

2) The Informal Strategy of Fraction Division

Empson and her colleagues (2011, 2015) found a variety of strategies in fraction division. Most students used direct modeling or using fraction relationship strategies when they solve story problems of fraction division. First, the direct modeling strategy in fraction division is to represent the relationship between the quantity and the situation with drawings. If allowed to use objects or drawing, most students – with little help- solve problems without formal instruction. Second, using fraction relationship solves the problems by iterative subtraction, addition, and multiplication with a fraction. Comparing to direct modeling, this strategy requires more abstract thoughts and symbolic representations. For example, when students solve a fraction division problem such as "Sheila drinks ¾ of a cup of water for every mile that she hikes. Her water bottle holds 5 cups of water. How far can she hike before her water runs out?" (Empson & Levi, 2011, p. 197), they approach the problem with many informal strategies. Some students might draw 5 circles to represent 5 cups of water, then count every group of ¾ in the partitioned circles with fourths. This is direct modeling by using their intuitive drawings to represent the situation. On the other hand, other students might solve the fraction division problem by adding fractions (3/4+3/4+3/4+3/4+3/4+3/4+2/4), then figure out six times of ¾ as 6 miles and 2/4 as 2/3 mile. As a result, the answer would be 6 2/3 miles. This group of students uses proportional reasoning to find the partial value of 2/3 mile, *1 mile: ¾ cup=□mile: 2/4 cup*.

III. METHODOLOGY

1. PARTICIPANTS

To take a snapshot of how students who didn't have formal instruction solve fraction division tasks with the informal understanding and knowledge, 6 fifth-graders (10-11 years old) were selected, Carrie, Lilly, Jung-Mi, Min-Hwang, Eun-Ji, and Hyun-Ji (pseudonyms) in a Midwestern city in the US. All the students were in the same school district and volunteered to participate in the interviews. Carrie and Lilly were native students and others were English language learners (ELL). All children were in the first semester of the school year and they were expected not to have formal instruction about fraction division prior the participation.

2. INTERVIEW PROTOCOL

1) Clinical Interview

To describe the complex cognitive system relevant to informal fraction knowledge, a standardized clinical interview method (Ginsburg, 1997) was employed. The interview protocol was developed by the assistance of a mathematics education researcher and another doctoral student (see Appendix). The other doctoral student, the second researcher, and I performed a clinical interview separately to collect the interview data. During the interviews, at first, we read a prompt of the task verbally, then handed the worksheet with tasks to the students. For exploring initial thoughts about tasks, students were asked, 'What do you see in your mind's eye when you think about this problem?' Students were supposed to articulate their prior knowledge, experience, and strategy in the previous problem-solving experience. Then, they solved tasks with the ways that work best for them. To probe students' ideas, they were asked supporting, extending, and justifying questions (e.g. Can you tell me your strategy? How did you decide to…?).

2) Interview Tasks

There are 5 story problems set of fraction division (Table 1). The participants were engaged in 2 partitive (tasks a and d) and 3 measurements (tasks b, c, and e) division problems. In the beginning, if students had struggled in the first fraction division problem ($4\frac{1}{2} \div 3$), they were asked to solve the whole number partitive division (e.g., $5 \div 3$, $12 \div 3$) in Table 2. The word problems were revised from the prior research about fraction division and equal sharing (Bulgar, 2003; Empson et al., 2005; Sharp & Adams, 2002). When revising the tasks, we considered children's context such as popular snacks and plausible social events and mathematical conceptions of divisions. To connect to children's experience, brownie, pizza, licorice, cake, and chocolate candy situations were illustrated in the interview protocol. Such real-life contexts enhanced fraction thinking and knowledge and facilitated the development of the whole number division algorithm (Sharp & Adams, 2002; Streefland, 1991, 1993). On the other hand, in terms of the division, the tasks were categorized as measurement division (determining the number of groups) and partitive division (determining the size of each group) (Warrington, 1997; Sinicrope et al., 2002; Carpenter, Fennema, Franke, Levi & Empson, 2015). Both division conceptions are similar but distinct. When students dealt with these two different types of fraction division tasks, they were supposed to apply their informal knowledge (Siebert, 2002).

3) Number Choices

In fraction learning, specific values of fractions might impact students' informal understanding (Pothier & Sawada, 1983; Sharp & Adams, 2002). Earlier age students might be comfortable with the whole number, 1/2, 1/4, and 1/8 rather than 1/3, 1/5, and 1/6. Such even-sized denominators are directly related to halving strategy, whereas students could model an odd number denominators after they begin to develop strategies for equal sharing into any number of pieces (Sharp & Adams, 2002).

Table 1. Main fraction division tasks

Indicated Division (division type)		Problems
(a)	$4\frac{1}{2} \div 3 = 1\frac{1}{2}$ (partitive)	There are 4 ½ brownies left after a birthday party. If 3 friends share the brownies evenly, how much brownie does each friend receive?
(b)	$4 \div \frac{2}{3} = 6$ (measurement)	Alberto wants to invite some friends to the pizza shop for his birthday. His mom gave him enough money to buy 4 pizzas. If he needs to make sure that each child, including himself, will have ⅔ of a pizza to eat, how many children can he invite?
(c)	$3\frac{1}{2} \div \frac{1}{4} = 14$ (measurement)	Sara has a licorice rope (or fruit roll) that is 3 ½ feet long. Each serving of licorice is ¼ of a foot. How many servings can Sara make with the licorice?
(d)	$\frac{1}{2} \div 3 = \square$ (partitive)	Three friends share half of a cake equally. How much of the original cake does each friend get?
(e)	$\frac{3}{4} \div \frac{1}{2} = 1\frac{1}{2}$ (measurement)	Maura is making holiday cookies for her family. She has ¾ pound of chocolate candies. Each batch of cookies requires ½ pound of chocolate candies. How many batches of cookies can she make?

The first task (a) is a relatively easy one between complex fractions and the whole number. 4 1/2 could be modeled as 9/2, and students might divide 3 groups. Because 9 pieces of half are multiple of 3 without remainder, students were anticipated to have easy access to a halving strategy. For struggling students, with this task, we prepared 2 supplementary tasks of the partitive division with the whole number (a-1, a-2). The difference between these two tasks is a sharing situation and the number type of quotient. The former task is a similar situation with the task (a), and the quotient is the fractional amount. However, in the latter case, the quotient is the whole number. If the student struggled with task a-1), then the interviewer was able to use task (a-2).

Table 2. Supplementary tasks for struggling students at task (a)

Indicated Division (division type)		Problems
(a-1)	5 ÷ 3= 5/3 (partitive)	There are 5 cookies. Three friends want to share the cookies. How many cookies can each friend get?
(a-2)	12 ÷ 3= 4 (partitive)	Seth has 12 flowers to put into some vases. He wants to put all the flowers into vases so that 3 flowers are in each vase. How many vases will he need?

The second task (b) is the relationship between the whole number and fraction, as the opposite relationship with the first one. The task is finding the number of the person who could invite from Alberto. If students model the denominator 3 with drawings or misunderstand the meaning of 2/3, the interviewers could substitute 2/3 to 1/2. Depending on the selection of fractions, students could divide 12 pieces of pizza respectively to 6 friends and 8. It was not surprising for students to misunderstand 'including himself' part. To get a correct solution, students had to subtract one child, Alberto. They might also be struggling with a bigger quotient than with a dividend because the dividend is always bigger than quotient in the whole number division.

The third task (c) is the measurement division problem with fractions. In the first and second tasks, pictorial representation would be rectangle or circle. However, 'licorice rope or fruit roll' is modeled as a linear representation. The ability to use different representations with real-life contexts could be revealed from this task. In addition, the denominator is 2 and 4, that is students continue to use halving strategy in the process of direct modeling.

The fourth task (d) is a partitive division, but the dividend is smaller than 1. Since students need to think part of the whole, this task might be not easy. To understand this problem, they were required to divide a half with three equal shares. Then, the interpretation of quotient could be described from the whole original cake.

The fifth task (e) is a measurement division as the most challenging one. This task has the remainder in the context. After subtracting one batch from 3/4, the remainder is 1/4. In this situation, I tried to find answers to the following questions: how do students deal with the remainder, how do they interpret with the problem context, and how do they use equivalent fractions (e.g., 1/2 = 2/4) to solve the task?

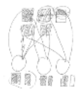
3. DATA SOURCE

To certify the validity and reliability of this study, I triangulated the different sources of data (Merriam & Tisdell, 2016; Yin, 2014). Field notes, transcribed interview videos, reflection notes, and students' worksheets were collected. First, as the field note data, we wrote about any important idea which comes up within the ongoing interview on the spot. Second, we recorded an interview with a video. When reviewing the video, verbatim written transcripts were produced from the video-recordings. Third, reflection notes were used for observing a child's behaviors very carefully such as fingers, whispers, attention, facial expression and gesture (Ginsburg, 1997). Fourth, we collected the paper worksheet as artifacts.

4. DATA ANALYSIS

To analyze the transcript data, a two-level coding cycle (Saldana, 2009) was used. During the First Cycle coding process, I did open coding line by line (Corbin & Strauss, 2008). From the transcribed interview, I found out preliminary patterns and categorized what I coded as the following Table 3. In the Second Cycle coding process, the final coding scheme was refined: (1) initial thinking, (2) prior knowledge, (3) explaining of strategy, (4) mathematical justification, and (5) others. Students' solutions were also coded as (a) type of strategy, (b) type of representation, and (c) fraction terminology. These codes were also drawn from prior literature to analyze the children's strategies (Empson, 1999; Empson et al., 2005).

Table 3. Final coding scheme

Code	Description	Example
First cycle		
Initial thinking	Students interpreted the problem context as numbers, operations, and a detailed strategy.	*I pictured 4 pizzas. And Alberto drawing it out, splitting 3 pieces. Cause said 2/3 which means the whole has 3 pieces.*
Prior knowledge	Students had informal knowledge about fraction division from school and out-of-school.	*Divide always makes smaller.* *4 brownies and half-eaten brownie. And 4…3 kids are trying to split it.*
Explaining strategy	Students explained their written representation after solving the problem.	*I colored 2 pieces of one of 4 pizzas. And little one on tops means one person then I color one left over one. one more from second pizza. Then I keep doing that until there was the number of people he can invite.*
Justification	Students verbally explained the mathematical meaning from the interviewer's question.	*I: What is the meaning of these lines?* *S: This line is dividing the 1 foot of licorice and this line is fraction line.*
Others	Not related to problem-solving	*"How many problems are left?"*
Second cycle		
Type of strategy	Students solve the problem with a specific strategy.	 *1 1/2 + 1 1/2 + 1 1/2 = 4 1/2*
Type of representation	Students utilized pictorial, symbolic, verbal representations when solving the problem.	
Fraction terminology	Students said the formal word about fractions when explaining their thinking.	*This problem has a remainder.* *I changed denominator 4 to make the same denominator.*

IV. FINDINGS

We analyzed fifth grader's informal knowledge and strategies about fraction division by examining how they solve the story problems. In terms of informal knowledge, students represent their idea about fraction division with verbal and written representation. Before formal instruction of fraction division, students applied informal strategies, such as a) initial strategy, b) direct modeling, c) number relationship, and d) ratio strategy. In the following sections, I evidence children's distinct informal knowledge and strategies on the basis of how they solved the given fraction division problems.

1. INFORMAL KNOWLEDGE

In order to solve the fraction division problem, students were required to use prior knowledge about fractions as well as division. About fraction knowledge, students showed their understanding of unitizing and the ability to use fractional words. They also showed the prior knowledge of whole number operation such as subtraction related to division.

1) Informal Knowledge of Fractions

Unitizing. When students read the story problems, the most common knowledge to share the quantity was how to make the unit as a fraction. In the partitive division, when considering 'if 3 friends share 4 1/2 brownies, how much does each friend receive?', for example, Jung-mi was able to conceptualize 1/2 as a unit to distribute to the whole. In the measurement division problem, she was asked to figure out the number of children to invite a birthday with 4 pizzas when each share was 2/3. She approached this problem by counting how many 2/3 units in 4. With different division context, she consistently applied the concept of unitizing, and, similarly, other students moved forward to the next phase of the procedure by partitioning the unit fraction, when using this knowledge.

Fraction language. 3 students represented their idea about fractions with drawings. They drew a divided circle, rectangle and line segments. Depending on the problem situation, discrete and continuous representations were drawn. The number of divided pieces was affected by the denominator. For example, Lilly had trouble with representing 2/3 with a circle representation. With halving strategy, she lined three times, and it made 8 pieces of pizza. Finally, she arrived at V-shaped partitioning to divide three-part as a denominator 3.

When the participants justified their strategy, 2 students couldn't read the fraction with the proper word. For example, Min-Hwang read 1/2 as one-two and 2/3 as two-three. Even though interviewees read loudly the problems before solving, they didn't notice the fraction words. In other words, students interpreted from the written fraction number rather than from fraction words. Even though inadequate fraction words were used, this rarely affected problem-solving because they justified their understanding with mathematically valid knowledge.

2) *Informal Knowledge of Division*

The whole number operations. Students proved their knowledge about the whole number, decimal and fraction operation with diverse strategies such as adding-up, iterative subtraction, and multiplication. To deal with the pizza problem (Task b), for example, Eun-Ji used iterative subtraction, "4 - 2/3 = 3 1/3 - 2/3 = 2 2/3 - 2/3 = 2 - 2/3 = 1 1/3 - 2/3 = 2/3 - 2/3 =0. I counted the number of 2/3, so the answer is 6." When the student was asked about 12 ÷ 2, she said, "Because 12 – 2 – 2 – 2 – 2 – 2 – 2 = 0, so the answer is 6." Confronted with a fraction division problem, students initially translated the fraction into the whole number. This example stemmed from the whole-number operation with a similar structure of thinking.

2. INFORMAL STRATEGY OF FRACTION DIVISION

1) *Initial Strategy with Mind's Eye*

Based on the existing informal knowledge, students tended to use an already known strategy. When students were asked about the initial thinking about the problem, they contextualized the problem situation with a variety of levels. Some students focused on the operation or fraction numbers, the others concentrated on the detailed situation with an initial strategy. For example, as soon as Lilly heard the brownie problem (Task a), she said, "3 friends are sitting in the table, and middle of the table there are 4 and 1/2 brownies. And then, each person gets one….". In the original word problem, a table and a place for the brownie were not illustrated. However, she connected her experience and imagined the situation of sharing brownies with friends. At first, she said each person got one brownie by mentally decomposing 4 1/2 as 3 and 1 ½. Then, she could divide the leftover 1 1/2 with 3 friends again. Based on the detailed description and an intuitive initial strategy, she solved the problems with robust conceptual understanding and justification afterward. This initial strategy could play a role as guiding posts for the procedure of problem-solving.

2) *Direct Modeling*

As the initial response to the problem, students showed a consistent tendency of using strategy. Carrie, Jung-Mi, and Lilly used the direct modeling strategy, representing all quantities with drawings based on the problem situation. This strategy reflects children's informal understanding of fractional quantities as well as the division with counting (Carpenter et al., 2015).

Partitive division. The brownie problem (Tasks a) was categorized as a partitive division type. In this problem type, students needed to find the share of each person. When students solved this partitive fraction division problem, they modeled in the two-part. One is the dividend and the other is the divisor. Then, a piece of the dividend was corresponding with each divisor. For example, Jung-Mi represented each brownie with a rectangular shape (Figure 1). She then partitioned each shape into half and shaded 4 and a half to show the given quantity. To determine how many shares could be made by 4 1/2 brownies, Jung-mi delivered each half to a circle as a friend. 9 half brownies divided to 3 persons and they got 1 1/2 each as a share. She drew 1 1/2 brownies below each person, then she wrote, 'Each get 1 1/2 brownies'. This strategy reveals her understanding of decomposing the whole unit to the partitioned unit (a half) and reconstructing the partial units into new fractional quantity (1 ½).

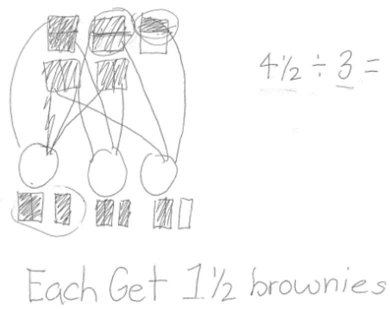


Figure 1. Jung-Mi's direct modeling of Task (a)

Measurement division. The pizza problem (Task b) was categorized as a measurement division type of problem. The purpose of a measurement division problem is to find out how many groups are in a given quantity. When students solved this type of division problem, they usually represented the whole quantity as a model. Also, the difference from the partitive division is using the counting strategies. For example, Jung-mi represented 4 circles as 4 pizzas (Figure 2). She partitioned each circle into 3 pieces. She shaded 2 pieces out of 3 of the first pizza, then shaded a third of the first pizza and other 1/3 piece in second. In order, she shaded 2 pieces five times, then finally shaded the sixth pizza. Every time she shaded two pieces, she numbered 1, 2, …., 6 as the number of children. This strategy reveals that she understood each pizza is a whole unit and a serving of 2/3 can be decomposed into 1/3 and 1/3. Her solution, with understanding and representation of the situation, helped determine that 6 children could come to the birthday party.

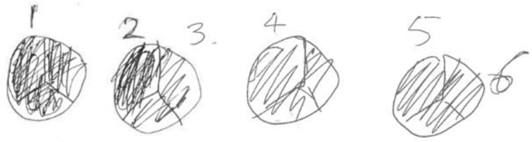


Figure 2. Jung-Mi's direct modeling of problem (b)

Partitioning. When students used the direct modeling strategy, the entry point of problem-solving is how to partition the whole with equal shares. In the case of one fractional quantity (Tasks a and b), students followed by the denominator (e.g., 1/2 was divided by 2 pieces and 2/3 by 3 pieces). In the case of two fractional quantities, they compared two quantities and selected the one fractional unit with the bigger denominator as the final unit. For example, in Task c, when two fractions (1/2 and ¼) were presented, the students partitioned either the whole unit into 4 pieces or half into 2 pieces. The other approach of partitioning is using the knowledge of equivalent fraction, which has the same number value with different partitioned pieces. For example, Hyun-Ji resolved the cake problem (Task d) by using an equivalent fraction (Figure 3). At first, she halved a circle as a half cake, and wrote, '1/2'. After a few seconds, she wrote '3/6' next to '1/2', and partitioned a whole circle into total 6 pieces. Then she justified her strategy with the following figure 3. First, she shaded a half to represent a half cake and wrote '1/2'. Next, she wrote '× 2, = 2/4' and drew a circle and shaded 2 pieces out of 4. Finally, she wrote '× 3, = 3/6' and drew a circle and shaded 3 pieces out of 6. She recognized the shaded half of the whole was invariant with a sequence of equivalent fractions (1/2, 2/4, 3/6).

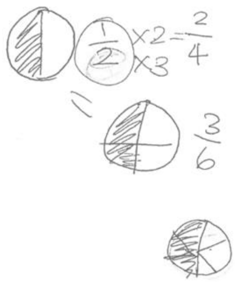


Figure 3. Hyun-Ji's partitioning with equivalent fractions of Task (d)

3) Number Relationship

The majority of strategies of Min-Hwang, Eun-Ji, and Hyun-Ji were represented by numerical symbols such as the whole number and fraction. Based on the whole number situation, they utilized iterative addition and subtraction, decimal division, multiplication, and/or proportional strategy, which need a more abstract understanding of the relationship between numbers than direct modeling.

Iterative addition. Ma (1999) suggested a model of knowledge package for understanding fraction division and the most fundamental knowledge was the meaning of addition. To use the number relationship, students set the entry and exit points. In fraction division, the former would be a divisor and the latter is a dividend. To make the whole quantity, a divisor as a unit was repeatedly added in order. For example, Eun-Ji solved the licorice problem (Task c) with iterative addition strategy (Figure 4). She represented the fractional quantity with fraction symbols. She transformed 3 1/2 into 3 2/4 with the equivalent fraction, then changed from mixed number to improper fraction as 14/4. Setting the exit point as 14/4, she added 1/4 fourteen times. Her transformation and addition reveal her understanding between the whole number and fraction. She understood how to change from mixed fraction to improper fraction because she knew that a whole was the same with 4 groups of 1/4. When she wrote fourteen 1/4 and calculated as 14/4, she knew that the addition of unit fraction was related to the numerator without changing the denominator. Also, she used the knowledge of the relationship between whole number and fraction: the meaning of red circle (1/4 + 1/4 + 1/4 + 1/4) was a whole (1), and the blue circle (1/4 + 1/4) was a half (1/2).

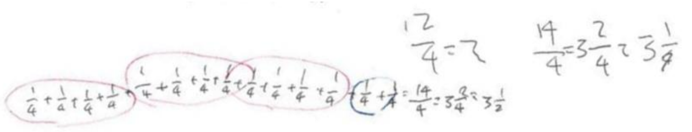


Figure 4. Eun-Ji's iterative addition strategy of #3 problem

Proportional strategy. In the pizza problem (Task b), children's approach had diverse strategies: direct modeling, iterative addition, and subtraction. However, this iterative addition has a different meaning. To solve the problem, Min-Hwang and Hyun-Ji began by using doubling strategy, adding 2 groups of 2/3 as 4/3(= 1 1/3). They understood and reasoned about the fractional quantities with re-unitizing 2 groups of 2/3 from the unit as 2/3. However, from the next step, they followed different ways of underpinning proportional thinking (Figure 5). Min-Hwang calculated 4 children and 6 children, then found 4 pizzas (1 2/3 + 1 2/3 +1 2/3 = 4). Therefore, the number of child is 6 (2 + 2 + 2 = 6). On the other hand, Hyun-Ji calculated 3 children, then found 2 pizzas (4/3 + 2/3 = 1 3/3 = 2). She doubled the 2 pizzas again to make 4 pizzas. Finally, she got 6 children (3 × 2 =

6). Both students applied proportional thinking by coordinating between the set of share and the set of children.

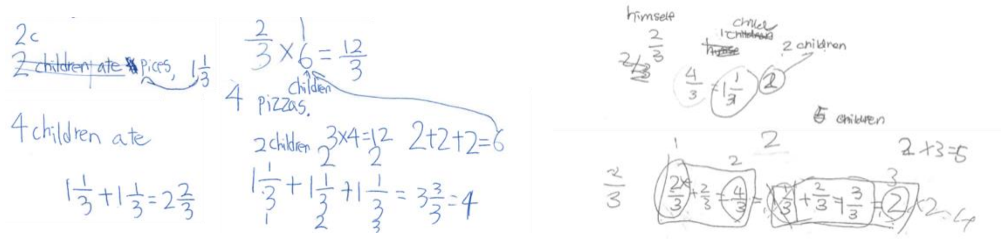


Figure 5. Doubling strategy of Task (b) (Left: Min-Hwang, Right: Hyun-Ji)

3. WHAT MAKES FRACTION DIVISION DIFFICULT?

To solve the fraction division meaningfully without formal instruction, a variety of informal knowledge worked together. However, fraction division is still difficult content for students. In this paper, significant evidence indicates the importance of rigorous informal knowledge and problem context.

1) The Relationship between Dividend and Quotient

Student's intuitive knowledge of fractions and division is a rich store to solve the story problems. However, this knowledge can hinder a student's conceptual understanding of fraction division. For example, after Jung-Mi solved the licorice rope problem (Task c) with drawings, she was asked to represent the situation with number expression. She counted 14 servings from the drawing and wrote '3 1/2 ÷ 1/4 = 14' at first. However, after a while, she changed from 14 to 14/4. She explained the reason, 'Cause I thought it couldn't work. … (14 is) bigger than 3 and dividing is just getting smaller.' when she solved the problem within context, she clearly understood what she got for the answer. However, when connected to the conception of the whole number division, she was confused with a larger quotient than the dividend.

2) Remainder

Even though students have a strong background in the whole number arithmetic rules, it might be not sure that they have an adequate understanding of fractions (Streefland, 1993). For example, when resolving the fraction division problem with the remainder (Task e), students tended to report that the remainder was 1/4 of candy. The contexts of the tasks were designed to push students toward thinking of remainders as "How many batches can you make with the remainder?" students couldn't answer easily and struggled the meaning of the remainder. Within the context of the problem, I should have asked as "How full is the last batch with the remainder?" To answer this question, the student would feel more comfortable with focusing on the unit itself (batch), rather than translation from the remainder to the unit (candy to batch).

V. CONCLUSION AND DISCUSSION

The purpose of this study was to explore how students solve the fraction division problems with informal thinking. The findings show that most of the fifth-graders were engaged to understand the meaning of fraction division before their formal instruction at school. In order to solve the story problem, the informal thinking related to the knowledge of fractions as well as division was actively utilized in student's strategies and justification.

Children have the ability to develop their own strategies to solve the problem based on context with an informal strategy (Carpenter et al., 2015; Saxe, 1988; Steffe, 1994). Direct modeling is the most basic idea to intuitively visualize the quantity with pictorial representation within a problem context. To successfully represent fractional quantity, partitioning the whole unit affects the student's application of the unitizing scheme (Mack, 2001; Steffe, 2003). From the whole quantity, students subtracted the same parts as the unit. Therefore, before the instruction of fraction division, students need more opportunities to exposure to these experiences to decide which unit is proper for the problem situation.

Based on prior knowledge, students constructed number relationships to solve the problem. The students generated a diverse number of expressions in the problem situation with quantity and operator. The property of fractions and operations turns as one of the strategies and becomes like the object to manipulate. Especially, the students tended to make the whole partitioned into the same parts as the unit. This informal strategy, of course, is not clear when solving the problems, but easy to make sense of in the context of the problem (Sharp & Adams, 2001). The rich connection with addition, subtraction, and multiplication also contributed to a conceptual understanding of fraction division (Ma, 1999; Kent et al., 2015).

Within limited participants and data, it might be hard to generalize the findings. In this study, all the participants were able to solve more than half of the tasks. This achievement is not subtle. Our children have the ability to solve such tasks based on prior informal knowledge and strategies. However, why our children are still not comfortable to solve fraction division tasks? One possible reason is a collision between informal knowledge and formal knowledge at school. Children, who have plentiful informal knowledge from out-of-school experience, had trouble with formal instruction (Nunes, Schilemann, & Carraher, 1993; Saxe, 1988). Teachers, teacher educators, researchers, and textbook authors should focus on unpacking student's informal thinking and connecting to formal instructions, rather than transmitting knowledge structures constructed by adults.

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Appendix

◆ **Interview protocol**

Overarching Question: How do children, prior to formal instruction, reason about fraction division story problems?

- Do they recognize these problems as division problems?
- Do they visualize (mentally construct) the situation?
- Do they make pictorial representations that accurately represent the situation?
- What mathematical strategies do children chose to solve the problems?
- How do visualizations connect with students' strategies for solving?
- Do they connect a number sentence to the problem?

tasks	a	b	C	d	e
Number type	Fraction ÷Whole number	Whole number ÷Fraction	Fraction ÷Fraction	Fraction ÷Whole number	Fraction ÷Fraction
equivalent	-	-	-	O	O
remainder	-	-	-	-	O
Expression	$4\frac{1}{2} \div 3 = 1\frac{1}{2}$	$4 \div \frac{2}{3} = 6$	$3\frac{1}{2} \div \frac{1}{4} = 14$	$\frac{1}{2} \div 3 = \frac{1}{6}$	$\frac{3}{4} \div \frac{1}{2} = 1\frac{1}{2}$
Division type	partitive	measure ment	measure ment	partitive	measure ment
literature	Empson et al., 2005	Bulgar, 2003	Sharp & Adams, 2002	Empson et al., 2005	Sharp & Adams, 2002

Introduction script (The purpose here is to introduce the purpose and expectations, and to establish rapport with the child.This can be flexible, but be clear that you are interested in how the child THINKS about the problems more than how correct (s)he is.

- What grade are you in? How do you like that?
- I'm interested in how children solve story problems. I'd like to ask you to solve a few story problems and I'll also ask you questions about what you are thinking. By doing this, I might be able to help other kids learn to solve story problems better. Of course, what you do here with me won't affect your grades in any way. Actually, I'm interested in how you are thinking about the problems and not whether or not you get them correct. It's ok to change your mind, it's ok if you get stuck, and you can ask questions anytime you want.

Lower level problems. If a child is not successful with problem (a) in the main set, the interviewer should try problem a-1 with the student. If the child is successful, try problem (a) again. If the child is not successful with a-1, the interviewer should try problem a-2 with the child. If the child is still not successful, the interview should be finished. If the child IS successful with a-2, go back to a-1, then proceed until the child is unsuccessful again. (A successful attempt is defined as a reasoned attempt in which the child is able to explain his/her thinking or procedure, not necessarily a correct answer.)

Partitive, sharing between friends, whole numbers with fraction result

a-1. There are 5 cookies. Three friends want to share the cookies. How much cookie can each friend get?

Partitive, whole numbers with whole number result

a-2. Seth has 12 flowers to put into some vases. He wants to put all the flowers into vases so that 3 flowers are in each vase. How many vases will he need?

Main problem set

If the child is unsuccessful with number 1, follow the instructions on the previous set of lower level problems.

Partitive, no equivalency, no remainder

a. There are 4 ½ brownies left after a birthday party. If 3 friends share the brownies evenly, how much brownie does each friend receive?

Measurement, no equivalency, no remainder

b. Alberto wants to invite some friends to the pizza shop for his birthday. His mom gave him enough money to buy 4 pizzas. If he needs to make sure that each child, including himself, will have ⅔ of a pizza to eat, how many children can he invite?

Measurement, no equivalency, no remainder

c. Sara has a licorice rope (or fruit roll) that is 3 ½ feet long. Each serving of licorice is ¼ of a foot. How many servings can Sara make with the licorice?

Partitive, equivalency, no remainder

d. Three friends share half of a cake equally. How much of the original cake does each friend get?

Measurement, equivalency, remainder

e. Maura is making holiday cookies for her family. She has ¾ pound of chocolate candies. Each batch of cookies requires ½ pound of chocolate candies. How many batches of cookies can she make?

Protocol Questions (for each problem that is presented to the child):

- (Read the question to the child. After initial reading, hand the question to the child on paper.) First of all, are there any words in the problem that you need help understanding? This is to make sure that the student has context familiarity.
- What do you think of when you hear this problem?
- What do you see in your mind's eye when you think about this problem? These questions openly explore what the child's initial thinking is about the problem. Does he/she have thoughts about problem type? connecting with experience?...forming some mental representation?...relate it to other math problems he/she has done in the past?
- I'd like you to solve the problem using the method that works best for you. You can record your work on the paper.
- Ask clarifying questions about the student's strategy.
    - Tell me about your strategy.
    - How did you decide to…?
    - What does (part of picture, etc.) represent?
- If student solves using a standard algorithm, ask the student to explain:
    - Tell me about your strategy.
    - How did you decide that this was a (division, etc.) problem?
    - How did you decide to form this number sentence?
    - Can you solve this problem different way?
- At the end of the problem set, if the student has NOT used a number sentence: Can you write a number sentence to describe these situations? (If short on time, just pick one or two.)