A FIXED POINT APPROACH TO THE STABILITY OF 3-LIE HOMOMORPHISMS AND 3-LIE DERIVATIONS

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of 3-Lie homomorphisms and 3-Lie derivations in 3-Lie algebras for Cauchy-Jensen functional equation.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [15] concerning the stability of group homomorphisms. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

A Lie algebra is a Banach algebra endowed with the Lie product

$$[x, y] := \frac{(xy - yx)}{2}.$$

Similarly, a 3-Lie algebra is a Banach algebra endowed with the product

$$\Big[[x, y], z\Big] := \frac{[x, y]z - z[x, y]}{2}.$$

Let A and B be two 3-Lie algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a 3-Lie homomorphism if

$$H([[x, y], z]) = [[H(x), H(y)], H(z)]$$

for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a 3-Lie derivation if

$$D\big([[x, y], z]\big) = [[D(x), y], z] + [[x, D(y)], z] + [[x, y,], D(z)]$$

for all $x, y, z \in A$ (see [17]).

We recall a fundamental result in fixed point theory.

Theorem 1.1 ([2, 5])**.** *Let* (X, d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $\alpha < 1$. *Then for each given element* $x \in X$, *either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [9] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4, 6, 12, 13]).

Throughout this paper, we suppose that A and B are two 3-Lie algebras. For convenience, we use the following abbreviation for a given mapping $f : A \to B$

$$D_\mu f(x, y, z) := f(\frac{\mu x + \mu y}{2} + \mu z) + f(\frac{\mu x + \mu z}{2} + \mu y) + f(\frac{\mu y + \mu z}{2} + \mu x)$$
$$- 2\mu f(x) - 2\mu f(y) - 2\mu f(z)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z \in A$.

Throughout this paper, assume that A is a 3-Lie algebra with norm $\|\cdot\|$ and that B is a 3-Lie algebra with norm $\|\cdot\|$.

2. Stability of 3-Lie Homomorphisms in 3-Lie Algebras

We need the following lemmas which have been given in for proving the main results.

Lemma 2.1 ([16])**.** *Let X be a uniquely 2-divisible abelian group and Y be linear space. A mapping $f : X \to Y$ satisfies*

(2.1) $$f(\frac{x+y}{2} + z) + f(\frac{x+z}{2} + y) + f(\frac{y+z}{2} + x) = 2[f(x) + f(y) + f(z)]$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is additive.

Lemma 2.2 ([10])**.** *Let X and Y be linear spaces and let $f : X \to Y$ be a mapping such that*

(2.2) $$D_\mu f(x, y, z) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then the mapping $f : X \to Y$ is $\mathbb{C}$-linear.

Using the fixed point method, we investigate the Hyers-Ulam stability of 3-Lie homomorphisms in 3-Lie algebras associated to the functional equation (2.1).

Theorem 2.3. *Let $\varphi : A^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

(2.3) $$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in A$. Suppose that $f : A \to B$ is a mapping satisfying

(2.4) $$\|D_\mu f(x, y, z)\| \leq \varphi(x, y, z),$$

(2.5) $$\|f([[x, y], z]) - [[f(x), f(y)], f(z)]\| \leq \varphi(x, y, z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that

(2.6) $$\|f(x) - H(x)\| \leq \frac{1}{6(1-L)}\varphi(x, x, x)$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x = y = z$ in (2.4), we get

(2.7) $$\|3f(2x) - 6f(x)\| \leq \varphi(x, x, x)$$

for all $x \in A$.

Consider the set

$$S := \{g : A \to B, \quad g(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x, x, x), \ \forall x \in A\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [11]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in A$.

It follows from (2.7) that

$$\left\| g(x) - \frac{1}{2}g(2x) \right\| \leq \frac{1}{6}\varphi(x, x, x)$$

for all $x \in A$.

Let $g, h \in S$ be given such that $d(g, h)) = \varepsilon$. Then

$$\| g(x) - h(x) \| \leq \varepsilon\varphi(x, x, x)$$

for all $x \in A$. Since

$$\left\| \frac{1}{2}g(2x) - \frac{1}{2}h(2x) \right\| \leq \frac{1}{2}\varepsilon\varphi(2x, 2x, 2x) \leq L\varepsilon\varphi(x, x, x)$$

for all $x \in A$, $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.7) that $d(f, Jf) \leq \frac{1}{6}$.

By Theorem 1.1, there exists a mappings $H : A \to A$ satisfying the following:

(1) H is a fixed point of J, i.e.,

(2.8) $$H(x) = \frac{1}{2}H(2x)$$

for all $x \in A$. The mapping H is a unique fixed point of J. This implies that H is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\| f(x) - H(x) \| \leq \mu\varphi(x, x, x)$$

for all $x \in A$;

(2) $d(J^l f, H) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l \to \infty} \frac{1}{2^l} f\left(2^l x \right) = H(x)$$

for all $x \in A$;

(3) $d(f, H) \leq \frac{1}{1-L}d(f, Jf)$, which implies

$$\| f(x) - H(x) \| \leq \frac{1}{6(1 - L)}\varphi(x, x, x)$$

for all $x \in A$. Thus we get the inequality (2.6).

It follows from (2.3) that

$$\|D_\mu H(x,y,z)\| = \lim_{n\to\infty} \frac{1}{2^n} \|D_\mu f(2^n x, 2^n y, 2^n z)\|$$

$$\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}^1$. So $D_\mu H(x,y,z) = 0$ for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By Lemma 2.2, the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.5) that

$$\|H([[x,y],z]) - [[H(x), H(y)], H(z)]\|$$

$$= \lim_{n\to\infty} \frac{1}{8^n} \|f([[2^n x, 2^n y], 2^n z]) - [[f(2^n x), f(2^n y)], f(2^n z)]\|$$

$$\leq \lim_{n\to\infty} \frac{1}{8^n} \varphi(2^n x, 2^n y, 2^n z) \leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0$$

for all $x, y, z \in A$. Thus

$$H([[x,y],z]) = [[H(x), H(y)], H(z)]$$

for all $x, y, z \in A$.

Therefore, the mapping $H : A \to B$ is a 3-Lie homomorphism. □

Corollary 2.4. *Let θ, r be positive real numbers with $r < 1$. Suppose that $f : A \to B$ is a mapping such that*

$$(2.9) \qquad \|D_\mu f(x,y,z)\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r),$$

$$(2.10) \qquad \|f([[x,y],z]) - [[f(y), f(z)], f(x)]\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that

$$(2.11) \qquad \|f(x) - H(x)\| \leq \frac{\theta}{2 - 2^r} \|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by taking $L = 2^{r-1}$ and $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. □

Theorem 2.5. *Let $\varphi : A^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$(2.12) \qquad \varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2} \varphi(x, y, z)$$

for all $x, y, z \in A$. Suppose that $f : A \to B$ is a mapping such that

$$\|D_\mu f(x, y, z)\|_B \leq \varphi(x, y, z),$$

$$\|f([[x, y], z]) - [[f(x), f(y)], f(z)]\| \leq \varphi(x, y, z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that

(2.13)
$$\|f(x) - H(x)\| \leq \frac{L}{6(1 - L)}\varphi(x, x, x)$$

for all $x \in A$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in A$.

It follows from (2.7) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{1}{3}\varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{6}\varphi(x, x, x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.3. $\square$

Corollary 2.6. *Let θr and q_3 be non-negative real numbers with $r > 1$. Suppose that $f : A \to B$ is a mapping satisfying (2.9) and (2.10). Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that*

(2.14)
$$\|f(x) - H(x)\| \leq \frac{\theta}{2^r - 2}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.5 by taking $L = 2^{1-r}$ and $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. $\square$

3. Stability of 3-Lie Derivations on 3-Lie Algebras

Using the fixed point method, we investigate the Hyers-Ulam stability of 3-Lie derivations in 3-Lie algebras associated to the functional equation (2.1).

Theorem 3.1. *Let* $\varphi : A^3 \rightarrow [0, \infty)$ *be a function satisfying* (2.3). *Suppose that* $f : A \rightarrow A$ *is a mapping satisfying*

$$\|D_\mu f(x, y, z)\| \leq \varphi(x, y, z),$$

(3.1) $\qquad \|f([[x, y], z]) - [[f(x), y], z] - [[x, f(y)], z] - [[x, y], f(z)]\| \leq \varphi(x, y, z)$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z \in A$. *Then there exists a unique 3-Lie derivation* $D : A \rightarrow A$ *satisfying* (2.6).

Proof. By the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $D : A \rightarrow A$ satisfying (2.6) and

$$D(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. It follows from (3.1) that

$$\|D([[x, y], z]) - [[D(x), y], z] - [[x, D(y)], z] - [[x, y], D(z)]\|$$

$$= \lim_{n \to \infty} \frac{1}{8^n} \|f([[2^n x, 2^n y], 2^n z]) - [[f(2^n x), 2^n y], 2^n z] - [[2^n x, f(2^n y)], 2^n z] - [[2^n x, 2^n x], f(2^n z)]\|$$

$$\leq \lim_{n \to \infty} \frac{1}{8^n} \varphi(2^n x, 2^n y, 2^n z) = 0$$

for all $x, y, z \in A$. So

$$D([[x, y], z]) = [[D(x), y], z] + [[x, G(y)], z] + [[x, y], D(z)]$$

for all $x, y, z \in A$. Therefore, the mapping $D : A \rightarrow A$ is a 3-Lie derivation. $\qquad \square$

Corollary 3.2. *Let* θ, r *be positive real numbers with* $r < 1$. *Suppose that* $f : A \rightarrow A$ *is a mapping satisfying* (2.9) *and* (2.10). *Then there exists a unique 3-Lie derivation* $D : A \rightarrow A$ *satisfying* (2.11).

Theorem 3.3. *Let* $\psi : A^3 \rightarrow [0, \infty)$ *be a function satisfying* (2.12). *Suppose that* $f : A \rightarrow A$ *is a mapping satisfying*

$$\|D_\mu f(x, y, z)\| \leq \psi(x, y, z),$$

$$\|f([[x, y], z]) - [[f(x), y], z] - [[x, f(y)], z] - [[x, y], f(z)]\| \leq \psi(x, y, z)$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z \in A$. *Then there exists a unique 3-Lie derivation* $D : A \rightarrow A$ *satisfying* (2.13).

Proof. By the proof of Theorem 2.5, there exists a unique $\mathbb{C}$-linear mapping $D : A \rightarrow A$ satisfying (2.13) and

$$D(x) := \lim_{n \to \infty} 2^n f(\frac{x}{2^n})$$

for all $x \in A$.

The rest of proof is similar to the proof Theorem 3.1. $\qquad\qquad\square$

Corollary 3.4. *Let θ, r be non-negative real numbers with $r > 1$. Suppose that $f : A \to B$ is a mapping satisfying (2.9) and (2.10). Then there exists a unique 3-Lie derivation $D : A \to A$ satisfying (2.14).*

Acknowledgments

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