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UTILIZING ISOTONE MAPPINGS UNDER MIZOGUCHI-TAKAHASHI CONTRACTION TO PROVE MULTIDIMENSIONAL FIXED POINT THEOREMS WITH APPLICATION

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Abstract. We study the existence and uniqueness of fixed point for isotone mappings of any number of arguments under Mizoguchi-Takahashi contraction on a complete metric space endowed with a partial order. As an application of our result we study the existence and uniqueness of the solution to integral equation. The results we obtain generalize, extend and unify several very recent related results in the literature.

1. Introduction

The concept of multidimensional fixed/coincidence point was introduced by Roldan et al. in [16], which is an extension of Berzig and Samet's notion given in [2]. For more details one can consult [1, $5-10$, 12, $14-21$].

Recently Ciric et al. [4] proved coupled fixed point theorems for mixed monotone mappings satisfying a generalized Mizoguchi-Takahashi condition in the setting of ordered metric spaces. Main results of Ciric et al. [4] extended and generalized the results of Gnana-Bhaskar and Lakshmikantham [3], Du [11] and Harjani et al. [13].

In this paper, we study the existence and uniqueness of fixed point for isotone mappings of any number of arguments under Mizoguchi-Takahashi contraction on a complete metric space endowed with a partial order. As an application we study the existence and uniqueness of the solution to integral equation. Our results improve, generalize and sharpen the results of Ciric et al. [4], Du [11], Harjani et al. [13] and several classical and very recent related results in the literature in metric spaces.

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2. Preliminaries

First of all, we recall the following notions. If X is a non-empty set, then we denote $X \times X \times ... \times X$ (n times) by X^n, where $n \in \mathbb{N}$ with $n \geq 2$. If elements x, y of a partially ordered set $(X, \preceq)$ are comparable that is, $x \preceq y$ or $y \preceq x$, then we will write $x \asymp y$. Let $\{A, B\}$ be a partition of the set $\Lambda_n = \{1, 2, ..., n\}$, that is, A and B are non-empty subsets of Λ_n such that $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$. We will denote

$$\begin{aligned}\Phi_{A,\,B} &= \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B\}\\ \Phi'_{A,\,B} &= \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A\}.\end{aligned}$$

Henceforth, let $\sigma_1, \sigma_2, ..., \sigma_n$ be n mappings from Λ_n into itself and let Υ be the $n-$tuple $(\sigma_1, \sigma_2, ..., \sigma_n)$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. For brevity, $g(x)$ will be denoted by gx.

A partial order $\preceq$ on X can be extended to a partial order $\sqsubseteq$ on X^n in the following way. If $(X, \preceq)$ be a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, then

$$x \preceq_i y \Rightarrow \begin{cases} x \preceq y, & \text{if } i \in A, \\ x \succeq y, & \text{if } i \in B. \end{cases} \tag{2.1}$$

Consider the following partial order on the product space X^n,

$$Y \sqsubseteq V \Leftrightarrow y_i \preceq_i v_i, \tag{2.2}$$

for all $Y = (y_1, y_2, ..., y_i, ..., y_n)$ and $V = (v_1, v_2, ..., v_i, ..., v_n) \in X^n$. Two points Y and V are comparable, if $Y \sqsubseteq V$ or $V \sqsubseteq Y$. Obviously, $(X^n, \sqsubseteq)$ is a partially ordered set.

Definition 2.1 ([15, 17, 18]). A point $(x_1, x_2, ..., x_n) \in X^n$ is called a $\Upsilon-$*fixed point* of the mapping $F : X^n \to X$ if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, ..., x_{\sigma_i(n)}) = x_i, \text{ for all } i \in \Lambda_n. \tag{2.3}$$

If we represent a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, that is, $\sigma = (\sigma(1), \sigma\ (2), ..., \sigma\ (n))$, then

(i) Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when $n = 2$, $\sigma_1 = (1, 2)$ and $\sigma_2 = (2, 1)$,

(ii) Berinde and Borcut's tripled fixed points are associated with $n = 3$, $\sigma_1 = (1, 2, 3)$, $\sigma_2 = (2, 1, 2)$ and $\sigma_3 = (3, 2, 1)$, (*iii*) Karapinar's quadruple fixed points are considered when $n = 4$, $\sigma_1 = (1, 2, 3, 4)$, $\sigma_2 = (2, 3, 4, 1)$, $\sigma_3 = (3, 4, 1, 2)$ and $\sigma_4 = (4, 1, 2, 3)$.

These cases consider A as the odd numbers in $\{1, 2, ..., n\}$ and B as its even numbers. However, Berzig and Samet [2] use $A = \{1, 2, ..., m\}$, $B = \{m + 1, ..., n\}$ and arbitrary mappings.

Definition 2.2 ([16])**.** Let $(X, \preceq)$ be a partially ordered space. We say that F has the *mixed monotone property* if F is monotone non-decreasing in arguments of A and monotone non-increasing in arguments of B, that is, for all $x_1, x_2, ..., x_n, y, z \in X$ and all i

$$y \preceq z \Rightarrow F(x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_n) \preceq_i F(x_1, ..., x_{i-1}, z, x_{i+1}, ..., x_n).$$

Definition 2.3 ([18, 21])**.** Let (X, d) be a metric space and define $\Delta_n, \rho_n : X^n \times X^n \to [0, +\infty)$, for $Y = (y_1, y_2, ..., y_n)$, $V = (v_1, v_2, ..., v_n) \in X^n$, by

$$\Delta_n(Y, V) = \frac{1}{n}\sum_{i=1}^{n} d(y_i, v_i) \text{ and } \rho_n(Y, V) = \max_{1\leq i\leq n} d(y_i, v_i).$$

Then Δ_n and ρ_n are metric on X^n and (X, d) is complete if and only if (X^n, Δ_n) and (X^n, ρ_n) are complete. It is easy to see that

$$\begin{aligned} \Delta_n(Y^k, Y) &\to 0 \Leftrightarrow d(y_i^k, y_i) \to 0 \text{ (as } k \to \infty) \\ \text{and } \rho_n(Y^k, Y) &\to 0 \Leftrightarrow d(y_i^k, y_i) \to 0 \text{ (as } k \to \infty),\ i \in \Lambda_n, \end{aligned}$$

where $Y^k = (y_1^k, y_2^k, ..., y_n^k)$ and $Y = (y_1, y_2, ..., y_n) \in X^n$.

Definition 2.4 ([21])**.** Let $(X, \preceq)$ be a partially ordered set and T be a self-mapping on X^n. It is said that T has an isotone property if, for any $Y_1, Y_2 \in X^n$, we have

$$Y_1 \preceq Y_2 \Rightarrow T(Y_1) \preceq T(Y_2).$$

Lemma 2.1 ([18, 20, 21])**.** Let $(X, d, \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Phi_{A, B}$ if $i \in A$ and $\sigma_i \in \Phi'_{A, B}$ if $i \in B$. Define $F_\Upsilon, G : X^n \to X^n$, for all $y_1, y_2, ..., y_n \in X$, by

$$\begin{aligned} F_\Upsilon(y_1, y_2, ..., y_n) &= \begin{pmatrix} F(y_{\sigma_1(1)}, y_{\sigma_1(2)}, ..., y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)}, y_{\sigma_2(2)}, ..., y_{\sigma_2(n)}), \\ ..., F(y_{\sigma_n(1)}, y_{\sigma_n(2)}, ..., y_{\sigma_n(n)}) \end{pmatrix}, \\ \text{and } G(y_1, y_2, ..., y_n) &= (gy_1, gy_2, ..., gy_n). \end{aligned}$$

(1) If F has the mixed $(g, \preceq)-$monotone property, then F_Υ is monotone $(G, \sqsubseteq)-$non-decreasing.

(2) If F is $d-$continuous, then F_Υ is also Δ_n-continuous and ρ_n-continuous.

(3) If g is $d-$continuous, then G is Δ_n-continuous and ρ_n-continuous.

(4) A point $(y_1, y_2, ..., y_n) \in X^n$ is a $\Upsilon-$fixed point of F if and only if $(y_1, y_2, ..., y_n)$ is a fixed point of F_Υ.

(5) A point $(y_1, y_2, ..., y_n) \in X^n$ is a $\Upsilon-$coincidence point of F and g if and only if $(y_1, y_2, ..., y_n)$ is a coincidence point of F_Υ and G.

(6) If $(X, d, \preceq)$ is regular, then $(X^n, \Delta_n, \sqsubseteq)$ and $(X^n, \rho_n, \sqsubseteq)$ are also regular.

(7) If there exists $y_0^1, y_0^2, ..., y_0^n \in X$ verifying $y_0^i \preceq_i F(y_0^{\sigma_i(1)}, y_0^{\sigma_i(2)}, ..., y_0^{\sigma_i(n)})$, for $i \in \Lambda_n$, then there exists $Y_0 \in X^n$ such that $Y_0 \sqsubseteq F_\Upsilon(Y_0)$.

(8) If F is a mixed monotone mapping, then F_Υ is an isotone mapping.

(9) If for each $i \in \Lambda_n$ and $y_i, v_i \in X$ there exists $z_i \in X$ which is $\preceq_i$ −comparable to y_i and v_i, then there exists $Z \in X^n$ which is $\sqsubseteq$ −comparable to Y and V.

3. MAIN RESULTS

Ciric et al. [4] introduced the family Ψ of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_ψ) ψ is non-decreasing,

(ii_ψ) $\psi(t) = 0 \Leftrightarrow t = 0$,

(iii_ψ) $\limsup_{t\to 0+} \frac{t}{\psi(t)} < \infty$.

Also Φ denote the family of all functions $\varphi : [0, +\infty) \to [0, 1)$ which satisfies $\lim_{r\to t+} \varphi(r) < 1$ for all $t \geq 0$.

Theorem 3.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $T : X \to X$ be a non-decreasing mapping for which there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d(Tx,\ Ty)) \leq \varphi(\psi(d(x,\ y)))\psi(d(x,\ y)), \tag{3.1}$$

for all $x, y \in X$ with $x \preceq y$. Suppose either

(a) T is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exists $x_0 \in X$ such that $x_0 \asymp Tx_0$, then T has a fixed point. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is $\preceq$ −comparable to x and y then the fixed point is unique.

Proof. Let $x_0 \in X$ be such that $x_0 \asymp Tx_0$. Take $x_1 \in X$ be such that $x_1 = Tx_0$, that is, $x_0 \asymp x_1$. Take $x_2 = Tx_1$, we have $Tx_0 \asymp Tx_1$, that is, $x_1 \asymp x_2$. Again, we have $Tx_1 \asymp Tx_2$. Proceeding by induction, we obtain a sequence $\{x_n\}_{n\geq 0}$ such that $x_{n+1} = Tx_n$ and $x_n \asymp x_{n+1}$ for each $n \geq 0$, that is,

$$x_0 \asymp x_1 \asymp x_2 ... \asymp x_n \asymp ..., \tag{3.2}$$

that is,

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq ... \text{ or } x_0 \succeq x_1 \succeq x_2 \succeq ... \succeq x_n \succeq ... \tag{3.3}$$

If $x_n = x_{n+1}$ for some $n \geq 0$, then T has a fixed point and the proof is complete. Assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Then, by using (3.1) and by the monotonicity of ψ, we have

$$\begin{aligned} \psi(d(x_n,\ x_{n+1})) &= \psi(d(Tx_{n-1},\ Tx_n)) \\ &\leq \varphi(\psi(d(x_{n-1},\ x_n)))\psi(d(x_{n-1},\ x_n)), \end{aligned}$$

which, by the fact that $\varphi < 1$, implies

$$\psi((d(x_n,\ x_{n+1}))) \leq \psi(d(x_{n-1},\ x_n)). \tag{3.4}$$

Thus (3.4) exhibit that the sequence $\{\psi(d(x_n, x_{n+1}))\}$ is non-increasing. Therefore, there exists some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \psi(d(x_n,\ x_{n+1})) = \delta. \tag{3.5}$$

Since $\varphi \in \Phi$, we have $\lim_{r\to\delta+} \varphi(r) < 1$ and $\varphi(\delta) < 1$. Then there exist $\alpha \in [0,\ 1)$ and $\varepsilon > 0$ such that $\varphi(r) \leq \alpha$ for all $r \in [\delta,\ \delta + \varepsilon)$. From (3.5), we can take $n_0 \geq 0$ such that $\delta \leq \psi(d(x_n, x_{n+1})) \leq \delta + \varepsilon$ for all $n \geq n_0$. Then from contractive condition (3.1) and by the monotonicity of ψ, for all $n \geq n_0$, we have

$$\begin{aligned} \psi(d(x_n,\ x_{n+1})) &= \psi(d(Tx_{n-1},\ Tx_n)) \\ &\leq \varphi(\psi(d(x_{n-1},\ x_n)))\psi(d(x_{n-1},\ x_n)) \\ &\leq \alpha\psi(d(x_{n-1},\ x_n)). \end{aligned}$$

Thus, we have

$$\psi(d(x_n,\ x_{n+1})) \leq \alpha\psi(d(x_{n-1},\ x_n)), \text{ for all } n \geq n_0. \tag{3.6}$$

Letting $n \to \infty$ in the above inequality and using (3.5), we obtain that $\delta \leq \alpha\delta$. Since $\alpha \in [0,\ 1)$, $\delta = 0$. Thus

$$\lim_{n\to\infty} \psi(d(x_n,\ x_{n+1})) = 0. \tag{3.7}$$

Since $\{\psi(d(x_n,\, x_{n+1}))\}$ is a non-increasing sequence and ψ is non-decreasing, $\{d(x_n,\, x_{n+1})\}$ is also a non-increasing sequence of positive numbers. This implies that there exists $\theta \geq 0$ such that

$$\lim_{n\to\infty} d(x_n,\ x_{n+1}) = \theta. \tag{3.8}$$

Since ψ is non-decreasing, we have

$$\psi(d(x_n,\ x_{n+1})) \geq \psi(\theta).$$

Letting $n \to \infty$ in this inequality and by using (3.7), we get $0 \geq \psi(\theta)$. It follows, by (ii_ψ), that $\theta = 0$. Thus, by (3.8), we obtain

$$\lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0. \tag{3.9}$$

Suppose that $d(x_n,\, x_{n+1}) = 0$, for some $n \geq 0$. Then, we have $x_n = x_{n+1} = Tx_n$, that is, x_n is a fixed point of T. Now, suppose that $d(x_n,\, x_{n+1}) \neq 0$, for all $n \geq 0$. Let

$$a_n = \psi(d(x_n,\ x_{n+1})),\ \text{for all } n \geq 0.$$

From (3.6), we have

$$a_n \leq \alpha a_{n-1},\ \text{for all } n \geq n_0.$$

Then, we have

$$\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{n_0} a_n + \sum_{n=n_0+1}^{\infty} \alpha^{n-n_0} a_{n_0} < \infty. \tag{3.10}$$

On the other hand, by (iii_ψ), we have

$$\limsup_{n\to\infty} \frac{d(x_n,\ x_{n+1})}{\psi\left(d(x_n,\ x_{n+1})\right)} < \infty. \tag{3.11}$$

Thus, by (3.10) and (3.11), we have $\sum d(x_n,\, x_{n+1}) < \infty$. It follows that $\{x_n\}$ is a Cauchy sequence in X. As X is complete and so there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x. \tag{3.12}$$

Suppose that (a) holds, that is, T is continuous. Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx$, that is, x is a fixed point of T.

Suppose now that (b) holds. Since $x_n \to x$, $x_n \asymp x$, therefore by (3.1) and by the monotonicity of ψ, we obtain

$$\psi(d(x_{n+1},\ Tx)) = \psi(d(Tx_n,\ Tx)) \leq \varphi(\psi(d(x_n,\ x)))\psi(d(x_n,\ x)),$$

which, by the fact that $\varphi < 1$, implies

$$\psi(d(x_{n+1},\ Tx)) \leq \psi(d(x_n,\ x)).$$

Since ψ is non-decreasing, we have

$$d(x_{n+1},\ Tx) \leq d(x_n,\ x).$$

Taking $n \to \infty$ in the above inequality and by using (3.12), we get $d(x, Tx) = 0$, that is, x is a fixed point of T.

Finally, we will prove the uniqueness of the fixed point. Suppose T has another fixed point y. From the assumption, there exists $z \in X$ such that $x \asymp z$ and $y \asymp z$. If $z = x$ or $z = y$, it is trivial. So let us now suppose that $z \neq x$ and $z \neq y$. Take $z_0 = z$ and $z_1 \in X$ such that $z_1 = Tz_0$. Then we have $z_0 \asymp x$, which implies that $Tz_0 \asymp Tx$, that is, $z_1 \asymp x$. Again, we have $Tz_1 \asymp Tx$, that is, $z_2 \asymp x$. Inductively, we can obtain $z_{n+1} = Tz_n$ and $z_n \asymp x$. Assume $x \neq z_n$ for all $n \geq 0$. Similarly, we have $z_n \asymp y$ and $z_n \neq y$ for all $n \geq 0$. By (3.1) and by the monotonicity of ψ, we obtain

$$\begin{aligned} \psi(d(z_{n+1},\ x)) &= \psi(d(Tz_n,\ Tx)) \\ &\leq \varphi(\psi(d(z_n,\ x)))\psi(d(z_n,\ x)), \end{aligned}$$

which, by the fact that $\varphi < 1$, implies

$$\psi(d(z_{n+1},\ x)) \leq \psi(d(z_n,\ x)). \tag{3.13}$$

Thus (3.13) display that $\{\psi(d(z_n,\ x))\}$ is a non-increasing sequence. Hence there exists some $\Delta \geq 0$ such that

$$\lim_{n\to\infty} \psi(d(z_{n+1},\ Tx)) = \Delta. \tag{3.14}$$

Since $\varphi \in \Phi$, we have $\lim_{r\to\Delta+} \varphi(r) < 1$ and $\varphi(\Delta) < 1$. Then there exist $\beta \in [0, 1)$ and $\varepsilon > 0$ such that $\varphi(r) \leq \beta$ for all $r \in [\Delta, \Delta+\varepsilon)$. From (3.14), we can take $n_0 \geq 0$ such that $\Delta \leq \psi(d(z_{n+1}, Tx)) \leq \Delta + \varepsilon$ for all $n \geq n_0$. Then from (3.1) and by the monotonicity of ψ, for all $n \geq n_0$, we have

$$\begin{aligned} \psi(d(z_{n+1},\ x)) &= \psi(d(Tz_n,\ Tx)) \\ &\leq \varphi(\psi(d(z_n,\ x)))\psi(d(z_n,\ x)) \\ &\leq \beta\psi(d(z_n,\ x)). \end{aligned}$$

Thus

$$\psi(d(z_{n+1},\ x)) \leq \beta\psi(d(z_n,\ x)), \text{ for all } n \geq n_0.$$

Letting $n \to \infty$ in the above inequality and by using (3.14), we obtain $\Delta \leq \beta\Delta$. As $\beta \in [0,\, 1)$ and so $\Delta = 0$. Thus by (3.14), we get

$$\lim_{n\to\infty} \psi(d(z_{n+1},\ x)) = 0. \tag{3.15}$$

Since $\{\psi(d(z_{n+1}, x))\}$ is a non-increasing sequence and ψ is non-decreasing, $\{d(z_{n+1}, x)\}$ is also a non-increasing sequence of positive numbers. This implies that there exists $\xi \geq 0$ such that

$$\lim_{n\to\infty} d(z_{n+1},\ x) = \xi. \tag{3.16}$$

Since ψ is non-decreasing, we have

$$\psi(d(z_{n+1},\ x)) \geq \psi\left(\xi\right).$$

Letting $n \to \infty$ in this inequality and by using (3.15), we get $0 \geq \psi(\xi)$. It follows, by (ii_ψ), that $\xi = 0$. Thus, by (3.16), we get

$$\lim_{n\to\infty} d(z_{n+1},\ x) = 0,$$

which implies that, we get $x = \lim_{n\to\infty} z_{n+1}$. Similarly, we can show that $y = \lim_{n\to\infty} z_{n+1}$. Thus $x = y$, that is, the fixed point of T is unique. □

Example 3.1. Suppose that $X = \mathbb{R}$, furnished with the usual metric $d : X \times X \to [0,\, +\infty)$ with the natural ordering of real numbers $\leq$. Let $T : X \to X$ be defined as

$$Tx = \ln\left(1 + x\right), \text{ for all } x \in X.$$

Define $\psi : [0,\, +\infty) \to [0,\, +\infty)$ by

$$\psi(t) = \begin{cases} \ln(1+t), \text{ for } t \neq 1, \\ \frac{3}{4}, \text{ for } t = 1, \end{cases}$$

and $\varphi : [0,\, +\infty) \to [0,\, 1)$ defined by

$$\varphi(t) = \frac{\psi(t)}{t}, \text{ for all } t \geq 0.$$

First, we shall show that the contractive condition (3.1) holds for the mapping T. Let $x,\, y \in X$ such that $x \preceq y$, we have

$$\begin{aligned} d(Tx,\ Ty) &= |Tx - Ty| \\ &= |\ln(1+x) - \ln(1+y)| \\ &= \left|\ln\frac{1+x}{1+y}\right| \\ &= \left|\ln\left(1+\frac{x-y}{1+y}\right)\right| \\ &\leq \ln(1+|x-y|) \\ &\leq \ln(1+d(x,\ y)), \end{aligned}$$

which implies that

$$\begin{aligned} \psi(d(Tx,\ Ty)) &= \ln(d(Tx,\ Ty)+1) \\ &\leq \ln(1+\ln(1+d(x,\ y))) \\ &\leq \frac{\ln(1+\ln(1+d(x,\ y)))}{\ln(1+d(x,\ y))}\ln(1+d(x,\ y)) \\ &\leq \varphi(\psi(d(x,\ y)))\psi(d(x,\ y)). \end{aligned}$$

This exhibit that the contractive condition (3.1) holds. In addition, all the other conditions of Theorem 3.1 are satisfied and $z = 0$ is a unique fixed point of T.

If we put $\varphi(t) = 1 - \frac{\widetilde{\varphi}(t)}{t}$ for all $t \geq 0$ in Theorem 3.1, then we get the following result:

Corollary 3.2. *Let $(X,\ d,\ \preceq)$ be a partially ordered complete metric space and $T : X \to X$ be a non-decreasing mapping for which there exist $\psi \in \Psi$ and $\widetilde{\varphi} \in \Phi$ such that*

$$\psi(d(Tx,\ Ty)) \leq \psi(d(x,\ y)) - \widetilde{\varphi}(\psi(d(x,\ y))),$$

for all $x,\ y \in X$ with $x \preceq y$. Suppose either

(a) T is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exists $x_0 \in X$ such that $x_0 \asymp Tx_0$, then T has a fixed point. Moreover, if for each $x,\ y \in X$ there exists $z \in X$ which is $\preceq -$comparable to x and y then the fixed point is unique.

If we put $\psi(t) = 2t$ for all $t \geq 0$ in Theorem 3.1, then we get the following result:

Corollary 3.3. *Let $(X,\ d,\ \preceq)$ be a partially ordered complete metric space and $T : X \to X$ be a non-decreasing mapping for which there exists $\varphi \in \Phi$ such that*

$$d(Tx,\ Ty) \leq \varphi(2d(x,\ y))d(x,\ y),$$

for all $x,\ y \in X$ *with* $x \preceq y$. *Suppose either*

(*a*) T *is continuous or*

(*b*) $(X,\ d,\ \preceq)$ *is regular.*

If there exists $x_0 \in X$ *such that* $x_0 \asymp Tx_0$, *then* T *has a fixed point. Moreover, if for each* $x,\ y \in X$ *there exists* $z \in X$ *which is* $\preceq -$*comparable to* x *and* y *then the fixed point is unique.*

If we put $\varphi(t) = k$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 3.3, then we get the following result:

Corollary 3.4. *Let* $(X,\ d,\ \preceq)$ *be a partially ordered complete metric space and* $T : X \to X$ *be a non-decreasing mapping such that*

$$d(Tx,\ Ty) \leq kd(x,\ y),$$

for all $x,\ y \in X$ *with* $x \preceq y$, *where* $0 < k < 1$. *Suppose either*

(*a*) T *is continuous or*

(*b*) $(X,\ d,\ \preceq)$ *is regular.*

If there exists $x_0 \in X$ *such that* $x_0 \asymp Tx_0$, *then* T *has a fixed point. Moreover, if for each* $x,\ y \in X$ *there exists* $z \in X$ *which is* $\preceq -$*comparable to* x *and* y *then the fixed point is unique.*

4. Multidimensional Fixed Point Results

Next we give an $n-$dimensional fixed point theorem for mixed monotone mappings. For brevity, $(y_1,\ y_2,\ ...,\ y_n)$, $(v_1,\ v_2,\ ...,\ v_n)$ and $(y_0^1,\ y_0^2,\ ...,\ y_0^n)$ will be denoted by Y, V and Y_0 respectively.

Theorem 4.1. *Let* $(X,\ d,\ \preceq)$ *be a complete partially ordered metric space. Let* $\Upsilon = (\sigma_1,\ \sigma_2,\ ...,\ \sigma_n)$ *be an* $n-$*tuple of mappings from* Λ_n *into itself verifying* $\sigma_i \in \Psi_{A,B}$ *if* $i \in A$ *and* $\sigma_i \in \Psi'_{A,B}$ *if* $i \in B$. *Let* $F : X^n \to X$ *be a mixed monotone mapping for which there exist* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that*

$$\begin{aligned} &\psi(d(F(y_1,\ y_2,\ ...,\ y_n),\ F(v_1,\ v_2,\ ...,\ v_n))) \\ \leq\ &\varphi\left(\psi\left(\max_{1\leq i\leq n} d(y_i,\ v_i)\right)\right)\psi\left(\max_{1\leq i\leq n} d(y_i,\ v_i)\right), \end{aligned} \tag{4.1}$$

for all $y_1,\ y_2,\ ...,\ y_n,\ v_1,\ v_2,\ ...,\ v_n \in X$ *with* $y_i \preceq_i v_i$, *for* $i \in \Lambda_n$. *Also, suppose that either* F *is continuous or* $(X,\ d,\ \preceq)$ *is regular. If there exists* $y_0^1,\ y_0^2,\ ...,\ y_0^n \in X$ *verifying* $y_0^i \preceq_i F(y_0^{\sigma_i(1)},\ y_0^{\sigma_i(2)},\ ...,\ y_0^{\sigma_i(n)})$, *for* $i \in \Lambda_n$, *then* F *has a* $\Upsilon-$*fixed* *point. Moreover, if for each $i \in \Lambda_n$ and $y_i,\ v_i \in X$ there exists $z_i \in X$ which is $\preceq_i -$comparable to y_i and v_i. Then F has a unique $\Upsilon-$fixed point.*

Proof. For fixed $i \in A$, we have $y_{\sigma_i(t)} \preceq_t v_{\sigma_i(t)}$ for $t \in \Lambda_n$. Thus by using (4.1), we have

$$\begin{aligned} &\psi(d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}))) \\ \le\ &\varphi(\psi(\max_{1\le i\le n} d(y_i,\ v_i)))\psi(\max_{1\le i\le n} d(y_i,\ v_i)), \end{aligned} \tag{4.2}$$

for all $i \in A$. Similarly, for fixed $i \in B$, we have $y_{\sigma_i(t)} \succeq_t v_{\sigma_i(t)}$ for $t \in \Lambda_n$. It follows from (4.1) that

$$\begin{aligned} &\psi(d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}))) \\ \le\ &\psi(d(F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}),\ F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}))) \\ \le\ &\varphi(\psi(\max_{1\le i\le n} d(y_i,\ v_i)))\psi(\max_{1\le i\le n} d(y_i,\ v_i)), \end{aligned} \tag{4.3}$$

for all $i \in B$. By (2.1), (2.2), (4.2), (4.3) and by the monotonicity of ψ, we have

$$\psi(\rho_n(F_\Upsilon(Y),\ F_\Upsilon(V))) \le \varphi(\psi(\rho_n(Y,\ V)))\psi(\rho_n(Y,\ V)),$$

for all $Y,\ V \in X^n$ with $Y \sqsubseteq V$. Thus it is only required to apply Theorem 3.1 to the mappings $T = F_\Upsilon$ in the ordered metric space $(X^n,\ \rho_n,\ \sqsubseteq)$ and taking all items of Lemma 2.1. □

In a similar way, one can state the results identical to Corollary 3.2, Corollary 3.3 and Corollary 3.4.

5. Applications

In this section, we present an application to our results. Consider the integral equation

$$u(t) = \int_0^T K(t,\ s,\ u(s))ds + g(t),\ t \in [0,\ T], \tag{5.1}$$

where $T > 0$. We introduce the following space:

$$C[0,\ T] = \{u : [0,\ T] \to \mathbb{R} : u \text{ is continuous on } [0,\ T]\},$$

endowed with the metric

$$d(x,\ y) = \sup_{t\in[0,\ T]} |x(t) - y(t)|\,,\ \text{for each } x,\ y \in C[0,\ T].$$

It is noticeable that $(C[0,T], d, \preceq)$ is a regular complete metric space. Furthermore, $C[0,T]$ can be furnished with the partial order $\preceq$ as follows:

$$x \preceq y \Longleftrightarrow x(t) \leq y(t), \text{ for each } x,\ y \in C[0,T] \text{ and } t \in [0,\ T].$$

Theorem 5.1. *We assume that the following hypotheses hold:*

(i) $K : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous,*
(ii) for all $s,\ t,\ u,\ v \in C[0,T]$ *with* $v \preceq u$, *we have*

$$K(t,\ s,\ v(s)) \leq K(t,\ s,\ u(s)),$$

(iii) there exists a continuous function $G : [0,T] \times [0,T] \to [0,+\infty)$ *such that*

$$|K(t,\ s,\ x) - K(t,\ s,\ y)| \leq G(t,\ s) \ln\left(1 + |x - y|\right),$$

for all $s,\ t \in C[0,T]$ *and* $x,\ y \in \mathbb{R}$ *with* $x \succeq y$,
(iv) $\sup_{t\in[0,\ T]} \int_0^T G(t,\ s)^2 ds \leq \frac{1}{T}$.
Then the integral equation (5.1) *has a solution* $u^* \in C[0,T]$.

Proof. Define $\psi : [0,+\infty) \to [0,+\infty)$ by

$$\psi(t) = \begin{cases} \ln(t+1), & \text{for } t \neq 1, \\ \frac{3}{4}, & \text{for } t = 1, \end{cases}$$

and $\varphi : [0,+\infty) \to [0,1)$ as follows

$$\varphi(t) = \frac{\psi(t)}{t}, \text{ for all } t \geq 0.$$

and also define $F : C[0,T] \to C[0,T]$ by

$$Fu(t) = \int_0^T K(t,\ s,\ u(s))ds + g(t), \text{ for } t \in [0,\ T] \text{ and } u \in C[0,\ T].$$

We first prove that F is non-decreasing. Assume that $v \preceq u$. From (ii), for all s, $t \in [0,T]$, we have $K(t, s, u(s)) \leq K(t, s, v(s))$. Thus, we get

$$Fv(t) = \int_0^T K(t,\ s,\ v(s))ds + g(t) \leq \int_0^T K(t,\ s,\ u(s))ds + g(t) = Fu(t).$$

Now, for all $u,\ v \in C[0,\ T]$ with $v \preceq u$, due to (iii) and by using Cauchy-Schwarz inequality, we formulate that

$$\begin{aligned}
&|Fu(t) - Fv(t)| \\
\leq\ & \int_0^T |K(t,\ s,\ u(s)) - K(t,\ s,\ v(s))|\, ds \\
\leq\ & \int_0^T G(t,\ s) \ln\left(1 + |u(s) - v(s)|\right) ds \\
\leq\ & \left(\int_0^T G(t,\ s)^2 ds\right)^{\frac{1}{2}} \left(\int_0^T \left(\ln\left(1 + |u(s) - v(s)|\right)\right)^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$|Fu(t) - Fv(t)| \leq \left(\int_0^T G(t,\ s)^2 ds\right)^{\frac{1}{2}} \left(\int_0^T \left(\ln\left(1 + |u(s) - v(s)|\right)\right)^2 ds\right)^{\frac{1}{2}}. \tag{5.2}$$

Taking (iv) into account, we estimate the first integral in (5.2) as follows:

$$\left(\int_0^T G(t,\ s)^2 ds\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{T}}. \tag{5.3}$$

For the second integral in (5.2) we proceed in the following way:

$$\left(\int_0^T \left(\ln\left(1 + |u(s) - v(s)|\right)\right)^2 ds\right)^{\frac{1}{2}} \leq \sqrt{T} \ln(1 + d(u,\ v)). \tag{5.4}$$

Combining (5.2), (5.3) and (5.4), we conclude that

$$|Fu(t) - Fv(t)| \leq \ln(1 + d(u,\ v)).$$

It yields

$$d(Fu,\ Fv) \leq \ln(1 + d(u,\ v)),$$

which implies that

$$\begin{aligned}
\psi(d(Fu,\ Fv)) &= \ln(d(Fu,\ Fv) + 1) \\
&\leq \ln(1 + \ln(1 + d(u,\ v))) \\
&\leq \frac{\ln(1 + \ln(1 + d(u,\ v)))}{\ln(1 + d(u,\ v))} \ln(1 + d(u,\ v)) \\
&\leq \varphi(\psi(d(u,\ v)))\psi(d(u,\ v)).
\end{aligned}$$

for all $u,\ v \in C[0,\ T]$ with $v \preceq u$. Hence, all the hypotheses of Theorem 3.1 are satisfied. Thus, F has a fixed point $u^* \in C[0,\ T]$ which is a solution of (5.1).

References

1. S.A. Al-Mezel, H. Alsulami, E. Karapinar & A. Roldan: Discussion on multidimensional coincidence points via recent publications. Abstr. Appl. Anal. **2014**, Article ID 287492.
2. M. Berzig & B. Samet: An extension of coupled fixed point's concept in higher dimension and applications. Comput. Math. Appl. **63** (2012), no. 8, 1319-1334.
3. T.G. Bhaskar & V. Lakshmikantham: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65** (2006), no. 7, 1379-1393.
4. L. Ćirić, B. Damjanović, M. Jleli & B. Samet: Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications. Fixed Point Theory Appl. **2012**, 51.
5. B. Deshpande & A. Handa: Coincidence point results for weak $\psi - \varphi$ contraction on partially ordered metric spaces with application. Facta Universitatis Ser. Math. Inform. **30** (2015), no. 5, 623-648.
6. B. Deshpande, A. Handa & C. Kothari: Coincidence point theorem under Mizoguchi-Takahashi contraction on ordered metric spaces with application. IJMAA **3** (4-A) (2015), 75-94.
7. B. Deshpande, A. Handa & S.A. Thoker: Existence of coincidence point under generalized nonlinear contraction with applications. East Asian Math. J. **32** (2016), no. 1, 333-354.
8. B. Deshpande & A. Handa: On coincidence point theorem for new contractive condition with application. Facta Universitatis Ser. Math. Inform. **32** (2017), no. 2, 209-229.
9. B. Deshpande & A. Handa: Multidimensional coincidence point results for generalized (ψ, θ, φ)-contraction on ordered metric spaces. J. Nonlinear Anal. Appl. **2017** (2017), no. 2, 132-143.
10. B. Deshpande & A. Handa: Utilizing isotone mappings under Geraghty-type contraction to prove multidimensional fixed point theorems with application. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **25** (2018), no. 4, 279-95.
11. W.S. Du: Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi-Takahashi's condition in quasi ordered metric spaces. Fixed Point Theory Appl. **2010**, 9 (2010) Article ID 876372.
12. I.M. Erhan, E. Karapinar, A. Roldan & N. Shahzad: Remarks on coupled coincidence point results for a generalized compatible pair with applications. Fixed Point Theory Appl. **2014**, 207.
13. J. Harjani, B. Lopez & K. Sadarangani: Fixed point theorems for mixed monotone operators and applications to integral equations. Nonlinear Anal. **74** (2011), 1749-1760.

14. E. Karapinar, A. Roldan, C. Roldan & J. Martinez-Moreno: A note on N-Fixed point theorems for nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl. **2013**, Article ID 310.
15. E. Karapinar, A. Roldan, J. Martinez-Moreno & C. Roldan: Meir-Keeler type multidimensional fixed point theorems in partially ordered metric spaces. Abstr. Appl. Anal. **2013**, Article ID 406026.
16. A. Roldan, J. Martinez-Moreno & C. Roldan: Multidimensional fixed point theorems in partially ordered metric spaces. J. Math. Anal. Appl. **396** (2012), 536-545.
17. A. Roldan & E. Karapinar: Some multidimensional fixed point theorems on partially preordered G^*-metric spaces under $(\varphi, \psi)-$contractivity conditions. Fixed Point Theory Appl. **2013**, Article ID 158.
18. A. Roldan, J. Martınez-Moreno, C. Roldan & E. Karapinar: Some remarks on multidimensional fixed point theorems. Fixed Point Theory **15** (2014), no. 2, 545-558.
19. F. Shaddad, M.S.M. Noorani, S.M. Alsulami & H. Akhadkulov: Coupled point results in partially ordered metric spaces without compatibility. Fixed Point Theory Appl. **2014**, 204.
20. S. Wang: Coincidence point theorems for G-isotone mappings in partially ordered metric spaces. Fixed Point Theory Appl. **2013**, 96.
21. S. Wang: Multidimensional fixed point theorems for isotone mappings in partially ordered metric spaces. Fixed Point Theory Appl. **2014**, 137.

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