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$(L, *, \odot)$-QUASIUNIFORM CONVERGENCE SPACES

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Abstract. In this paper, we define the notion of $(L, *, \odot)$-quasiuniform convergence spaces on ecl-premonoid. From $(L, *, \odot)$-quasiuniform structures, we can obtain various $(L, *, \odot)$-quasiuniform convergence structures and give their examples.

1. Introduction

Gäher [2,3] introduced the notions of L-filters in a frame. Höhle and Sostak [4] introduced the concept of L-filters for a complete quasimonoidal lattice L. For the case that the lattice is a stsc quantale, L-filters were introduced in [12]. Jäger [5-6] developed stratified L-convergence structures based on the concepts of L-filters where L is a complete Heyting algebra. Yao [14] extended stratified L-convergence structures to complete residuated lattices and investigated between stratified L-convergence structures and L-fuzzy topological spaces. As an extension of Yao [14], Fang [7-11] introduced L-ordered convergence structures and (pre, quasi,semi) uniform convergence spaces on L-filters and investigated their relations.

In this paper, we define the $(L, *, \odot)$-quasiuniform convergence spaces as an extension of Fang's uniform convergence spaces on ecl-premonoid in Orpen's sense [13]. From $(L, *, \odot)$-quasiuniform structures, we can obtain various $(L, *, \odot)$-quasiuniform convergence structures and give their examples.

2. Preliminaries

Definition 2.1 ([13])**.** A complete lattice $(L, \leq, \perp, \top)$ is called a *GL-monoid* $(L, \leq, *, \perp, \top)$ with a binary operation $* : L \times L \to L$ satisfying the following conditions:

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(G1) $a * \top = a$, for all $a \in L$,
(G2) $a * b = b * a$, for all $a, b \in L$,
(G3) $a * (b * c) = (a * b) * c$, for all $a, b \in L$,
(G4) if $a \leq b$, there exists $c \in L$ such that $b * c = a$,
(G5) $a * \bigvee_{i \in \Gamma} b_i = \bigvee_{i \in \Gamma}(a * b_i)$.
We can define an implication operator:

$$a \Rightarrow b = \bigvee\{c \mid a * c \leq b\}.$$

Example 2.2 ([1, 4, 13])**.** (1) A continuous t-norm $([0,1], \leq, *)$ is a GL-monoid.
(2) A frame $(L, \leq, \wedge)$ is a GL-monoid.

Definition 2.3 ([1, 4, 13])**.** A complete lattice $(L, \leq, \perp, \top)$ is called a *cl-premonoid* $(L, \leq, \odot)$ with a binary operation $\odot : L \times L \to L$ satisfying the following conditions:
(CL1) $a \leq a \odot \top$ and $a \leq \top \odot a$, for all $a \in L$,
(CL2) if $a \leq b$ and $c \leq d$, then $a \odot c \leq b \odot d$,
(CL3) $a \odot \bigvee_{i \in \Gamma} b_i = \bigvee_{i \in \Gamma}(a \odot b_i)$ and $\bigvee_{j \in \Gamma} a_j \odot b = \bigvee_{j \in \Gamma}(a_j \odot b)$.
We can define an implication operator:

$$a \to b = \bigvee\{c \mid a \odot c \leq b\}.$$

Example 2.4 ([1, 4, 13])**.** (1) Every GL-monoid $(L, \leq, *)$ is a cl-premonoid.
(2) Defines maps $\odot_i : [0,1] \times [0,1] \to [0,1]$ as follows:

$$x \odot_1 y = x^{\frac{1}{p}} \cdot y^{\frac{1}{p}}(p \geq 1), x \odot_2 y = (x^p + y^p) \wedge 1 (p \geq 1).$$

Then $(L, \leq, \odot_i)$ is a cl-premonoid for $i = 1, 2$.

Definition 2.5 ([1, 4, 13])**.** A complete lattice $(L, \leq, \perp, \top)$ is called an *ecl-premonoid* $(L, \leq, \odot, *)$ with a GL-monoid $(L, \leq, *)$ and a cl-premonoid $(L, \leq, \odot)$ which satisfy the following condition:
(D) $(a \odot b) * (c \odot d) \leq (a * c) \odot (b * d)$, for all $a, b, c, d \in L$.
An ecl-premonoid $(L, \leq, \odot, *)$ is called an M-ecl-premonoid if it satisfiesthe following condition:
(M) $a \leq a \odot a$ for all $a \in L$.

In this paper, we always assume that $(L, \leq, \odot, *)$ is an ecl-premonoid unless otherwise specified.

Example 2.6 ([1, 4, 13])**.** (1) Let $(L, \leq, *)$ be a GL-monoid and $(L, \leq, \wedge)$ is a cl-premonoid. Then $(L, \leq, \wedge, *)$ is an M-ecl-premonoid.

(2) Let $(L,\leq,*)$ be a GL-monoid. Then $(L,\leq,*,*)$ is an ecl-premonoid. If $*=\cdot$, $0.5\not\leq 0.5\cdot 0.5=0.25$. $(L,\leq,\cdot,\cdot)$ is not an M-ecl-premonoid.

(3) Let $(L,\leq,\cdot)$ be a GL-monoid. Define a map $\odot:[0,1]\times[0,1]\to[0,1]$ as $x\odot y=(x+y)\wedge 1$. Then $(L,\leq,\odot,\cdot)$ is not an M-cl-premonoid because

$$0.7=(0.3\odot 0.4)\cdot(0.5\odot 0.7)\not\leq(0.3\cdot 0.5)\odot(0.4\cdot 0.7)=0.15+0.28=0.43$$

(4) Let $(L,\leq,\cdot)$ be a GL-monoid. Define a map $\odot:[0,1]\times[0,1]\to[0,1]$ as $x\odot y=x^{\frac{1}{3}}\cdot y^{\frac{1}{3}}$. Then $(L,\leq,\odot,\cdot)$ is an *M-cl-premonoid.*

Lemma 2.7 ([1, 4, 13])**.** *Let $(L,\leq,\odot,*)$ be an ecl-premonoid. For each $a,b,c,d,a_i,b_i\in L$ and for $\uparrow\in\{\to,\Rightarrow\}$, we have the following properties.*

(1) *If $b\leq c$, then $a\odot b\leq a\odot c$ and $a*b\leq a*c$.*

(2) *$a\odot b\leq c$ iff $a\leq b\to c$. Moreover, $a*b\leq c$ iff $a\leq b\Rightarrow c$.*

(3) *If $b\leq c$, then $a\uparrow b\leq a\uparrow c$ and $c\uparrow a\leq b\uparrow a$.*

(4) *$a\leq b$ iff $a\Rightarrow b=\top$.*

(5) *$a*b\leq a\odot b$, $a\to b\leq a\Rightarrow b$ and $a*(b\odot c)\leq(a*b)\odot c$.*

(6) *$(a\uparrow b)\odot(c\uparrow d)\leq(a\odot c)\uparrow(b\odot d)$.*

(7) *$(b\uparrow c)\leq(a\odot b)\uparrow(a\odot c)$.*

(8) *$(b\uparrow c)\leq(a\uparrow b)\uparrow(a\uparrow c)$ and $(b\uparrow a)\leq(a\uparrow c)\uparrow(b\uparrow c)$.*

(9) *$(b\to c)\leq(a\uparrow b)\to(a\uparrow c)$ and $(b\uparrow a)\leq(a\to c)\to(b\uparrow c)$*

(10) *$a_i\uparrow b_i\leq(\bigwedge_{i\in\Gamma}a_i)\uparrow(\bigwedge_{i\in\Gamma}b_i)$.*

(11) *$a_i\uparrow b_i\leq(\bigvee_{i\in\Gamma}a_i)\uparrow(\bigvee_{i\in\Gamma}b_i)$.*

(12) *$(c\uparrow a)*(b\to d)\leq(a\to b)\to(c\uparrow d)$.*

Definition 2.8 ([4, 13])**.** A mapping $\mathcal{F}:L^X\to L$ is called an $(L,*)$-*filter* on X if it satisfies the following conditions:

(F1) $\mathcal{F}(1_\emptyset)=\bot$ and $\mathcal{F}(1_X)=\top$, where $1_\emptyset(x)=\bot, 1_X(x)=\top$ for $x\in X$.

(F2) $\mathcal{F}(f*g)\geq\mathcal{F}(f)*\mathcal{F}(g)$, for each $f,g\in L^X$,

(F3) if $f\leq g$, $\mathcal{F}(f)\leq\mathcal{F}(g)$.

An $(L,*)$-filter is called *stratified* if

(S) $\mathcal{F}(\alpha*f)\geq\alpha*\mathcal{F}(f)$ for each $f\in L^X$ and $\alpha\in L$.

The pair $(X,\mathcal{F})$ is called an (resp. a stratified)$(L,*)$-*filter space*. We denote by $F_*(X)$ (resp. $F_*^s(X)$) the set of all (resp. stratified) $(L,*)$-filters on X.

Let $(X,\mathcal{F}_1)$ and $(Y,\mathcal{F}_2)$ be two $(L,*)$-filter spaces and $\phi:X\to Y$ called an L-filter map if $\mathcal{F}_2(g)\leq\mathcal{F}_1(\phi^{\leftarrow}(g))$ for all $g\in L^Y$ where $\phi^{\leftarrow}(g)=g\circ\phi$.

Example 2.9 ([4, 13]). (1) Define a map $[x] : L^X \to L$ as $[x](f) = f(x)$. Then $[x]$ is a stratified $(L,*)$-filter on X.

(2) Define a map inf $: L^X \to L$ as $\inf(f) = \bigwedge_{x\in X} f(x)$. Then inf is a stratified $(L,*)$-filter on X.

3. $(L,*,\odot)$-quasiuniform Convergence Spaces

Theorem 3.1. *Let $\mathcal{U},\mathcal{V} \in F_*(X\times X)$. We define $\mathcal{U}\circ_\odot \mathcal{V} : L^{X\times X} \to L$ as follows:*

$$(\mathcal{U}\circ_\odot \mathcal{V})(w) = \bigvee\{\mathcal{U}(u)\odot\mathcal{V}(v) \mid u\circ v \le w\}$$

where $u\circ v(x,z) = \bigvee_{y\in X}(u(x,y)*v(y,z))$.

(1) *$u\circ v = \bot$ implies $\mathcal{U}(u)\odot\mathcal{V}(v) = \bot$ iff $(\mathcal{U}\circ_\odot\mathcal{V}) \in F_*(X\times X)$.*

(2) *If $u\circ v = \bot$ implies $\mathcal{U}(u)\odot\mathcal{V}(v) = \bot$ and $\mathcal{U}\in F^s_*(X\times X)$ or $\mathcal{V}\in F^s_*(X\times X)$, then $\mathcal{U}\circ_\odot\mathcal{V}\in F^s_*(X\times X)$.*

(3) *If $\mathcal{U}(1_\triangle) = \top$ where $1_\triangle(x,x) = \top$ and $1_\triangle(x,y) = \bot$ for $x\ne y\in X$, $\mathcal{U}\circ\mathcal{U}\ge\mathcal{U}$.*

(4) *$\mathcal{U}\circ_\odot[(x,x)]\in F^s_*(X\times X)$, $\mathcal{U}\circ_\odot[(x,x)]\ge\mathcal{U}$.*

(5) *$[(x,x)]\circ_*[(x,x)] = [(x,x)]$.*

(6) *$\bigwedge_{x\in X}[(x,x)]\circ_*\bigwedge_{x\in X}[(x,x)] = \bigwedge_{x\in X}[(x,x)]$.*

(7) *$\mathcal{U}\circ_*\mathcal{U}^{-1}\in F_*(X\times X)$.*

(8) *$(\mathcal{U}\circ_\odot\mathcal{V})^{-1} = \mathcal{V}^{-1}\circ_\odot\mathcal{U}^{-1}$.*

Proof. (1) Since $(u_1*u_2)\circ(v_1*v_2)\le(u_1\circ v_1)*(u_2\circ v_2)$,

$$\begin{aligned}
&(\mathcal{U}\circ_\odot\mathcal{V})(u)*(\mathcal{U}\circ_\odot\mathcal{V})(v)\\
&= \bigvee_{u_1\circ v_1\le u}(\mathcal{U}(u_1)\odot\mathcal{V}(v_1)) * \bigvee_{u_2\circ v_2\le v}(\mathcal{U}(u_2)\odot\mathcal{V}(v_2))\\
&\le \bigvee_{(u_1\circ v_1)*(u_2\circ v_2)\le u*v}\Big((\mathcal{U}(u_1)\odot\mathcal{V}(v_1))*(\mathcal{U}(u_2)\odot\mathcal{V}(v_2))\Big)\\
&\le \bigvee_{(u_1\circ v_1)*(u_2\circ v_2)\le u*v}\Big((\mathcal{U}(u_1)*\mathcal{U}(u_2))\odot(\mathcal{V}(v_1)*\mathcal{V}(v_2))\Big)\\
&\le \bigvee_{(u_1*u_2)\circ(v_1*v_2)\le u*v}(\mathcal{U}(u_1*u_2)\odot\mathcal{V}(v_1*v_2))\\
&\le (\mathcal{U}\circ_\odot\mathcal{V})(u*v).
\end{aligned}$$

Other cases are easily proved.

(2) Let $\mathcal{U}\in F^s_*(X\times X)$. Since $a*(b\odot c)\le(a\odot\top)*(b\odot c)\le(a*b)\odot(\top*c) = (a*b)\odot c$, we have

$$\begin{aligned}
\alpha*(\mathcal{U}\circ_\odot\mathcal{V})(u) &= \alpha*\bigvee_{u_1\circ v_1\le u}(\mathcal{U}(u_1)\odot\mathcal{V}(v_1))\\
&= \bigvee_{u_1\circ v_1\le u}(\alpha*(\mathcal{U}(u_1)\odot\mathcal{V}(v_1)))\\
&\le \bigvee_{(u_1\circ v_1)\le u}((\alpha*\mathcal{U}(u_1))\odot\mathcal{V}(v_1))\\
&\le \bigvee_{((\alpha*u_1)\circ v_1)\le\alpha*u}(\mathcal{U}(\alpha*u_1)\odot\mathcal{V}(v_1))\\
&\le (\mathcal{U}\circ_\odot\mathcal{V})(\alpha*u)
\end{aligned}$$

(3) For $u \circ 1_\triangle = u$, $\mathcal{U} \circ_\odot \mathcal{U}(u) \geq \mathcal{U}(u) \odot \mathcal{U}(1_\triangle) = \mathcal{U}(u)$.

(4) Since $[(x,x)](\alpha * u) = \alpha * u(x,x) = \alpha * [(x,x)](u)$, $[(x,x)] \in F_*^s(X \times X)$. For $u \circ 1_\triangle = u$, we have

$$(\mathcal{U} \circ_\odot [(x,x)])(u) \geq \mathcal{U}(u) \odot [(x,x)](1_\triangle) = \mathcal{U}(u).$$

(5) For $u_1 \circ u_2 \leq u$, we have

$$([(x,x)] \circ_* [(x,x)])(u) = \bigvee_{x \in X} ([(x,x)](u_1) * [(x,x)](u_2)) \leq u(x,x) = [(x,x)](u).$$

(6) For $u \circ 1_\triangle = u$, we have

$$\begin{aligned}(\textstyle\bigwedge_{x\in X}[(x,x)] \circ_* \bigwedge_{x\in X}[(x,x)])(u) \ &\geq \textstyle\bigwedge_{x\in X}[(x,x)](u) * [(x,x)](1_\triangle) \\ &= \textstyle\bigwedge_{x\in X}[(x,x)](u).\end{aligned}$$

For $u \circ v \leq w$,

$$\begin{aligned}&(\textstyle\bigwedge_{x\in X}[(x,x)](u)) * (\bigwedge_{x\in X}[(x,x)](v)) = \bigwedge_{x\in X} u(x,x) * \bigwedge_{x\in X} v(x,x) \\ &\leq \textstyle\bigwedge_{x\in X}[(x,x)](u \circ v) \leq \bigwedge_{x\in X}[(x,x)](w).\end{aligned}$$

(7) For $u \circ v = \perp$, we have $\mathcal{U}(u) * \mathcal{U}^{-1}(v) \leq \mathcal{U}(u * v^{-1}) = \perp$ because $u * v^{-1}(x,y) \leq u \circ v(x,x) = \perp$.

(8) Since $(v \circ u)^{-1} = u^{-1} \circ v^{-1}$, we have

$$\begin{aligned}\mathcal{V}^{-1} \circ_\odot \mathcal{U}^{-1}(w) \ &= \textstyle\bigvee\{\mathcal{V}^{-1}(v) \odot \mathcal{U}^{-1}(u) \mid v \circ u \leq w\} \\ &= \textstyle\bigvee\{\mathcal{V}(v^{-1}) \odot \mathcal{U}(u^{-1}) \mid u^{-1} \circ v^{-1} \leq w^{-1}\} \\ &= \mathcal{U} \circ_\odot \mathcal{V}(w^{-1}) = (\mathcal{U} \circ_\odot \mathcal{V})^{-1}(w).\end{aligned}$$

Definition 3.2. A subset $\mathcal{U}$ of $F_*(X \times X)$ is called an $(L, *, \odot)$-*quasiuniform structure* on X if it satisfies the following conditions:

(QU1) $\mathcal{U} \leq [(x,x)]$, for each $x \in X$.

(QU2) $\mathcal{U} \leq \mathcal{U} \circ_\odot \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called an $(L, *, \odot)$ *quasiuniform space.*

An $(L, *, \odot)$-quasiuniform space is called an $(L, *, \odot)$-*uniform space* if it satisfies the following condition;

(U) $\mathcal{U} \leq \mathcal{U}^{-1}$.

Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ be $(L, *, \odot)$-quasiuniform spaces. A map $\psi : (X, \mathcal{U}_X) \to (Y, \mathcal{U}_Y)$ is called *quasiuniformly continuous* if for all $u \in L^{Y \times Y}$, $\mathcal{U}_Y(u) \leq \mathcal{U}_Y((\psi \times \psi)^{\leftarrow}(u))$.

Example 3.3. Let $X = \{a, b, c\}$ be a set and $(L = [0,1], \leq, \wedge, *, 0, 1)$ an M-ecl-premonoid with $a * b = (a + b - 1) \vee 0$. Put $u, v \in [0,1]^{X \times X}$ as follows:

$$u(a,a) = u(b,b) = u(c,c) = 1, \ \ u(a,b) = u(b,a) = 0.6,$$

$$u(a,c) = u(c,a) = 0.5, u(b,c) = u(c,b) = 0.4.$$

$$v(a,a) = v(b,b) = 1, v(c,c) = 0.4, \ \ v(a,b) = v(b,a) = 0.6,$$

$$v(a,c) = v(c,a) = 0.5, v(b,c) = v(c,b) = 0.4.$$

(1) Define a $([0,1],*)$-filter as $\mathcal{U} : [0,1]^{X\times X} \to [0,1]$ as follows:

$$\mathcal{U}(w) = \begin{cases} 1, & \text{if } w = 1_{X\times X}, \\ 0.6, & \text{if } u \leq w \neq 1_{X\times X}, \\ 0.3, & \text{if } u * u \leq w \not\geq u, \\ 0, & \text{otherwise.} \end{cases}$$

Since $u \circ u = u$, we obtain $\mathcal{U} = \mathcal{U} \circ_\wedge \mathcal{U} = \mathcal{U}^{-1}$ and

$$(\mathcal{U} \circ_* \mathcal{U})(w) = \begin{cases} 1, & \text{if } w = 1_{X\times X}, \\ 0.2, & \text{if } u \leq w \neq 1_{X\times X}, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, $\mathcal{U}(w) \leq [(x,x)](w)$, for each $x \in X$, $w \in L^{X\times X}$. Hence $\mathcal{U}$ is an $(L,*,\wedge)$-uniform structure on X but not an $(L,*,*)$-uniform structure on X because $0.6 = \mathcal{U}(u) \not\leq (\mathcal{U} \circ_* \mathcal{U})(u) = 0.2$.

(2) Define $[0,1]$-filter as $\mathcal{V} : [0,1]^{X\times X} \to [0,1]$ as follows:

$$\mathcal{V}(w) = \begin{cases} 1, & \text{if } w \geq 1_\triangle, \\ 0.6, & \text{if } v \leq w \not\geq 1_\triangle, \\ 0.3, & \text{if } v * v \leq w \not\geq v, \\ 0, & \text{otherwise.} \end{cases}$$

Since $v \circ 1_\triangle = v$, we obtain $\mathcal{V} \circ_* \mathcal{V} = \mathcal{V} \circ_\wedge \mathcal{V} = \mathcal{V} = \mathcal{V}^{-1}$. But $0.6 = \mathcal{V}(v) \not\leq [(c,c)](v) = 0.4$. Hence $\mathcal{V}$ is neither an $(L,*,\wedge)$-uniform structure nor an $(L,*,*)$-uniform structure on X.

Definition 3.4. A map $\Lambda : F_*(X \times X) \to L$ is called an $(L,*,\odot)$-*quasiuniform convergence structure* on X if it satisfies the following conditions:

(QC1) $\Lambda([(x,x)]) = \top$, for each $x \in X$.

(QC2) If $\mathcal{U} \leq \mathcal{V}$, then $\Lambda(\mathcal{U}) \leq \Lambda(\mathcal{V})$.

(QC3) $\Lambda(\mathcal{U}) \odot \Lambda(\mathcal{V}) \leq \Lambda(\mathcal{U} \odot \mathcal{V})$.

(QC4) $\Lambda(\mathcal{U}) \odot \Lambda(\mathcal{V}) \leq \Lambda(\mathcal{U} \circ_\odot \mathcal{V})$ where $\mathcal{U} \circ_\odot \mathcal{V} \in F_*(X \times X)$.

The pair (X,Λ) is called an $(L,*,\odot)$-*quasiuniform convergence space.*

An $(L,*,\odot)$-quasiuniform convergence space is called an $(L,*,\odot)$-*uniform convergence space* if it satisfies the following condition;

(U) $\Lambda(\mathcal{U}) \leq \Lambda(\mathcal{U}^{-1})$.

We say Λ_1 is *finer* than Λ_2 (or Λ_2 is *coarser* than Λ_1) iff $\Lambda_1 \leq \Lambda_2$.

We define $\Lambda_\top, \Lambda_\bot : F_*(X\times X) \to [0,1]$ as follows:

$$\Lambda_\top(\mathcal{W}) = \begin{cases} \top, & \text{if } \mathcal{W} \geq [(x,x)], \forall x \in X \\ \bot, & \text{otherwise.} \end{cases} \qquad \Lambda_\bot(\mathcal{W}) = \top,\ \forall \mathcal{W} \in F(X\times X)$$

Then $\Lambda_\top$ (resp. $\Lambda_\bot$) is the finest (resp. coarsest) $(L,*,\odot)$-quasiuniform convergence structure.

Let (X,Λ_X) and (Y,Λ_Y) be $(L,*,\odot)$-quasiuniform convergence spaces. A map $\psi : (X,\Lambda_X) \to (Y,\Lambda_Y)$ is called *quasiuniformly continuous* if for all $\mathcal{U} \in F_*(X\times X)$, $\Lambda_X(\mathcal{U}) \leq \Lambda_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))$.

Theorem 3.5. *Let (X,Λ_X) be an $(L,*,\odot)$-quasiuniform convergence space. We define a map $\Lambda_X^{-1} : F_*(X\times X) \to L$ as*

$$\Lambda_X^{-1}(\mathcal{U}) = \Lambda_X(\mathcal{U}^{-1})$$

Then

(1) (X,Λ_X^{-1}) *is an $(L,*,\odot)$-quasiuniform convergence space.*

(2) *If $\psi : (X,\Lambda_X) \to (Y,\Lambda_Y)$ is quasiuniformly continuous, then $\psi : (X,\Lambda_X^{-1}) \to (Y,\Lambda_X^{-1})$ is quasiuniformly continuous.*

Proof. (1) (QC1) It is easy because $[(x,x)]^{-1} = [(x,x)]$.

(QC2) If $\mathcal{U} \leq \mathcal{V}$, then $\mathcal{U}^{-1} \leq \mathcal{V}^{-1}$. Thus $\Lambda_X^{-1}(\mathcal{U}) = \Lambda_X(\mathcal{U}^{-1}) \leq \Lambda_X(\mathcal{V}^{-1}) = \Lambda_X^{-1}(\mathcal{V})$.

(QC3) $\Lambda_X^{-1}(\mathcal{U})\odot\Lambda_X^{-1}(\mathcal{V}) = \Lambda_X(\mathcal{U}^{-1})\odot\Lambda_X(\mathcal{V}^{-1}) \leq \Lambda_X(\mathcal{U}^{-1}\odot\mathcal{V}^{-1}) = \Lambda_X^{-1}(\mathcal{U}\odot\mathcal{V})$.

(QC4)

$$\begin{aligned} \Lambda_X^{-1}(\mathcal{U})\odot\Lambda_X^{-1}(\mathcal{V}) &= \Lambda_X^{-1}(\mathcal{V})\odot\Lambda_X^{-1}(\mathcal{U}) = \Lambda_X(\mathcal{V}^{-1})\odot\Lambda_X(\mathcal{U}^{-1}) \\ &\leq \Lambda_X(\mathcal{V}^{-1}\circ_\odot \mathcal{U}^{-1}) = \Lambda_X((\mathcal{U}\circ_\odot\mathcal{V})^{-1}) \\ &= \Lambda_X^{-1}(\mathcal{U}\circ_\odot\mathcal{V}). \end{aligned}$$

(2) $\Lambda_X^{-1}(\mathcal{U}) = \Lambda_X(\mathcal{U}^{-1}) \leq \Lambda_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{U}^{-1})) = \Lambda_Y(((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))^{-1}) = \Lambda_Y^{-1}((\psi\times\psi)^{\Rightarrow}(\mathcal{U}))$.

Example 3.6. Let $X = \{a,b,c\}$ be a set, $(L=[0,1],\leq,\odot,*,0,1)$ an ecl-premonoid with $a*b = a\cdot b$, $a\odot b = a^{\frac{1}{3}}\cdot b^{\frac{1}{3}}$ and $u \in [0,1]^{X\times X}$ defined as follows:

$$u(a,a) = u(b,b) = u(c,c) = 1,\ \ u(a,b) = 0.5, u(b,a) = 0.6,$$

$$u(a,c) = u(c,a) = 0.5, u(b,c) = 0.6, u(c,b) = 0.4.$$

Define $[0,1]$-filter as $\mathcal{U} : [0,1]^{X\times X} \to [0,1]$ as follows:

$$\mathcal{U}(w) = \begin{cases} 1, & \text{if } w = 1_{X\times X}, \\ 0.6^n, & \text{if } u^n \le w \not\ge u^{n-1}, n \in N, \\ 0, & \text{otherwise.} \end{cases}$$

where $u^{n+1} = u^n * u$ and $u^0 = 1_{X\times X}$.

Since $u^n \circ u^n = u^n$, we obtain

$$(\mathcal{U} \circ_{\odot} \mathcal{U})(w) = \begin{cases} 1, & \text{if } w = 1_{X\times X}, \\ 0.6^n \odot 0.6^n, & \text{if } u^n \le w \not\ge u^{n-1}, n \in N,, \\ 0, & \text{otherwise.} \end{cases}$$

$$(\mathcal{U} \odot \mathcal{U})(w) = \begin{cases} 1, & \text{if } w = 1_{X\times X}, \\ 0.6^n \odot 0.6^n, & \text{if } u^n \le w \not\ge u^{n-1}, n \in N, \\ 0, & \text{otherwise.} \end{cases}$$

We define $\Lambda : F_*(X \times X) \to [0,1]$ as follows:

$$\Lambda(\mathcal{W}) = \begin{cases} 1, & \text{if } \mathcal{W} \ge [(x,x)], x \in X \\ 0.5^{[n]}, & \text{if } \mathcal{U}^{[n]} \le \mathcal{W} \not\ge \mathcal{U}^{[n+1]}, n \in N \\ 0, & \text{otherwise.} \end{cases}$$

where $\mathcal{U}^{[n+1]} = \mathcal{U}^{[n]} \odot \mathcal{U}$ and $0.5^{[n+1]} = 0.5^{[n]} \odot 0.5$.

Then Λ is an $(L, *, \odot)$-quasiuniform convergence structure on X.

We obtain $\Lambda^{-1} : F(X \times X) \to [0,1]$ as follows:

$$\Lambda^{-1}(\mathcal{W}) = \begin{cases} 1, & \text{if } \mathcal{W} \ge [(x,x)], x \in X \\ 0.5^{[n+1]}, & \text{if } \mathcal{V}^{[n]} \le \mathcal{W} \not\ge \mathcal{V}^{[n+1]}, \\ 0, & \text{otherwise.} \end{cases}$$

where $\mathcal{V} = \mathcal{U}^{-1}$, $\mathcal{V}^{[n+1]} = \mathcal{V}^{[n]} \odot \mathcal{V}$ and $0.5^{[n+1]} = 0.5^{[n]} \odot 0.5$. Then Λ^{-1} is an $(L, *, \odot)$-quasiuniform convergence structure on X.

Example 3.7. Let $X = \{a,b,c\}$, $([0,1], *)$, $u \in [0,1]^{X\times X}$ and $\mathcal{U}$ as defined in Example 12. We define $\Lambda : F_*(X \times X) \to [0,1]$ as follows:

$$\Lambda(\mathcal{W}) = \begin{cases} 1, & \text{if } \mathcal{W} \ge [(x,x)], \\ 0.6, & \text{if } \mathcal{U} \le \mathcal{W} \not\ge [(x,x)], \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathcal{U} \circ_{\wedge} \mathcal{U} = \mathcal{U} \wedge \mathcal{U} = \mathcal{U} = \mathcal{U}^{-1}$, Λ is an $(L, *, \wedge)$-uniform convergence structure.

Theorem 3.8. *Let $(L, \le, \odot, *)$ be an M-ecl-premonoid. Let $\mathcal{U}$ be a quasiuniform structure on X. We define a map $\Lambda^{\mathcal{U}} : F(X \times X) \to L$ as follows:*

$$\Lambda^{\mathcal{U}}(\mathcal{W}) = \bigwedge_{u \in L^{X\times X}} (\mathcal{U}(u) \to \mathcal{W}(u)).$$

Then

(1) $\Lambda^{\mathcal{U}}$ *is an* $(L,*,\odot)$ *quasiuniform convergence structure.*

(2) *If* $\psi:(X,\mathcal{U}_X)\to(Y,\mathcal{U}_Y)$ *is quasiuniformly continuous, then* $\psi:(X,\Lambda^{\mathcal{U}}_X)\to(Y,\Lambda^{\mathcal{U}}_Y)$ *is quasiuniformly continuous.*

Proof. (QC1) Since $\mathcal{U}\le[(x,x)]$,

$$\Lambda^{\mathcal{U}}([(x,x)])=\bigwedge_{u\in L^{X\times X}}(\mathcal{U}(u)\to[(x,x)](u))=\top.$$

(QC3)

$$\begin{aligned}
&\Lambda^{\mathcal{U}}(\mathcal{W}_1)\odot\Lambda^{\mathcal{U}}(\mathcal{W}_2)\\
&=\Big(\bigwedge\nolimits_{u\in L^{X\times X}}(\mathcal{U}(u)\to\mathcal{W}_1(u))\Big)\odot\Big(\bigwedge\nolimits_{v\in L^{X\times X}}(\mathcal{U}(v)\to\mathcal{W}_2(v))\Big)\\
&\le\bigwedge\nolimits_{u\in L^{X\times X}}\Big((\mathcal{U}(u)\to\mathcal{W}_1(u))\odot(\mathcal{U}(u)\to\mathcal{W}_2(u))\Big)\\
&\le\bigwedge\nolimits_{u\in L^{X\times X}}\Big((\mathcal{U}(u)\odot\mathcal{U}(u))\to(\mathcal{W}_1(u)\odot\mathcal{W}_2(u))\Big)\\
&\le\bigwedge\nolimits_{u\in L^{X\times X}}\Big(\mathcal{U}(u)\to(\mathcal{W}_1\odot\mathcal{W}_2)(u)\Big)\\
&=\Lambda^{\mathcal{U}}(\mathcal{W}_1\odot\mathcal{W}_2).
\end{aligned}$$

(QC4)

$$\begin{aligned}
&\Lambda^{\mathcal{U}}(\mathcal{V}\circ_\odot\mathcal{W})\\
&=\bigwedge\nolimits_{u\in L^{X\times X}}\Big(\mathcal{U}(u)\to(\mathcal{V}\circ_\odot\mathcal{W})(u)\Big)\\
&\ge\bigwedge\nolimits_{u\in L^{X\times X}}\Big((\mathcal{U})\circ_\odot\mathcal{U})(u)\to(\mathcal{V}\circ_\odot\mathcal{W})(u)\Big)\\
&\ge\bigwedge\nolimits_{u\in L^{X\times X}}\Big(\bigvee\nolimits_{u_1\circ u_2\le u}(\mathcal{U}(u_1)\odot\mathcal{U}(u_2))\to(\mathcal{V}\circ_\odot\mathcal{W})(u))\Big)\\
&=\bigwedge\nolimits_{u\in L^{X\times X}}\bigwedge\nolimits_{u_1\circ u_2\le u}\Big(\mathcal{U}(u_1)\odot\mathcal{U}(u_2)\to(\mathcal{V}\circ_\odot\mathcal{W})(u)\Big)\\
&\ge\bigwedge\nolimits_{u\in L^{X\times X}}\bigwedge\nolimits_{u_1\circ u_2\le u}\Big((\mathcal{U}(u_1)\odot\mathcal{U}(u_2))\to(\mathcal{V}(u_1)\odot\mathcal{W}(u_2))\Big)\\
&\ge\bigwedge\nolimits_{u_1\in L^{X\times X}}\bigwedge\nolimits_{u_2\in L^{X\times X}}\Big((\mathcal{U}(u_1)\to\mathcal{V}(u_1))\odot(\mathcal{U}(u_2)\to\mathcal{W}(u_2))\Big)\\
&\ge\Big(\bigwedge\nolimits_{u_1\in L^{X\times X}}(\mathcal{U}(u_1)\to\mathcal{V}(u_1))\Big)\odot\Big(\bigwedge\nolimits_{u_2\in L^{X\times X}}(\mathcal{U}(u_2)\to\mathcal{W}(u_2))\Big)\\
&=\Lambda^{\mathcal{U}}(\mathcal{V})\odot\Lambda^{\mathcal{U}}(\mathcal{W}).
\end{aligned}$$

(2)

$$\begin{aligned}
&\Lambda^{\mathcal{U}_X}_X(\mathcal{W})\to\Lambda^{\mathcal{U}_Y}_Y((\psi\times\psi)^{\Rightarrow}(\mathcal{W}))\\
&\ge\Big(\bigwedge\nolimits_{u\in L^{X\times X}}(\mathcal{U}_X(u)\to\mathcal{W}(u))\Big)\\
&\to\Big(\bigwedge\nolimits_{v\in L^{Y\times Y}}(\mathcal{U}_Y(v)\to(\psi\times\psi)^{\Rightarrow}(\mathcal{W})(v))\Big)\\
&\ge\Big(\bigwedge\nolimits_{v\in L^{Y\times Y}}(\mathcal{U}_X((\psi\times\psi)^{\leftarrow}(v))\to\mathcal{W}((\psi\times\psi)^{\leftarrow}(v)))\Big)\to\\
&\Big(\bigwedge\nolimits_{v\in L^{Y\times Y}}(\mathcal{U}_Y(v)\to(\psi\times\psi)^{\Rightarrow}(\mathcal{W})(v))\Big)
\end{aligned}$$

$$\geq \bigwedge_{v\in L^{Y\times Y}} \Big(\mathcal{U}_X((\psi\times\psi)^{\leftarrow}(v)) \to \mathcal{U}((\psi\times\psi)^{\leftarrow}(v))) \to (\mathcal{U}_Y(v) \to \mathcal{U}((\psi\times\psi)^{\leftarrow}(v)))\Big)$$
$$\geq \bigwedge_{v\in L^{Y\times Y}} \Big(\mathcal{U}_Y(v) \to \mathcal{U}_X((\psi\times\psi)^{\leftarrow}(v))\Big).$$

Example 3.9. Let $X = \{a,b,c\}$, $([0,1],*)$, $u \in [0,1]^{X\times X}$ and $\mathcal{U}$ as defined in Example 12. Since $(X,\mathcal{U})$ is an $(L,*,\wedge)$ is uniform structure and $(L,\leq,\wedge,*)$ is an M-ecl-premonoid, we obtain an $(L,*,\wedge)$-quasiuniform convergence structure $\Lambda^{\mathcal{U}} : F_*(X\times X) \to [0,1]$ as follows:

$$\begin{aligned}\Lambda^{\mathcal{U}}(\mathcal{W}) &= \textstyle\bigwedge_{v\in L^{X\times X}} (\mathcal{U}(v) \to \mathcal{W}(v)) \\ &= (0.6 \to \mathcal{W}(u)) \wedge (0.3 \to \mathcal{W}(u*u))\end{aligned}$$

where $(a \to b) = 1$ if $a \leq b$ and $(a \to b) = b$, otherwise.

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